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On Almost Increasing Sequences For Generalized Absolute Summability

W. T. Sulaiman[1]

Abstract

A general result concerning absolute summability of infinite series by quasi-power increasing sequence is proved. Our result gives correction and improvement to the result of Savas and Sevli [2].

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1 Introduction

Let $\sum a_n$ be an infinite series with partial sum (s_n), A denote a lower triangular matrix . The series $\sum a_n$ is said to be absolutely A-summable of order $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty,$$

where

$$T_n = \sum_{v=0}^{n} a_{nv} s_v .$$

[1] Department of Computer Engineering College of Engineering University of Mosul, Iraq, email: waadsulaiman@hotmail.com

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The series $\sum a_n$ is summable $|A,\delta|_k$, $k \geq 1,\ \delta \geq 0,$ if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |T_n - T_{n-1}|^k < \infty. \tag{1}$$

A positive sequence $\gamma = (\gamma_n)$ is said to be a quasi-β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$K\, n^{\beta} \gamma_n \geq m^{\beta} \gamma_m \tag{2}$$

Holds for all $n \geq m \geq 1$. It may be mentioned that every almost increasing sequence is a quasi -β-power increasing sequence for any nonnegative β, but the converse need not be true.

Two lower triangular matrices $\overline{A}$ and $\hat{A}$ are associated with A as follows

$$\overline{a}_{nv} = \sum_{r=v}^{n} a_{nr}, \quad n, v = 0,1,..... , \tag{3}$$

$$\hat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \quad n = 1, 2, ..., \qquad \hat{a}_{00} = \overline{a}_{00} = a_{00}\,.$$

Savas and Sevli [2] proved the following result.

Theorem1.1. Let A be a lower triangular matrix with nonnegative entries satisfying

$$a_{n-1,v} \geq a_{n,v} \quad \text{for} \quad n \geq v+1,$$

$$\overline{a}_{n0} = 1, \quad n = 0,1,.... ,$$

$$na_{nn} = O(1),\ 1/na_{nn} = O(1), \quad \text{as } n \to \infty,$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{v,n+1} = O(a_{nn}), \tag{4}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k} |\Delta_v \hat{a}_{nv}| = O(v^{\delta k} a_{vv}), \tag{5}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k} \hat{a}_{n,v+1} = O(v^{\delta k}), \tag{6}$$

and let (β_n) and (λ_n) be sequences such that

$$\left|\Delta\lambda_n\right| \le \beta_n , \tag{7}$$

$$\beta_n \to 0 \text{ as } n \to \infty, \tag{8}$$

If (X_n) is a quasi - β -increasing sequence satisfying

$$\sum_{n=1}^{m} n^{\delta k-1}\left|s_n\right|^k = O(X_m), \; m \to \infty, \tag{9}$$

$$\sum_{n=1}^{\infty} nX_n\left|\Delta\beta_n\right| < \infty, \tag{10}$$

$$\left|\lambda_n\right| X_n = O(1), \tag{11}$$

then the series $\sum a_n\lambda_n$ is summable $\left|A,\delta\right|_k$, $k \ge 1,\; 0 < \delta \le 1/k$.

We name the following condition

$$\sum_{n=1}^{m} \frac{n^{\delta k-1}}{X_n^{k-1}}\left|s_n\right|^k = O(X_m), \; m \to \infty . \tag{12}$$

Remark 1. It may be mentioned that in the proof of theorem 1.1, an incorrect step through the estimation of I_2. The author consider $(v\beta_v)$ is bounded regarding this follows from the fact $v\beta_v X_v = O(1)$. This not true, as for X_n is $\beta - quasi$, we may take $X_v = v^{-\beta}$, which implies via $v\beta_v X_v = O(1)$ that $(v\beta_v)$ is not bounded.

Therefore the proof of theorem 1.1 is not valid.

2 Lemmas

Lemma 2.1. Condition (12) is weaker than (9) when X_n is non-decreasing.

Proof. If (9) holds, then we have

$$\sum_{n=1}^{m} \frac{\left|s_n\right|^k}{nX_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m}\frac{1}{n}\left|s_n\right|^k = O\left(X_m\right),$$

while if (12) is satisfied then,

$$\sum_{n=1}^{m}\frac{1}{n}|s_n|^k = \sum_{n=1}^{m}\frac{1}{nX_n^{k-1}}|s_n|^k X_n^{k-1}$$

$$= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{|s_v|^k}{vX_v^{k-1}}\right)\Delta X_n^{k-1} + \left(\sum_{n=1}^{m}\frac{|s_n|^k}{nX_n^{k-1}}\right)X_m^{k-1}$$

$$= \mathrm{O}(1)\sum_{n=1}^{m-1} X_n\left|\Delta X_n^{k-1}\right| + \mathrm{O}(X_m)X_m^{k-1}$$

$$= \mathrm{O}(X_{m-1})\sum_{n=1}^{m-1}\left(X_{n+1}^{k-1} - X_n^{k-1}\right) + \mathrm{O}\left(X_m^k\right)$$

$$= \mathrm{O}(X_{m-1})\left(X_m^{k-1} - X_1^{k-1}\right) + \mathrm{O}\left(X_m^k\right)$$

$$= \mathrm{O}\left(X_m^k\right).$$

Therefore (9) implies (12) but not conversely .

