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Basket Default Swaps Pricing Based on the Normal Inverse Gaussian Distribution

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Abstract

In this paper, a normal inverse Gaussian factor model is developed to describe the fat-tailed feature of the default distribution of reference entities in order to study basket default swaps pricing. Based on this model, the explicit formula for the distribution of the *k*th default time is accurately obtained by making use of order statistics, and the closed forms of the price of BDS at the *k*th default and *m* out of *n* default entities are calculated using the risk-neutral pricing principle.

Mathematics Subject Classification: 91B28, 91G10

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1 Introduction

Credit default swaps (CDS) provides an insurance against credit risks, such as bankruptcy, merge, in which the definition of a credit event has been standardized by ISDA. In CDS contracts a protection buyer pays premiums to a protection seller until credit event occurs or the end of the life of the contract and a protection buyer stops paying premiums to a protection seller. The protection buyer has the right to sell corporate bonds to the protection seller at its par value when the credit event occurs. Credit default swaps can be divided into two categories according to the number of reference entities, the standard CDS with only one reference entity and basket default swaps (BDS) with multiple reference entities.

The key problem about BDS pricing is to derive the joint default probability of reference entities, depending on the credit risk models including a structural model and a reduced-form model. Structural model is based on the option pricing theory from Merton [11], assumed that a firm's capital structure consists of assets and debts, a default event occurs when the value of the asset of the firm is less than the face value of debt. Black and Cox [3] extended the Merton's work from the view of a default trigger point depending on any time or not a maturity of the debt. Zhou [14] derived the closed form solution for BDS pricing using the first passage time of two correlated Brownian motions. Hull and White [6] studied BDS pricing with Monte Carlo simulation method, and got an approximate price of BDS.

In reduced form model, the default of reference entities is not decided by the firm's value, but is determined by external economic conditions of the firm. Duffie [5] developed the first reduced form credit risk model according to a default time and default intensity. Kijima [9] introduced the Vasicek intensity of related process in order to find the closed-form price of the BDS. Iscoe and Kreinin [8] constructed an iterative algorithm, derived the risk neutral probability of default times and the closed solution of the first to default swaps and *k*th to default swaps in the conditional independence framework. Choe and Jang [4] developed the *k*th default time distribution using order statistic for BDS pricing. O 'Kane and Schloegl [12] proposed one factor Gaussian copula to compute the default probability of portfolio credit risk.

However, one factor Gaussian copula-based approach ignores fat tails feature of the default distribution function based on the real market data. In order to make the model fit with the practical situation, Andersen and Sidenius [1] extended one factor Gaussian copula to the model with random recover rate and random loading, Hull and White [7] introduced double t-Copulas, Kalemanova et al.[10] suggested the normal inverse Gaussian (NIG) for CDO pricing, and compared it with normal Copula and double t-Copulas, found that the normal inverse Gaussian distribution could not only improve the computing speed, but also has an elastic dependent structure. Barndorff-Nielsen [2] provided an inverse Gaussian process.

According to the above analysis, this article describes fat-tailed phenomenon with the normal inverse Gaussian random variable in order to improve the computing speed and an elastic dependent structure of BDS' price. The main contribution in this paper is as follows: the first one is to introduce the normal inverse Gaussian distribution for describing fat-tailed character of the default time distribution; the second one is to use order statistic method proposed to calculate the theoretical price of BDS, which simplifies the process of pricing and provides investors with a new pricing method.

The rest of the article is organized as follows. In section 2, the normal inverse Gaussian distribution and its properties are introduced, and an inverse Gaussian factor model and order statistic of the *k*th default time are given. Section 3 considers the closed forms of the price of BDS at the *k*th default and *m* out of *n* reference entities.

2 Normal inverse Gaussian factor model and order statistics

Definition1. If the probability density function of nonnegative random variable Y with parameters $\alpha > 0$, $\beta > 0$ is

$$f_{IG}\left(y;\alpha,\beta\right)=\begin{cases}\dfrac{\alpha}{\sqrt{2\pi\beta}}y^{-\frac{3}{2}}\exp\left(-\dfrac{\left(\alpha-\beta y\right)^2}{2\beta y}\right), y>0 \\ 0, \qquad\qquad\qquad\qquad\qquad y\le 0\end{cases}, \tag{1}$$

then Y is called an inverse Gaussian random variable, and denoted as $Y \sim IG\left(\alpha,\beta\right)$.

