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Homotopy Perturbation Method and Variational Iteration Method for Voltra Integral Equations

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Abstract

In this work, homotopy perturbation method and variational iteration method are employed to solve Volterra integral equations. The results reveal that the homotopy perturbation method and variational iteration method are very effective and simple. But at the end it is observed that homotopy perturbation method in some of the issues lead to exact solution and variational iteration method lead to limit solution.

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1 Introduction

Non-linear phenomena, that appear in many application in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modeled by PDEs and by integral equations as well. In

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recent years, a large amount of literature developed concerning the modified decomposition method introduced by Wazwaz [8] by applying it to a large size of applications in applied sciences. A new perturbation method called homotopy perturbation method (HPM) was proposed by He in 1997 and systematical description in 2000 which is, in fact, coupling of the traditional perturbation method and homotopy in topology [5]. Until recently, the application of the HPM [8] in non-linear problems has been developed by scientists and engineers, because this method is the most effective and convenient ones for both weakly and strongly non-linear equations. Variational iteration method [10,11,12] is a powerful device for solving various kinds of equations, linear or non-linear. Volterra integral equations have been solved by classical numerical and theoretical methods [13,15]. In this paper, these methods are applied for the Voltra integral equations. The general form of these integral equations is given by

$$h(x)u(x) = f(x) + \int_0^x K(x,y)F(u(y))dy \ , \quad 0 \le x \le T$$

Where K(x, t) is the kernel of integral equation, f and F are known functions and u(x) is the unknown, solution of integral equation, which we are going to find it. We termed the second kind when h(x)=1,

$$\mathrm{u(x) = f(x) + \int_0^x K(x,y)F(u(y))dy} \quad , \ 0 \le \mathrm{x} \le \mathrm{T} \qquad (1.1)$$

and the first kind when h(x)=0 ,

$$-\mathrm{f(x) = \int_0^x K(x,y)F(u(y))dy} \quad , \ 0 \le \mathrm{x} \le \mathrm{T}$$

2 Preliminary Notes

2.1 Homotopy Perturbation Method

Consider a non-linear equation in the form

$$L_u + N_u = 0,$$

where *L* and *N* are linear operator and non-linear operator, respectively. In order to use the homotopy perturbation, a suitable construction of a homotopy equation is of vital importance. Generally, a homotopy can be constructed in the form

$$L_u + p(N_u + N_u - L_u) = 0,$$

where L can be a linear operator or a simple non-linear operator, and the solution of $L_u= 0$ with possible some unknown parameter can best describe the original non-linear system. For example, for a non-linear oscillator we can choose
$L_u = u + \omega^2 u$, where ω is the frequency of the non-linear oscillator.
The non-linear Voltra integral equations are given by

$$u(x) = f(x) + \int_0^x K(x,y)\{R(u(y)) + N(u(y))\}dy \quad , \; 0 \le x \le T \tag{2.1}$$

u(x) is a unknown function that will be determined, K(x,y) is the kernel of the integral equation, f(x) is an analytic function, R(u) and N(u) are linear and non-linear functions of u, respectively [3,4]. To illustrate the HPM, we consider (2.1) as

$$L(x) = u(x) - f(x) - \int_0^x K(x,y)\{R(u(y)) + N(u(y))\}dy = 0 \tag{2.2}$$

As a possible remedy, we can define $H(u,p)$ by

$$H(u,0) = F(u) \quad , \quad H(u,1) = L(u),$$

where F(u) is an integral operator with known solution u_0 , which can be obtained easily. Typically, we may choose a convex homotopy by

$$H(u,p) = (1- p)F(u) + pL(u) \tag{2.3}$$

and continuously trace an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(U,1)$. The embedding parameter p monotonically increase from zero to unit as the trivial problem L(u) = 0.
The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter.

The HPM uses the homotopy parameter p as expanding parameter to obtain

$$U = \lim_{p \to 1} u = u_0 + pu_1 + p^2 u_2 + \ldots \tag{2.4}$$

When $p \to 1$, (2.4) corresponds to (2.3) becomes the approximate solution of (2.2), i.e. ,

$$U = \lim_{p\to 1} u = u_0 + u_1 + u_2 + ... \tag{2.5}$$

The series (2.5) is convergent for most cases, and also the rate of convergent depends on L(u).

2.2 Variational Iteration Method

For solving equation (1.1) by variational iteration method, first we differentiate once from both sides of equation (1.1), with respect to *x*:

$$u'(x) = f'(x) + k(x,x)F(u(x)) + \int_0^x \frac{\partial K(x,y)}{\partial x} F(u(y))dy \tag{2.6}$$

Now we apply Variational iteration method to equation (2.6). According to this method correction functional can be written in the following form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(u_n'(s) - f'(s) - K(s,s)F(\tilde{u}_n(s)) - \int_0^s \frac{\partial K(s,y)}{\partial s} F(\tilde{u}_n(y))dy \right)ds \tag{2.7}$$

