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The Deficient Discrete Quartic Spline Interpolation

Y.P. Dubey[1], S.S. Rana[2] and R.K. Dubey[3]

Abstract

In the present paper, the existence, uniqueness and convergence properties of discrete quartic spline interpolation over non-uniform mesh have been studied which match the given functional values at mesh points, interior points and second difference at boundary points.

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1 Introduction

Let us consider a mesh P on [0, 1] which is defined by

$$P: 0 = x_0 < x_1 < .... < x_n = 1.$$

[1] Department of Mathematics, L.N.C.T. Jabalpur 482 003.
E-mail: ypdubey2006@rediffmail.com
[2] Department of Mathematics, R.D.University, Jabalpur.
[3] Department of Mathematics, Global Nature Care Sangathan's Group of Institutions, Jabalpur. E-mail: r.dubey005@gmail.com

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Let p_i be the length of the mesh interval $[x_{i-1}, x_i]$. Let $p = \max_i p_i$, and $p' = \min_i p_i$, throughout h will represent a given positive real number. Consider a real function $s(x,h)$ defined over [0, 1] which is such that its restriction s_i on $[x_{i-1}, x_i]$ is a polynomial of degree 4 or less for i=1, 2,....n. Then $s(x,h)$ defines a deficient discrete quartic spline if

$$D_h^{(j)} s_i\,(x_i, h) = D_h^{\{j\}} s_{i+1}\,(x_i, h)$$

for $j = 0,1,2,\ i\ = 1,2,\dots,n$. The class of such splines is denoted by $S(4,1,P,h)$.

Discrete splines have been introduced by Mangasarian and Schumaker [6] in connection with certain studies of minimization problems involving differences (See also [7]). Discrete cubic splines which interpolate given functional values at one intermediate point of a uniform mesh have been studied in [1]. These results were generalized by Dikshit and Rana [2] for non-uniform meshes. An asymptotic precise estimates of the difference between discrete cubic spline interpolant and the function interpolated have been obtained by Rana and Dubey [9] which are sometime used to smooth a histogram. Usefulness of the difficient splines is quite apparent from the fact that they require less smooth data. In the direction of some constructive aspects of discrete splines, we refer to Jia [8], Schumaker [10].

In the present paper, we shall study the existence, uniqueness and convergence properties of deficient discrete quartic spline interpolation, which matches the given functional values at mesh points, interior points and second difference at bounary points over non uniform mesh. Our result in particular includes results of Dubey and Shukla [3].

2 Main result

The difference operator $D_h^{\{i\}}$ for a function *f* is defined by

$$D_h^{(0)}\ f(x)=f(x),$$

$$D_h^{(1)}\ f(x)=\frac{f(x+h)-f(x-h)}{2h}, \tag{2.1}$$

$$D_h^{(2)}\ f(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2},$$

and $D_h^{(m+n)}\ f(x)=D_h^{(m)}\ D_h^{(n)}\ f(x), \qquad m,n\geq 0$.

Let $S^*(4,1,P,h)$ denotes the class of all deficient discrete quartic splines of deficiency 1 and degree 4 satisfying the boundary conditions,

$$D_h^{(2)}\ s(x_0,h)=D_h^{(2)} f(x_0,h)$$

$$D_h^{(2)}\ s(x_n,h)=D_h^{(2)}\ f(x_n,h). \tag{2.2}$$

For a given function *f*, we introduce the following interpolatory conditions,

$$s(x_i)=f(x_i),\ \ i=0,\ 1,\ldots,n,$$

$$s(\alpha_i)=f(\alpha_i),\ i=1,\ldots,n-1,$$

where

$$\alpha_i=x_i+\theta p_i,\ 0<\theta<1, \tag{2.3}$$

and pose the following.

Problem A : Given h > 0, for what restrictions on P does there exist a unique $s(x,h)\in S^*(4,1,P,h)$ which satisfy the interpolatory conditions (2.3) and boundary conditions (2.2).