Definition 2. If a random variable X follows the normal distribution under the condition of the inverse Gaussian random variable $Y = y$, then X is called a normal inverse Gaussian random variable, and denoted as $X \sim NIG\left(\alpha,\beta,\mu,\sigma\right)$, where $Y \sim IG\left(\sigma\gamma,\gamma^2\right)$ and these parameters $\gamma=\sqrt{\alpha^2-\beta^2}$, $0 \le |\beta| < \alpha$, $\sigma > 0$. The distribution function and probability density function of the normal inverse Gaussian random variable X are denoted by $F_{NIG}\left(\alpha,\beta,\mu,\sigma\right)$ and $f_{NIG}\left(\alpha,\beta,\mu,\sigma\right)$, respectively, and

$$f_{NIG}\left(x;\alpha,\beta,\mu,\sigma\right)=\frac{\sigma\alpha\cdot\exp\left(\sigma\gamma+\beta\left(x-\mu\right)\right)}{\pi\cdot\sqrt{\sigma^2+\left(x-\mu\right)^2}}K_1\left(\alpha\sqrt{\sigma^2+\left(x-\mu\right)^2}\right), \tag{2}$$

where $K_1\left(\omega\right)=\frac{1}{2}\int_0^{\infty}\exp\left(-\frac{1}{2}\omega\left(t+t^{-1}\right)\right)dt$ is the third kind of Bessel functions.

The main properties of the normal inverse random variable are given from the reference [13].

Proposition1. The expectation and variance of the random variable $X \sim NIG\left(\alpha,\beta,\mu,\sigma\right)$ are

$$E[X]=\mu+\sigma\frac{\beta}{\gamma},\quad \mathrm{var}[X]=\sigma\frac{\alpha^2}{\gamma^3}, \tag{3}$$

respectively.

Proposition2. If for any non-zero number $c\in R$, $X\sim NIG(\alpha,\beta,\mu,\sigma)$ and $Y\sim NIG(\alpha,\beta,\mu_2,\sigma_2)$, then

$$cX\sim NIG\left(\frac{\alpha}{c},\frac{\beta}{c},c\mu,c\sigma\right), \tag{4}$$

$$X+Y\sim NIG(\alpha,\beta,\mu_1+\mu_2,\sigma_1+\sigma_2). \tag{5}$$

The following factor model represents both systemic risk and idiosyncratic risk of the company.

Suppose that a random variable X_i is denoted by

$$X_i=\sqrt{\rho}Z+\sqrt{1-\rho}Z_i, \tag{6}$$

where the normal inverse Gaussian random variables

$$Z\sim NIG\left(\alpha,\beta,-\frac{\beta\gamma^2}{\alpha^2},\frac{\gamma^3}{\alpha^2}\right)$$

and

$$Z_i\sim NIG\left(\delta\alpha,\delta\beta,-\delta\frac{\beta\gamma^2}{\alpha^2},\delta\frac{\gamma^3}{\alpha^2}\right),$$

are the common factor representing system risk and the idiosyncratic factor representing the idiosyncratic risk, respectively, the parameters $\gamma=\sqrt{\alpha^2-\beta^2}$ and $\delta=\frac{\sqrt{1-\rho}}{\sqrt{\rho}},\rho>0$. Then the equation (6) is called a normal inverse Gaussian factor model.

According to Proposition 2, we get

$$X_i \sim NIG\left(\frac{1}{\sqrt{\rho}}\alpha, \frac{1}{\sqrt{\rho}}\beta, -\frac{1}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2}\right) \tag{7}$$

with the distribution function $F_{NIG(\frac{1}{\sqrt{\rho}})}(x)$ and the probability density function $f_{NIG(\frac{1}{\sqrt{\rho}})}(x)$.

Assume that $Y_1,\ldots,Y_n$ are the independent and identically distributed (i.i.d.) random variables with absolutely continuous distribution. If these variables are rearranged in an increasing order and if Y^k denotes the k th order statistic, $k=1,\ldots,n$, then we have $Y^1 < Y^2 < \cdots < Y^n$, which are dependent.

