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# Computation the Exponential Functions of Matrices And Similar Matrices

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**Abstract**

In this thesis we apply some methods of the similarity to compute the matrix exponential functions. Finally, new results of computation of the matrix exponential by using the similarity are obtained.

**Mathematics Subject Classification**: 15A18; 15A60; 26C10; 30C15
**Keywords**: Exponential function; Similar matrix; Computation matrix.

## 1 Introduction

The exponential function of matrices is a very important subclass of functions of matrices that has been studied extensively in the last 50 years.
The matrix exponential is a function on square matrices analogous to the ordinary exponential function. Let $A \in M_n$, The exponential of $A$ denoted by $e^A or \exp(A)$, is the $n \times n$ matrix given by the power series

$$e^A = \sum_{K=0}^{\infty} \frac{A^K}{K!}$$

The above series always converges, so the exponential of $A$ is well-defined. Note that if $A$ is $1 \times 1$ matrix, the matrix exponential of $A$ corresponds with the ordinary exponential of $A$ thought of as a number.

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The numerical evaluation of the exponential of a matrix is of some importance because of its occurrence in many physical, engineering, and economics applications.
In this paper we obtained new results of computation of the matrix exponential by using the similarity.

## 2 New Results

**Theorem 2.1**

Let $A,B \in M_n$ and $B = Q^{-1}AQ$, where $Q = diag(r_1,...,r_n)$ s.t $r_i \in \mathbb{R}^+ \ \forall i = 1,...,n$ . And the exponential function of $A$ is given by Hamilton method as follows

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k A^k \text{. Then } e^{Bt} = \sum_{k=0}^{n-1} \alpha_k B^k .$$

**Proof :**

Let $A = \left[a_{ij}\right] \in M_n, \ \forall i, j = 1,...,n$ . And it is given that $B = Q^{-1}AQ$ , where $Q = diag(r_1,...,r_n)$ s.t $r_i \in \mathbb{R}^+ \ \forall i = 1,...,n$

$$B = Q^{-1}AQ = \begin{pmatrix} \frac{1}{r_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{r_n} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_n \end{pmatrix}$$

$$B = \left[\frac{r_j}{r_i} a_{ij}\right]. .$$

Its clear that $B$ is similar to $A$ , so the eigenvalues of $B$ are the same eigenvalues of $A$ Say $\lambda_1, \lambda_2, \ldots, \lambda_n$ .

Since the matrix exponential is a simply one case of an analytic function as described in the Cayley-Hamilton method to determine the analytic functions of a matrix ,then

$$e^{Bt} = \sum_{k=0}^{n-1} \alpha_k B^K \ .$$

Where $\alpha_i$ 's are described from the equation gives by the eigenvalues of $B$ .

$$e^{\lambda_i t} = \sum_{K=0}^{n-1} \alpha_k \lambda_i^{\ k}$$

**Example** Let $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ , $Q = (r_1 \ , \ r_2)$ s.t $r_i \in \mathbb{R}^+$

$$\forall i = 1,2$$

Find $e^{Bt}$ s.t $B = Q^{-1} A Q$ .

**Solution :**

$$B = \begin{pmatrix} \frac{1}{r_1} & 0 \\ 0 & \frac{1}{r_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

$$B=\begin{pmatrix} 0 & \frac{r_2}{r_1} \\ \frac{-2r_1}{r_2} & -3 \end{pmatrix}$$

The characteristic equation is $s^2+3s+2=0$ , and the eigenvalues are $\lambda_1=-1$ , $\lambda_2=-2$.

$$e^{Bt}=\alpha_\circ I+\alpha_1 B$$

So, $\lambda_1=-1$ and $\lambda_2=-2$

$$e^{-t}=\alpha_\circ-\alpha_1$$
$$e^{-2t}=\alpha_\circ-2\alpha_1$$

Or $\alpha_\circ=(2e^{-t}-e^{-2t})$ and $\alpha_1=(e^{-t}-e^{-2t})$. Then

$$e^{Bt}=(2e^{-t}-e^{-2t})I+(e^{-t}-e^{-2t})B$$

$$e^{Bt}=\begin{pmatrix} 2e^{-t}-e^{-2t} & \frac{r_2}{r_1}e^{-1}-\frac{r_2}{r_1}e^{-2t} \\ \frac{-2r_1}{r_2}e^{-t}+\frac{2r_1}{r_2}e^{-2t} & -e^{-t}+2e^{-2t} \end{pmatrix} \quad \ldots\ldots(5)$$

**Corollary** If we let $r_1=1$ , $r_2=1$. With applying equation (5) , then

$$B=\begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix}.$$

And hence, $e^{Bt} = \begin{pmatrix} 2e^{-t} - e^{-2t} & 2e^{-1} - 2e^{-2t} \\ -e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$

**Corollary** If we let $r_1 = r_2 = r$ . With applying equation (4.2.1) , then

$$B = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \text{.And hence, } e^{Bt} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-1} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}.$$