Let P(t) be a discrete quartic polynomial on [0, 1], then we can show that

$$\begin{aligned} P(t)=&\,P(0)\,Q_1(t)+P(1)\,Q_2(t)+P(\theta)\,Q_3(t)\\ &+D_h^{(2)}\ P(0)\,Q_4(t)+D_h^{(2)}\ P(1)\,Q_5(t) \end{aligned} \tag{2.4}$$

Proof : Denoting $(x-x_i)/\,p_i$ by t, 0 ≤ t ≤ 1, we can write (2.4) in the form of the restriction $s_i\,(x,h)$ of the deficient discrete quartic spline s(x, h) on $[x_i,x_{i+1}]$ as follows:

$$\begin{aligned} s_i(x) &= f(x_i)\,Q_1(t) + f(x_{i+1})\,Q_2(t) + f(\alpha_i)\,Q_3(t) \\ &\quad + D_h^{(2)} s_i(x)\,Q_4(t) + D_h^{(2)}\, s_{i+1}(x)\,Q_5(t), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} Q_1(z) &= 1 + [\{-6\theta^4 - 6 + 12\theta^3 + 6h^2(\theta^2 - 1)\}t \\ &\qquad - 6(\theta-1)h^2t^2 - 12(\theta-1)t^3 + 6(\theta-1)t^4]/6A, \end{aligned}$$

$$Q_2(z) = [(6\theta^4 - 12\theta^3 - \theta^2h^2)t + 6h^2\theta t^2 + 12\theta t^3 - 6\theta t^4]/6A,$$

$$Q_3(z) = [6(1+h^2)t - 6h^2t^2 - 12t^3 + 6t^4]/6A,$$

$$\begin{aligned} Q_4(z) &= [\{-2\theta^4 + 5\theta^3 - 3\theta^2 + (\theta^3 - \theta^2)h^2\}t \\ &\quad + \{3(\theta^4 + \theta - 2\theta^3) + (\theta - \theta^3)h^2\}t^2 \\ &\quad + \{(-\theta^4 - 5\theta + 6\theta^2 - (\theta - \theta^3)\,h^2\}\,t^3 + (\theta^3 - 3\theta^2 + 2\theta)t^4]/6A, \end{aligned}$$

$$\begin{aligned} Q_5(z) &= [\{-\theta^4 + \theta^3 - h^2(\theta^3 - \theta^2)\}t + h^2\{(\theta^3 - \theta)t^2 \\ &\quad + \{\theta^4 - \theta + (\theta - \theta^2)h^2\}t^3 - (\theta^3 - \theta)t^4]/6A, \end{aligned}$$

where $A = \{\theta^4 + \theta - 2\theta^3 + (\theta - \theta^2)h^2\}$

Observing (2.5), it may easily be verified that $s_i(x,h)$ is a quartic on $[x_i, x_{i+1}]$ for i=o, 1,....n-1 satisfying (2.2) - (2.3).

Denoting $g(a,b) = a + bh^2$, $G_i(c,d) = cp_i^2 + dh^2$, where a, b, c and d are real numbers and we are set to answer the problem A in the following.

Theorem 2.1. For any h > 0 and $p' > h$ then there exist a unique dificient discrete quartic spline $s(x,h) \in S^*(4,1,P,h)$ which satisfies the condition (2.2) and (2.3).

Remark. In the case when $\theta = \frac{1}{2}$, and $p_i = p$ i.e. uniform mesh Theorem 2.1 gives the corresponding result for deficient discrete quartic spline studied in Dubey and Shukla [3].

Now applying continuity of first difference of $s_i(x,h)$ at x_i we get the following system of equations,

$$p_i\, G_{i-1}\{g(\theta^4-\theta+3\theta^2-3\theta^3,\theta^3-\theta^2),g(-\theta^4+3\theta-6\theta^2+4\theta^3;$$
$$(\theta^2-\theta)\}\,D_h^{\{2\}}s_{i-1}(x)+[\,p_iG_{i-1}\{g(2\theta^4-3\theta^3+\theta,\theta^3+\theta^2-2\theta),$$
$$g(\theta^4+3\theta-4\theta^3,\theta-\theta^2)\}$$
$$-p_{i-1}G_i\{g(-2\theta^4+5\theta^3-3\theta^2,\theta^3-\theta^2),g(-\theta^4-5\theta+6\theta^2,(-\theta+\theta^2))\}]$$
$$D_h^{\{2\}}s_i(x)+p_{i-1}G_i\{g(\theta^4-\theta^3,\theta^3-\theta^2),g(\theta-\theta^4,\theta^2-\theta)\}\,D_h^{(2)}s_{i+1}(x)=F_i\ \ (say)$$
$$i=1,2,\ldots,n\,. \tag{2.6}$$

where

$$F_i=\frac{-p_i}{p_{i-1}^2}[G_{i-1}\{g(-6\theta^4+6+12\theta^3-12,6\theta^2+6-12\theta),g(0,12(\theta-1))\}\,f_{i-1}$$
$$+G_{i-1}\{g(6\theta^4-12\theta^3+12\theta,(12\theta-6\theta^2),g(0,-12\theta)\}\,f_i$$
$$+G_{i-1}\{g(-6(1+h^2),0),g(0,12)\}\,f(\alpha_{i-1})$$
$$+\frac{p_{i-1}}{p_i^2}G_i\{g(-6\theta^4-6+12\theta^3,6(\theta^2-1),g(0,12(1-\theta)h^2\}\,f_i$$
$$+G_i\{g(6\theta^4-12\theta^3,-6\theta^2),g(0,12\theta)\}\,f_{i+1}$$
$$+G_i\{g(6\theta+h^2),6),g(0,-12)\}\,f(\alpha_i)]$$