Lemma 1. Suppose that $Y_1,\ldots,Y_n$ are the i.i.d. random variables with the distribution function G and the density function g. Then the density function $g^{(k)}$ of the k th order statistic Y^k is given by

$$g^{(k)} = \frac{n!}{(k-1)!(n-k)!} G(x)^{k-1}\left(1-G(x)\right)^{n-k} g(x) \quad . \tag{8}$$

3 Pricing of basket default swaps

The protection seller only pays a compensation to the protection buyer when k th to default swaps occurs and doesn't compensate for the losses of former $k-1$th times. Firstly, the closed form of the price of BDS at the k th default is given by the framework model in this section. Then, m of n the reference entities default is considered for the pricing of BDS.

3.1 The default time and its distribution

Let $\tau_1,\ldots,\tau_n$ be the default times of each one of n reference entities in BDS contract, which have the same and differentiable distribution function F. We use the equation (6) to compute the default distribution function.

Definition3. Let $\tau_i(z)$ ($1 \le i \le n$) denote the unordered default time of the reference entity i under the given condition $Z = z$, which is given by

$$\tau_i(z) = F^{-1}\left(F_{NIG(\frac{1}{\sqrt{\rho}})}\left(\sqrt{\rho}z + \sqrt{1-\rho}Z_i\right)\right). \tag{9}$$

Note that $\tau_1(z),\ldots,\tau_n(z)$ are i.i.d. random variables since $Z_1,\cdots Z_n$ are i.i.d. random variables when $Z = z$.

Theorem1. The distribution function F_z of $\tau_i(z)$, $1 \le i \le n$, is given by

$$F_z(t) = F_{NIG(\delta)}\left((F^{-1}_{NIG(\frac{1}{\sqrt{\rho}})}(F(t)) - \sqrt{\rho}z)/\sqrt{1-\rho}\right), \tag{10}$$

and its density function f_z is

$$f_z(t) = F'(t)\delta\rho\sqrt{\rho(1-\rho)}(\beta\mu - \beta z - \sigma\gamma) \quad . \tag{11}$$

Proof. Note that

$$\begin{aligned} F_z(t) &= P\left(F^{-1}\left(F_{NIG(\frac{1}{\sqrt{\rho}})}\left(\sqrt{\rho}z + \sqrt{1-\rho}Z_i\right)\right) \le t\right) \\ &= P\left(Z_i \le (F^{-1}_{NIG(\frac{1}{\sqrt{\rho}})}(F(t)) - \sqrt{\rho}z)/\sqrt{1-\rho}\right) \quad . \\ &= F_{NIG(\delta)}\left((F^{-1}_{NIG(\frac{1}{\sqrt{\rho}})}(F(t)) - \sqrt{\rho}z)/\sqrt{1-\rho}\right) \end{aligned} \tag{12}$$

Let

$$K(t) = \frac{F^{-1}_{NIG(\frac{1}{\sqrt{\rho}})}\left[F(t)\right] - \sqrt{\rho}z}{\sqrt{1-\rho}} . \tag{13}$$

Then the derivative of $K(t)$ is calculated as

$$\frac{d}{dt}K(t)=\frac{F'(t)}{\sqrt{1-\rho}\,\varphi_{NIG(\frac{1}{\sqrt{\rho}})}\left(F^{-1}_{NIG(\frac{1}{\sqrt{\rho}})}\left[F(t)\right]\right)}. \tag{14}$$

Hence, its density function f_z is obtain by

$$f_z(t)=\frac{d}{dt}F_z(t)=\frac{d}{dt}\int_{-\infty}^{K(t)}\varphi_{NIG(\delta)}(x)dx=\varphi_{NIG(\delta)}\left(K(t)\right)\frac{d}{dt}K(t) \quad , \tag{15}$$

where φ is the standard normal inverse Gaussian density.

Furthermore, submit the equation (13)-(14) into (15), we get the equation (11).

Theorem2. If $\tau^1,\ldots,\tau^n$ denote the order statistics of $\tau_1,\ldots,\tau_i$, then the distribution function F^k of τ^k is

$$F^k(t)=\upsilon\int_{-\infty}^{+\infty}\int_0^t F_z\left(1-F_z(u)\right)^{n-k}\ f_z(u)du\varphi_0(z)(u)^{k-1}dz\,, \tag{16}$$

where $\upsilon=\dfrac{n!}{(k-1)!(n-k)!}$.