Remark 2.

1. Condition (9) has been replaced by (12) which is better in the following sense
   (a). If X_n is non-decreasing, (12) is weaker than (9) (see lemma 2.1)
   (b) The more advantage of our conditions is to obtain the desired result without any loss of powers through estimations. As an example the proof via condition (9) impose to deal with $|\lambda_n|^k$ *as* $|\lambda_n|^k = |\lambda_n|^{k-1}|\lambda_n| = \mathrm{O}(|\lambda_n|)$, loosing $|\lambda_n|^{k-1}$ as considered to be $\mathrm{O}(1)$. We have no such case via condition (12).
2. Condition (4) is eliminated.

Lemma 2.2. Conditions (8) and (10) imply

$$mX_m\beta_m = \mathrm{O}(1), \; m \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty}\beta_n X_n = \mathrm{O}(1). \tag{14}$$

Proof. As $\beta_n \to 0$, and $n^{\beta} X_n$ is non-decreasing, we have

$$nX_n\beta_\nu = n^{1-\beta}\, n^{\beta}\, X_n \sum_{\nu=n}^{\infty} \Delta\beta_\nu$$

$$= \mathrm{O}(1)\, n^{1-\beta} \sum_{\nu=n}^{\infty} \nu^{\beta} X_\nu \left|\Delta\beta_\nu\right|$$

$$= \mathrm{O}(1) \sum_{\nu=n}^{\infty} \nu^{1-\beta}\, \nu^{\beta} X_\nu \left|\Delta\beta_\nu\right|$$

$$= \mathrm{O}(1) \sum_{\nu=n}^{\infty} \nu X_\nu \left|\Delta\beta_\nu\right| = \mathrm{O}(1).$$

This proves (13). To prove (14), we observe that

$$\sum_{\nu=1}^{m} X_\nu \beta_\nu = \sum_{\nu=1}^{m-1}\left(\sum_{r=1}^{\nu} X_r\right)\Delta\beta_\nu + \left(\sum_{\nu=1}^{m} X_\nu\right)\beta_m$$

$$= \mathrm{O}(1)\sum_{\nu=1}^{m-1}\left(\sum_{r=1}^{\nu} r^{-\beta}\, r^{\beta} X_r\right)\left|\Delta\beta_\nu\right| + \mathrm{O}(1)\left(\sum_{\nu=1}^{m} \nu^{-\beta}\, \nu^{\beta} X_\nu\right)\beta_m$$

$$= \mathrm{O}(1)\sum_{\nu=1}^{m-1} \nu^{\beta}\, X_\nu \left|\Delta\beta_\nu\right| \sum_{r=1}^{\nu} r^{-\beta-\epsilon}\, r^{\epsilon}$$

$$+ \mathrm{O}(1) m^{\beta} X_m\, \beta_m \sum_{\nu=1}^{m} \nu^{-\beta-\epsilon}\, \nu^{\epsilon}, \quad \epsilon < 1-\beta$$

$$= \mathrm{O}(1)\sum_{\nu=1}^{m-1} \nu^{\beta}\, X_\nu \left|\Delta\beta_\nu\right| \nu^{\epsilon} \sum_{r=1}^{\nu} r^{-\beta-\epsilon}$$

$$+ \mathrm{O}(1) m^{\beta} X_m\, \beta_m\; m^{\epsilon} \sum_{\nu=1}^{m} \nu^{-\beta-\epsilon}$$

$$= \mathrm{O}(1)\sum_{\nu=1}^{m} \nu^{\beta+\epsilon} X_\nu \left|\Delta\beta_\nu\right| \left(\int_1^{\nu} u^{-\beta-\epsilon} du\right) + \mathrm{O}(1) m^{\beta+\epsilon} X_m \beta_m \left(\int_1^{m} u^{-\beta-\epsilon} du\right)$$

$$= \mathrm{O}(1)\sum_{\nu=1}^{m} \nu\, X_\nu \left|\Delta\beta_\nu\right| + \mathrm{O}(1) m\, X_m \beta_m$$

$$= \mathrm{O}(1).$$

Lemma 2.3 [1]. Let A be as defined in theorem 1.1, then

$$\hat{a}_{n,\nu+1} \le a_{nn} \quad \text{for } n \ge \nu+1.$$

3 Main Result

Theorem 3.1. Suppose all conditions of theorem 1.1 are satisfied except condition (9) is replaced by condition (12), and condition (4) is removed, then the series $\sum a_n\lambda_n$ is summable $|A,\delta|_k$, $k \ge 1,\quad 0<\delta \le 1/k$.