**Theorem 2.2**

Let $A \in M_n$ diagonalizable matrix and $f$ analytic function on a domain that contains the eigenvalues of $A$ . Then

$f(A) = Xf(B)X^{-1}$.Where $A = XBX^{-1}$ and $f$ is defined by the Newton's divided difference interpolations

**Proof** : We have

$$(A - \lambda_j I) = (XBX^{-1} - \lambda_j I) = X(B - \lambda_j I)X^{-1}$$

Hence, $f(A) = \sum_{i=1}^{n} f(\lambda_1, \ldots, \lambda_i) \prod_{j=1}^{i-1} (A - \lambda_j I)$

$$= \sum_{i=1}^{n} f(\lambda_1, \ldots, \lambda_i) \prod_{j=1}^{i-1} (XBX^{-1} - \lambda_j I)$$

$$= \sum_{i=1}^{n} f(\lambda_1, \ldots, \lambda_i) \prod_{j=1}^{i-1} X(B - \lambda_j I)X^{-1}$$

$$= X\left(\sum_{i=1}^{n} f(\lambda_1, \ldots, \lambda_i) \prod_{j=1}^{i-1} (B - \lambda_j I)\right)X^{-1}$$

$f(A) = Xf(B)X^{-1}$.

**Example .**Let $A = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, X = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and

$f(x) = e^x$ .Find $e^A$ .

Solution :

Since $A = XBX^{-1}$, we can find $e^A$ by above theorem . The eigenvalues of $B$ are $4,3$

$f(4) = e^4$ and $f(3) = e^3$.

let $(\lambda_\circ, f(\lambda_\circ)) = (4, e^4)$ and $(\lambda_1, f(\lambda_1)) = (3, e^3)$

Then by definition of the Newton‘s divided difference interpolations

We have , $f(B) = f(\lambda_\circ)I + f(\lambda_\circ, \lambda_1)(B - \lambda_\circ I)$

$$e^B = \begin{pmatrix} e^4 & 0 \\ 0 & e^4 \end{pmatrix} + \frac{e^3 - e^4}{3-4}\left\{\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right\} = \begin{pmatrix} e^4 & 0 \\ 0 & e^3 \end{pmatrix}$$

$$e^A = Xe^B X^{-1}$$

$$e^A = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^4 & 0 \\ 0 & e^3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^4 - e^3 & e^4 - e^3 \\ -2e^4 + 2e^3 & -e^4 + 2e^3 \end{pmatrix}$$

**Theorem 2.3**

If $A,B \in M_n$ such that $A = XBX^{-1}$ , where $X$ is a nonsingular matrix , then

$$e^A = Xe^B X^{-1}.$$

**Proof** : We know that $e^A = \lim_{n\to\infty}(I + \frac{A}{n})^n$ and $A = XBX^{-1}$, where $X$ is a nonsingular matrix .

So, we have

$$(I + \frac{A}{n})^n = (I + \frac{XBX^{-1}}{n})^n = X\,(I + \frac{B}{n})^n X^{-1}$$

Hence,

$$\lim_{n\to\infty}(I + \frac{A}{n})^n = \lim_{n\to\infty}(I + \frac{XBX^{-1}}{n})^n = \lim_{n\to\infty} X\,(I + \frac{B}{n})^n X^{-1}$$
$$= X\,(\lim_{n\to\infty}(I + \frac{B}{n})^n)X^{-1}$$

$$\therefore e^A = Xe^B X^{-1}$$

We can evaluate the previous example by using the definition of $e^A$ which defined as a limit of power

$$e^A = \lim_{n\to\infty}(I + \frac{A}{n})^n$$

as the following example .

**Example**

Let $A=\begin{pmatrix}5 & 1\\ -2 & 2\end{pmatrix}, B=\begin{pmatrix}4 & 0\\ 0 & 3\end{pmatrix}, X=\begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}$ and $f(x)=e^x$

Find $e^A$.

**Solution** :

Since $A=XBX^{-1}$, we can find $e^A$ by above theorem .

The eigenvalues of $B$ are $4,3$

Hence, $B=\begin{pmatrix}4 & 0\\ 0 & 3\end{pmatrix}$

$$(I+\frac{Bt}{n})^n=\begin{pmatrix}(1+\frac{4t}{n})^n & 0\\ 0 & (1+\frac{3t}{n})^n\end{pmatrix}.$$

Then $$\lim_{n\to\infty}(I+\frac{Bt}{n})^n=\begin{pmatrix}\lim\limits_{n\to\infty}(1+\frac{4t}{n})^n & 0\\ 0 & \lim\limits_{n\to\infty}(1+\frac{3t}{n})^n\end{pmatrix}$$

$$e^{Bt}=\begin{pmatrix}e^{4t} & 0\\ 0 & e^{3t}\end{pmatrix}$$

So, $e^A=Xe^BX^{-1}$

$$e^A = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^4 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^4 - e^3 & e^4 - e^3 \\ -2e^4 + 2e^3 & -e^4 + 2e^3 \end{pmatrix}$$

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