Proof. Note that

$$F^k(t)=\mathrm{E}_Z\left[\mathrm{P}(\tau^k\le t\,|\,Z=z)\right]=\int_{-\infty}^{+\infty}\mathrm{P}(\tau^k\le t\,|\,Z=z)\varphi_0(z)dz \quad , \tag{17}$$

where $\varphi_0(z)$ is the density of Z.

According to Lemma 1, we have

$$\begin{aligned}&\mathrm{P}(\tau^k\le t\,|\,Z=z)=\mathrm{P}\left(\tau^k(z)\le t\right)\\&=\int_0^t\upsilon F_z(u)^{k-1}\left(1-F_z(u)\right)^{n-k}\ f_z(u)du\end{aligned}. \tag{18}$$

And submit the equation (18) into the equation (17), we get the equation (16).

Theorem3. Let τ^i and τ^j denote the i th and j th order statistics of $\tau_1,\ldots,\tau_n$, for $1\le i<j\le n$, then

$$P(\tau^i \le t, \tau^j > t) = F^i(t) - F^j(t) \quad . \tag{19}$$

Proof. Note that $P(\tau^i \le t, \tau^j \le t) = P(\tau^j \le t)$, $P(\tau^i > t, \tau^j > t) = P(\tau^i > t)$, and $P(\tau^i > t, \tau^j \le t) = 0$. Therefore

$$\begin{aligned} & P(\tau^i \le t, \tau^j > t) \\ & = 1 - P(\tau^i > t, \tau^j > t) - P(\tau^i \le t, \tau^j \le t) - P(\tau^i > t, \tau^j \le t) . \\ & = 1 - P(\tau^i > t) - P(\tau^j \le t) = F^i(t) - F^j(t) \end{aligned} \tag{20}$$

3.2 *K*th to default swaps

Let $0 = t_0 < t_1 < \cdots t_N = T$ be the premium payments time, T be the time-maturity of the BDS contract with the notional value M and the recovery rate R for each reference entity, $B(0,t) = \exp\left(-\int_0^t f(0,u)du\right)$ be the discount factor with the spot and forward interest rate $f(0,t)$, and $\Delta_{i-1,i} = t_i - t_{i-1}$ be the time period of premium payments in unit year. Let C_k and D_k be a premium leg and a default leg of kth default to BDS, respectively.

The present value of all the premiums paid by the buyer of BDS is

$$C_k = \sum_{i=1}^{N} s_k \Delta_{i-1,i} MB(0,t_i) I_{\{\tau^k > t_i\}} . \tag{21}$$

Then the expected present value of all the premiums is equal to

$$E\left[C_k\right] = \sum_{i=1}^{N} s_k \Delta_{i-1,i} MB(0,t_i)\left(1 - F^k(t_i)\right). \tag{22}$$

The present value of the default leg when the *k*th reference entity default is given by

$$D_k = (1-R) MB(0,\tau^k) I_{\{\tau^k \le T\}} . \tag{23}$$

Then the expected present value of the default leg is given by

$$\begin{aligned} E\left[D_k\right] &= (1-R)M\int_0^T \exp\left(-\int_0^t f(0,u)du\right)dF^k(t) \\ &= (1-R)M\left(B(0,T)F^k(t)+\int_0^T f(0,t)B(0,t)F^k(t)dt\right) \end{aligned}. \tag{24}$$

According to the risk-neutral pricing principle, the expected present value of the premium leg and the expected present value of the default leg should be equal, that is

$$E\left[C_k\right] = E\left[D_k\right]. \tag{25}$$

To submit the equation (22) and (24) into the equation (25), we get the pricing formula of k th to default swaps as follows

$$s_k = \frac{(1-R)M\left(B(0,T)F^k(t)+\int_0^T f(0,t)B(0,t)F^k(t)dt\right)}{\sum_{i=1}^{N}\Delta_{i-1,i}MB(0,t_i)\left(1-F^k(t_i)\right)}. \tag{26}$$

Based on submitting the equation (16) into the equation (26), the price of k th to default swaps can be calculated using the other parameters given in the above assumption.