Writing $D_h^{(2)}s(x_i,h)=m_i(h)=m_i$ (say) for all i, we can easily see that excess of the absolute value of the coefficient of m_i dominant the sum of the absolute values of the coefficient of m_{i-1} and m_{i+1} in (2.6) under the conditions of Theorem 2.1 and is given by

$$d_i(h)=[\theta p_iG_{i-1}\{g(\theta^3+2-3\theta,\theta-1),g(2\theta^3-8\theta^2+6\theta,2-2\theta)$$
$$+\theta P_{i-1}\,G_i\,\{g(\theta^3-4\theta^2+3\theta,2\theta-2\theta^2),g(2\theta^3+4-6\theta,2-2\theta))\}].$$

Therefore, the coefficient matrix of the system of equation (2.6) is diagonally dominent and hence invertible. Thus, the system of equations has a unique solution. This complete the proof of Theorem 2.1. □

3 Error bounds

It may be observed that system of equations (2.6) may be written as

$$A(h)M(h)=F \tag{3.1}$$

where $A(h)$ is coefficient matrix and $M(h)=m_i=D_h^{(2)}s(x_i,h)$ (say).

However, as already shown in the proof of Theorem 2.1. A(h) is invertible. Denoting the inverse of A(h) by $A^{-1}(h)$, we note that row max norm $\|A^{-1}(h)\|$ satisfies the following inequality

$$\|A^{-1}(h)\|\le y(h) \tag{3.2}$$

where $y(h)=\max\{d_i(h)\}^{-1}$.

For a given $h>0$, we introduce the set $R_h=\{jh: j \text{ is an integer}\}$ and define a discrete interval as follows: $[0,1]_h=[0,1]\cap R_h$.

For a function *f* and three distinct points x_1, x_2, x_3 in the domain, the first, and second divided difference are defined by

$[x_1,x_2]_f=\dfrac{f(x_1)-f(x_2)}{(x_1-x_2)}$, and $[x_1,x_2,x_3]f=\dfrac{[x_2,x_3]f-[x_1,x_2]f}{(x_3-x_1)}$, respectively.

For convenience, we write $f^{\{2\}}$ for $D_h^{\{2\}}f$ and *w(f,p)* is the modulus of continuity of *f*. The discrete norm of the function *f* over the intervals $[0,1]_h$ is defined by $\|f\|=\max\limits_{x\in[0,1]}|f(x)|$.

Without assuming any smoothness condition on the data *f*, we shall obtain in the following the bounds for the error function over the discrete interval $[0,1]_h$.

Theorem 3.1. Suppose $s(x,h)$ is the discrete quartic spline interpolant of Theorem 2.1. Then

$$\|e_i^{\{2\}}\|\le C_1(h)K(p,h)w(f,p) \tag{3.3}$$

$$\|e_i^{\{1\}}\|\le C_2(h)K_1(p,h)w(f,p) \tag{3.4}$$

and

$$\|e(x)\|\le p^2K^*(p,h)w(f,p) \tag{3.5}$$

where the K (p, h), K_1 (p, h) and K* (p,h) are positive constants of *p* and *h*.

Proof: To obtain the error estimate (3.3) - (3.5) first we replace $m_i(h)$ by error function $e^{\{2\}}(x_i) = D_h^{\{2\}} s(x_i, h) - f_i^{\{2\}} = L_i$ in (3.1) and need following Lemma due to Lyche [4,5].