To make the above correction functional stationary with respect to u_n, we have:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \lambda\left(u_n'(s) - f'(s) - K(s,s)F(\tilde{u}_n(s)) - \int_0^s \frac{\partial K(s,y)}{\partial s} F(\tilde{u}_n(y))dy \right)ds =$$

$$= \delta u_n(x) + \int_0^x \lambda(s)\delta(u_n'(s))ds = \delta u_n(x) + +\lambda(x)\delta u_n(x) + \int_0^x \lambda'(s)\delta u_n(s)ds = 0$$

From the above relation for any δu_n, we obtain the Euler-Lagrange equation:

$$\lambda'(s) = 0 \tag{2.8}$$

With the following natural boundary condition:

$$\lambda(x) + 1 = 0 \tag{2.9}$$

Using equations (2.8) and (2.9), Lagrange multiplier can be identified optimally as follows:

$$\lambda(s) = \ 1 \tag{2.10}$$

Substituting the identified Lagrange multiplier into equation (2.7) we obtain the following iterative relation:

$$u_{n+1}(x)=u_n(x)-\int_0^x\left(u_n'(s)-f'(s)-K(s,s)F(u_n(s))-\int_0^s\frac{\partial K(s,y)}{\partial s}F(u_n(y))dy\right)ds \qquad (2.11)$$

By starting from $u_0(x)$, we can obtain the exact solution or an approximate solution to the equation (1.1). Also in some Volterra integral equations by differentiating from integral equation, for example when the kernel is independent of *x*, we obtain a differential equation and we can solve differential equation by variational iteration method.

3 Numerical Examples

This section contained 2 example of linear and non-linear Volterra integral equation.

Example 1. Consider the following linear Volterra integral equation with the exact solution u(x) = sin x[13]

$$u(x)=x-\int_0^x(x-y)u(y)dy \qquad (3.1)$$

Homotopy perturbation method:

We define

$$F(u)=u(x)-x\ ,$$
$$L(u)=u(x)-x+\int_0^x(x-y)u(y)dy$$

and substituting F(u) and L(u) in (2.3) and equating the terms with identical power of p, we obtain

$$p^0\ :u_0(x)=x$$
$$p^1\ :u_1(x)=-\int_0^x(x-y)u_0(y)dy=-\int_0^x(x-y)ydy=-\frac{x^3}{3}$$
$$p^2\ :u_2(x)=-\int_0^x(x-y)u_1(y)dy=-\int_0^x(x-y)\left(-\frac{y^3}{3}\right)dy=-\frac{x^5}{5!}$$

$$p^3 : u_3(x) = -\int_0^x (x-y)u_2(y)dy = -\int_0^x (x-y)\left(-\frac{y^{53}}{5!}\right)dy = -\frac{x^7}{7!}$$

$\vdots$

$$p^k : u_k(x) = -\int_0^x (x-y)u_{k-1}(y)dy = \sum_{i=0}^{k} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$$

with using (2.5) we have:

$$U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + u_3(x) + ... = \sin(x)$$

Variational Iteration Method:

The corresponding iterative relation (2.11) for this example can be constructed as:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u'_n(s) - 1 + \int_0^s u_n(y)dy \right) ds \tag{3.2}$$

by taking $u_0(x) = x$, we derive the following results:

$$u_1(x) = x - \frac{x^3}{3!}$$

$$u_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$u_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

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.

$$u_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$$

Thus, we have

$$u(x) = \lim_{n \to \infty} u_n = \sin x$$

which is the exact solution

Example 2. Consider the following non-linear Volterra integral equation of first kind with the exact solution $u(x) = \sec x$[14]

$$u(x) = \sec x + \tan x + x - \int_0^x (1 + u^2(y))dy \tag{3.3}$$

Homotopy perturbation method:

We define

$$F(u) = u(x) - \sec x \ ,$$

$$L(u) = u(x) - \sec x - \tan x - x + \int_0^x (1 - u^2(y))dy$$

and substituting F(u) and L(u) in (2.3) and equating the terms with identical power of p , we obtain:

$$p^0 : u_0(x) = \sec x$$

$$p^1 : u_1(x) = -\tan x - x + \int_0^x (1 + u_0^2(y))dy = 0$$

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$$p^{k+2} : u_{k+2}(x) = -\tan x - x + \int_0^x (1 + u_{k+1}^2(y))dy = 0$$

such that $k \geq 0$.With using (2.5) we have

$$U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + u_3(x) + ... = \sec(x)$$

Variational Iteration Method:

By differentiating once from integral equation (3.3), we obtain the following differential equation

$$u'(x) + u^2(x) - \tan x \sec x - \tan^2 x - 1 = 0 \qquad (3.4)$$

by applying variational iteration method for equation (3.4), we derive the following iterative formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(s) + u_n^2(s) - \tan s \sec s - \tan^2 s - 1)ds \qquad (3.5)$$

Consider initial approximation $u_0(x) = \sec x$ and by the iterative formula (3.5), we get:

$$u_1(x) = \sec x$$

$$u_2(x) = \sec x$$

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therefore, the exact solution can be recognized easily.

4 Conclusion

In this work, homotopy perturbation method and variational iteration method have been successfully applied to find the solution of Volterra integral equations. The two methods can be concluded that the method is very powerful and efficient techniques in finding exact solutions or approximate solutions for wide classes of problems.

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