3.3 *M* out of *n* default swaps

To determine the price of m out of n default swaps, define C_n^m and D_n^m as the premium leg and the default leg of m out of n default swaps, respectively. First of all, the premium payment is calculated. The protection buyer stops to pay the premium to a protection seller when m reference entities default occur. Suppose that the seller of BDS pays the loss amount which is equal to the notional value minus the recovery amount for each default entity at default times $\tau^k,\ldots,\tau^{k+m-1}$. For an each payment date t_i, if $\tau^k > t_i$, the seller will protect against m reference entities in the contract. If $\tau^{k+m-1} \le t_i$, the contract is terminated. If

$\tau^{k+j-1} \le t_i$ and $\tau^{k+j} > t_i$, the seller has to protect reference entities to only $m-j\,(1 \le j \le m)$ names, and the premium is paid on the outstanding nominal. So, the present value of all the premium payments is given by

$$C_n^m = \sum_{i=1}^{N} \Delta_{i-1,i} s B(0,t_i) M \left(m \mathrm{I}_{\{\tau^k > t_i\}} + \sum_{j=1}^{m} (m-j) \mathrm{I}_{\{\tau^{k+j-1} \le t_i, \tau^{k+j} > t_i\}} \right) \tag{27}$$

where s is the spread of m out of n default swaps.

Therefore, the expected present of the premium leg is equal to

$$E\left[C_n^m\right] = \sum_{i=1}^{N} \Delta_{i-1,i} s B(0,t_i) M \left(m \mathrm{P}(\tau^k > t_i) + \sum_{j=1}^{m} (m-j) \mathrm{P}(\tau^{k+j-1} \le t_i, \tau^{k+j} > t_i)\right). \tag{28}$$

Furthermore, using the equation (18), we have

$$\mathrm{P}(\tau^{k+j-1} \le t_i, \tau^{k+j} > t_i) = F^{k+j-1}(t_i) - F^{k+j}(t_i). \tag{29}$$

Hence, we get

$$\begin{aligned} & m\mathrm{P}(\tau^k > t_i) + \sum_{j=1}^{m} (m-j) \mathrm{P}(\tau^{k+j-1} \le t_i, \tau^{k+j} > t_i) \\ &= m(1 - F^k(t_i)) + \sum_{j=1}^{m} (m-j)\left(F^{k+j-1}(t_i) - F^{k+j}(t_i)\right) = \sum_{j=0}^{m-1} \left(1 - F^{k+j}(t_i)\right) \end{aligned}. \tag{30}$$

To submit the equation (30) into (28), one has

$$E\left[C_n^m\right] = \sum_{i=1}^{N} \Delta_{i-1,i} s B(0,t_i) M \sum_{j=0}^{m-1} \left(1 - F^{k+j}(t_i)\right). \tag{31}$$

On the other hand, the present value of the default leg is written by

$$D_n^m = \sum_{j=0}^{m-1} (1-R) M (B(0,\tau^{k+j}) \mathrm{I}_{\{\tau^{k+j} < T\}}. \tag{32}$$

The expected present of the premium leg is equal to

$$E\left[D_n^m\right] = \sum_{j=0}^{m-1} (1-R) M \left(B(0,T) F^{k+j}(T) + \int_0^T f(0,t) B(0,t) F^{k+j}(t) dt\right). \tag{33}$$

To avoid the arbitrage, we have

$$E\left[C_n^m\right] = E\left[D_n^m\right]. \tag{34}$$

Submit the equation (31) and (33) into the equation (34), the value of m out of n default is obtained as

$$s=\frac{\sum_{j=0}^{m-1}(1-R)\left(B(0,T)F^{k+j}(T)+\int_0^T f(0,t)B(0,t)F^{k+j}(t)dt\right)}{\sum_{j=0}^{m-1}\sum_{i=1}^{N}\Delta_{i-1,i}B(0,t_i)\left(1-F^{k+j}(t_i)\right)} \quad . \tag{35}$$

Note that if the value of F^k and other relevant variables into (35), then we can derive the price of m out of n default swaps.

4 Conclusion

The default of a reference entity in the BDS contract could cause the default of another reference entity. The distribution of the default time shows fat-tailed character Because of the inaccurate data and incomplete information. We extend one factor Gaussian copula to one factor normal inverse Gaussian (NIG) model to depict the fat-tailed phenomenon. Based on one factor normal inverse Gaussian model, the explicit formula for the distribution of the *k*th default time is accurately obtained by making use of the method for calculating of order statistics. On this basis of the above analysis, the closed forms of the price of BDS at the *k*th default and m out of n reference entities are calculated by using the risk-neutral pricing principle. The model and the method proposed can describe default times' fat-tailed feature and simplify the progress of BDS pricing, which provides investors with a new pricing method.

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