Lemma 3.1. Let $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ be a given sequences of non-negative real numbers such that $\sum a_i = \sum b_j$. Then for any real value function *f* defined on a discrete interval $[0,1]_h$, we have

$$\left|\sum_{i=1}^{m} a_i [x_{i0}, x_{i1}, \ldots\ldots x_{ik}] f - \sum_{j=1}^{n} b_j [y_{j0}, y_{j1'} \ldots . y_{jk}] f \le w(f^{\{1\}}, |1-p|) \, f \sum a_i \, | k!\right.$$

where $x_{ik}, y_{jk} \in [0,1]_h$ for relavent values of i, j and k. It may be observed that the r.h.s. of (3.1) after replacing $m_i(h)$ by $e^{(2)}(x_i)$ is written as

$$|(L_i)| = \sum_{i=1}^{8} a_i [x_{i0}, x_{i1}] f - \sum_{j=1}^{7} b_j [y_{j0}, y_{j1}] f \,| \tag{3.6}$$

where

$$a_1 = p_{i-1} p_i \, g(6\theta^4 - 12\theta^3, -6\theta^2) = b_1$$

$$a_2 = p_{i-1} p_i \, g(6\theta, 6\theta) = b_2 = a_3$$

$$a_4 = \frac{p_i}{p_{i-1}} g(0, 12\theta) = b_3 = a_5 = b_4$$

$$a_6 = \frac{p_i}{2h} G_{i-1} \{ g(\theta^4 - \theta + 3\theta^2 - 3\theta^2), (\theta^3 - \theta)\}, g\{(-\theta^4 + 3\theta - 6\theta^2 + 4\theta^3) - (\theta - \theta^2)\} = b_5$$

$$a_7 = \frac{p_i}{2h} G_{i-1} [\{ g(2\theta^4 - 3\theta^2 + \theta, \theta^3 + \theta^2 - 2\theta), g(\theta^4 + 3\theta - 4\theta^3, \theta - \theta^2\}$$

$$- \frac{p_{i-1}}{2h} G_i \{ g(2\theta^4 - 5\theta^3 + 3\theta^2, \theta^2 - \theta^3), g(\theta^4 + 5\theta - 6\theta^2, \theta - \theta^2\} = b_6$$

$$a_8 = \frac{p_i - 1}{2h} \left[G_i \{ g(\theta^4 - \theta^3, \theta^3 - \theta^2), g\{(\theta - \theta^4, \theta^2 - \theta)\} \right] = b_7$$

and

$$x_{10} = x_i = y_{11} = x_{30} = y_{21} = x_{41} = y_{31} = x_{50} = y_{40} = y_{60} = x_{71},$$

$y_{10} = x_{i-1} = x_{21} = y_{20} = x_{40} = x_{61} = y_{50}$,

$x_{11} = x_{i+1} = x_{51} = y_{70} = x_{81} = y_{71} = x_{i+1} + h$,

$x_{20} = \alpha_{i-1} = y_{30}$,

$x_{31} = \alpha_i = y_{41}$,

$x_{60} = x_{i-1} - h$,

$y_{51} = x_{i-1} + h,$
$y_{61} = x_1 + h,$
$x_{70} = x_i - h,$
$x_{80} = x_{i+1} - h.$

We observe that

$$\sum_{i=1}^{8} a_i = \sum_{j=1}^{7} b_j = N(\theta, P, h) \text{ (say).}$$

Thus applying Lemma 3.1 in (3.6) for $m = 8, n = 7$ and k=1, we get

$$|L_i \| \le N(\theta, P, h)\, w(f^{(1)}, |p|). \tag{3.7}$$

Now using the equation (3.2) and (3.7), we get

$$|e^{\{2\}}(x_i)\| \le y(h)\, K(p,h)\, w(f^{\{1\}}, p) \tag{3.8}$$

where $K(p,h)$ is some positive function of *p* and *h* which completes the proof of (3.3).

We next proceed to obtain a upper bound for *e(x)*, replacing *$m_i(h)$* by $e_i^{\{2\}}$ and $s(x,h)$ by $e(x,h)$ in equation (2.5), we obtain

$$e(x,h) = p_{i-1}^2 Q_4(t)\, e_{i-1}^{(2)} + p_1^2 Q_5(t)\, e_i^{(2)} = M_i(f) \qquad \text{(Say)} \tag{3.9}$$

Now, we write $M_i(f)$ in term of divided difference as follows :

$$M_i(f) = \left| \sum_{i=1}^{3} u_i [x_{i0}, x_{i1}] f - \sum_{j=1}^{4} [v_j [y_{j0}, y_{j1}] f \right|, \tag{3.10}$$

where

$u_1 = p_i \{6\theta^4 - 12\theta^3 - 6\theta^2 h^2) t + 6\theta^2 t^2 h^2 + 12\theta t^3 - 6\theta t^4\}$,