Proof. Let x_n be the nth term of the *A*-transform of the series $\sum a_n\lambda_n$. By definition, we have

$$x_n = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \overline{a}_{nv}\lambda_v a_v ,$$

and hence

$$T_n := x_n - x_{n-1} = \sum_{v=0}^{n} \hat{a}_{nv}\lambda_v a_v .$$

Applying Abel's transformation,

$$T_n = a_{nn}\lambda_n s_n + \sum_{v=1}^{n-1} \Delta_v \hat{a}_{nv}\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v s_v = T_{n1} + T_{n2} + T_{n3} .$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}\left|T_{nj}\right|^k < \infty, \quad j = 1,2,3.$$

Applying Holder's inequality, we have

$$\sum_{n=1}^{m} n^{\delta k+k-1}\left|T_{n1}\right|^k = \sum_{n=1}^{m} n^{\delta k+k-1}\left|a_{nn}\lambda_n s_n\right|^k$$

$$\le \sum_{n=1}^{m} (na_{nn})^k \frac{n^{\delta k-1}}{X_n^{k-1}} |s_n|^k |\lambda_n| (|\lambda_n| X_n)^{k-1}$$

$$= \mathrm{O}(1) \sum_{n=1}^{m} \frac{n^{\delta k-1}}{X_n^{k-1}} |s_n|^k |\lambda_n|$$

$$= \mathrm{O}(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| \sum_{v=1}^{n} \frac{v^{\delta k-1}}{X_v^{k-1}} |s_v|^k + \mathrm{O}(1)\ |\lambda_m| \sum_{n=1}^{m} \frac{n^{\delta k-1}}{X_n^{k-1}} |s_n|^k$$

$$= \mathrm{O}(1) \sum_{n=1}^{m-1} \beta_n X_n + \mathrm{O}(1) |\lambda_m| X_m = \mathrm{O}(1).$$

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n2}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\Delta_v \hat{a}_{nv}\lambda_v s_v\right|^k$$

$$\le \sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}||\lambda_v|^k|s_v|^k\left(\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}||\lambda_v|^k|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m} n^{\delta k}|\Delta_v \hat{a}_{nv}|$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k}a_{vv}|\lambda_v|^k|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k-1}}{X_v^{k-1}}|s_v|^k|\lambda_v|(|\lambda_v|X_v)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k-1}}{X_v^{k-1}}|s_v|^k|\lambda_v|$$

$= O(1)$, as in the case of T_{n1}.

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n3}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v s_v\right|^k$$

$$\le \sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{v=1}^{n-1}(\hat{a}_{n,v+1})^k|\Delta\lambda_v||s_v|^k X_v^{1-k}\left(\sum_{v=1}^{n-1}|\Delta\lambda_v|X_v\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{v=1}^{n-1}(\hat{a}_{n,v+1})^k|\Delta\lambda_v||s_v|^k X_v^{1-k}$$

$$= O(1)\sum_{v=1}^{m}|\Delta\lambda_v||s_v|^k X_v^{1-k}\sum_{n=v+1}^{m+1} n^{\delta k+k-1}\hat{a}_{n,v+1}(\hat{a}_{n,v+1})^{k-1}$$

$$= O(1)\sum_{v=1}^{m}|\Delta\lambda_v||s_v|^k X_v^{1-k}\sum_{n=v+1}^{m+1} n^{\delta k+k-1}\hat{a}_{n,v+1}(a_{nn})^{k-1} \quad \text{(by lemma 2.3)}$$

$$= O(1)\sum_{v=1}^{m}|\Delta\lambda_v||s_v|^k X_v^{1-k}\sum_{n=v+1}^{m+1} n^{\delta k}\hat{a}_{n,v+1}(na_{nn})^{k-1}$$

$$=O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right|\left|s_v\right|^k X_v^{1-k}\sum_{n=v+1}^{m+1} n^{\delta k}\hat{a}_{n,v+1}$$

$$=O(1)\sum_{v=1}^{m} v^{\delta k}\left|\Delta\lambda_v\right|\left|s_v\right|^k X_v^{1-k}$$

$$=O(1)\sum_{v=1}^{m} v^{\delta k}\beta_v\left|s_v\right|^k X_v^{1-k}$$

$$=O(1)\sum_{v=1}^{m} v\beta_v\frac{v^{\delta k-1}}{X_v^{k-1}}\left|s_v\right|^k$$

$$=O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{r^{\delta k-1}}{X_r^{k-1}}\left|s_r\right|^k+O(1)\,m\beta_m\sum_{v=1}^{m}\frac{v^{\delta k-1}}{X_v^{k-1}}\left|s_v\right|^k$$

$$=O(1)\sum_{v=1}^{m}\beta_v X_v+O(1)\sum_{v=1}^{m} v\left|\Delta\beta_v\right|X_v+O(1)\,m\beta_m X_m$$

$$=O(1)$$

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