$v_1 = \theta p_i \{6(1+h^2)t + 6h^2t^2 + 12t^3 - 6t^4\}$,

$u_2 = \dfrac{p_{i-1}^2}{2h} Q_4(t) = v_2$,

$u_3 = \dfrac{p_i^2}{2h} \theta_5(t) = v_3$,

$v_4 = 6A\, p_i t$,

and

$x_{10} = x_i = x_{21}$,
$x_{11} = x_{i+1}$,
$x_{20} = x_i - h$,
$x_{30} = x_{i+1} - h$,

$x_{31} = x_{i+1}$,
$y_{10} = \alpha_i$,
$y_{11} = x_i = y_{20}$,

$y_{21} = x_i + h$,
$y_{30} = x_{i+1}$,
$y_{31} = x_{i+1} + h$,

$y_{40} = x,\ y_{41} = x_i$.

Hence

$$\sum_{i=1}^{3} u_i = \sum_{j=1}^{4} v_j = 6 p_i \theta t[\theta^3 - 2\theta^2 - \theta h^2 + h^2 + 2t^2 - t^3)] + \frac{1}{2h}\{p_{i-1}^2 Q_4(t) + p_i^2 Q_5(t)\} .$$

Again applying Lemma 3.1 in (3.10) for $m = 3, n = 4, k = 1$, we get

$$|M_i(f)| \le N^*(p,h)\, w(f^{(1)}, p) . \tag{3.11}$$

Thus, using (3.8) and (3.11) in (3.9) we get the following

$$\|e(x)\| \le p\, k^*(p,h)\, w(f^{(1)}, p) , \tag{3.12}$$

where $K^*(p,h)$ is a positive constant of p and h. This is the inequality (3.5) of Theorem 3.1.

We now proceed to obtain an upper bound of $e_i^{(1)}$. From equation (2.5), we get

$$6Ae_i^{(1)}(x,h) = p_{i-1}^2 e_{i-1}^{(2)} Q_4^{(1)}(t) + p_i^2 e_i^{(2)} Q_5^{(1)}(t) + U_i(f), \tag{3.13}$$

where

$$\begin{aligned} U_1(f) = [& f_i Q_1^{(1)}(t) + f_{i+1} Q_2^{(1)}(t) + f(\alpha_i) Q_3^{(1)}(t) \\ & + p_{i-1}^2 f_{i-1}^{(2)} Q_4^{(1)}(t) + p_1^2 \; f_i^{(2)} Q_5^{(1)}(t) - 6A\, f_i^{(1)}(x,h)]. \end{aligned} \tag{3.14}$$

By using Lemma 3.1 and first and second divided difference in $U_i(f)$ as follows:

$$|U_i(f)| \le w(f^{(1)}, p) \sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j \,, \tag{3.15}$$

where

$$a_1 = p_i \{6\theta^4 - 12\theta^3 - 6\theta^2 h^2 + 12h^2\theta t + 12\theta(h^2 + 3t^2) - 24\theta t(t^2 + h^2)\},$$

$$a_2 = p_i \{6(1+h^2) - 12h^2 + 12\theta(h^2 + 3t^2) + 24\theta t(t^2 + h^2)\},$$

$$a_3 = \frac{p_{i-1}^2}{2h} Q_4^{(1)}(t) = b_1,$$

$$a_4 = \frac{p_{i-1}^2}{2h} Q_5^{(1)}(t) = b_2,$$

$$b_3 = 6Ap_i,$$

and

$$x_{10} = x_i = x_{20},$$
$$x_{21} = \alpha_i,$$
$$x_{11} = x_{i+1} = x_{40} = y_{21},$$
$$x_{30} = x_i + h,$$
$$x_{31} = x_i = y_{30} = y_{11},$$
$$x_{40} = x_{i+1},\; x_{41} = x_{i+1} + h,$$
$$y_{10} = x_i - h, y_{11} = x_i \;,$$
$$y_{20} = x_{i+1} - h, y_{21} = x_{i+1},$$

$y_{30} = x_i,$

$y_{31} = x_i + h.$

Since $a_1 + a_2 = b_3$, so

$$\sum_{i=1}^{4} a_1 = \sum_{j=1}^{3} b_j = \left[6A\,p_i + \frac{p_{i-1}^2}{2h} Q_4^{(1)}(t) + \frac{p_i^2}{2h} Q_5^{(1)}(t) \right]. \tag{3.16}$$

From equation (3.8) putting the value of $e_i^{(2)}$ in (3.13) and using (3.15) we get upper bound of $e_i^{(1)}$. This completes the proof of Theorem 3.1. □

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