Journal of Applied Mathematics & Bioinformatics, vol.4, no.1, 2014, 45-70
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2014

Analysis of small oscillations of a rigid body elastically suspended and containing an almost-homogeneous incompressible liquid

Hilal Essaouini[1], Larbi Elbakkali[1] and Pierre Capodanno[2]

Abstract

We consider a 2d model for a rigid body with a cavity completely filled by a liquid and suspended by means of an elastic beam. The liquid is assumed to be "almost-homogeneous" and incompressible inviscid. From the equations of the system beam-container-liquid, we deduce the variational equation of the problem, and then two operatorial equations in a suitable Hilbert space. We show that the spectrum of the system is real and consists of a countable set of eigenvalues and an essential continuous spectrum filling an interval. The existence and uniqueness of the associated evolution problem are then proved using the weak formulation.

Mathematics Subject Classification: 76B03; 49R05; 47A75; 35Q35.

---

[1] Abdelmalek Essaadi Universty, Faculty of Science, M2SM, ER28/FS/05,93000 Tetuan, Morocco. E-mail: hilal_essaouini@yahoo.fr.
[2] Université Franche-Comté, 2B, Rue des jardins, 25000 Besançon, France.

Article Info: *Received* : December 28, 2013. *Revised* : February 3, 2014.
Published online : March 20, 2014.

Keywords: heterogeneous liquid; beam; small oscillations; variational method; operators pencils; spectral and evolution problems.

1 Introduction

The systems studied are close to various engineering applications, as for example in construction of tanks, of trucks for the companies of transport of liquids, etc...

The theoretical results are very important for numerical and experimental calculations of hydroelastic properties and dynamic characteristics of such structures.

The study of the classical case of a system formed by a rigid body containing a homogeneous ideal liquid, by means of the methods of functional analysis, has been the subject of very many works; see, for example, [15], [12].

On the other hand, the case of a heterogeneous incompressible liquid in a container was studied, first by Rayleigh and then, was the subject of limited number of works [13], [7], [1], [4].

The particular case of an “almost-homogeneous liquid”, i.e. a liquid whose density in equilibrium position is practically a linear function of the height differing a little bit from a constant, was treated in [2], [5], [6], [11].

The aim of this work is to extend the precedent works by studying the case of a container submitted to elastic constraints, reserving for another work the general case of an elastic container.

After writing the general equations of motion of the system, we linearize the problem assuming small displacements from an equilibrium position.

As a second step, and under the hypothesis that the liquid is almost-homogeneous, we reformulate the equations as a variational problem and, finally, as an operatorial problem involving non bounded linear operators on suitable Hilbert space.

Finally, we compute the spectrum of the relevant operator, showing that this is composed by a discrete part and an essential part filling an interval and corresponding physically to a domain of resonance: we argue that the presence of the essential part of the spectrum is due the hypothesis of almost-homogeneity, in contrast to the classical case in which the fluid is homogeneous and the spectrum is entirely discrete [3].

The existence and uniqueness of the associated evolution problem are then proved using the week formulation.

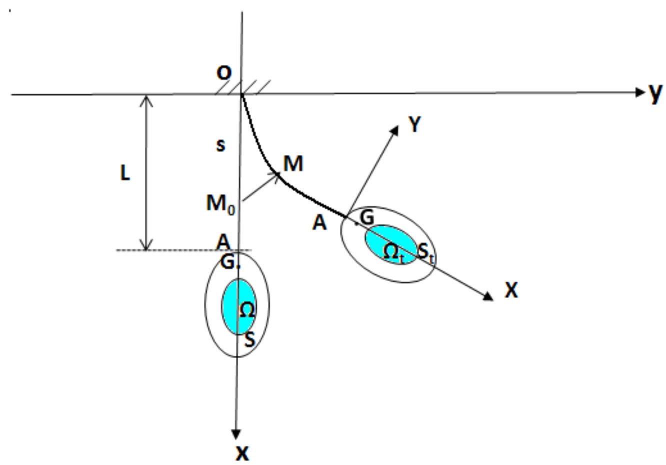

Figure1

2 Position of the problem

Let $OA = L$ the free part of the beam. This one is clamped in O in a fixed support and in A in the rigid body. We suppose that the tangent in O to the beam is the axis Ox directed vertically downwards and the tangent AX in A contains the centre of mass G of the body ($AG = a$). Oy is horizontal and AY

is perpendicular to AX. We denote by $\vec{x}$, $\vec{y}$ the unit vectors of Ox, Oy.

The body is homogeneous and has a cavity, and both are symmetrical with respect to AX. The cavity is completely filled by an almost-homogeneous incompressible inviscid liquid.

In the equilibrium position (resp. at the instant t), the liquid occupies the domain Ω (resp. Ω_t) limited by the wall S (resp. S_t). Obviously, Ω_t (resp. S_t) and Ω (resp. S) are isometric since the body is rigid.

We denote by $w(s,t)$, $u(s,t)$ the components on Ox, Oy respectively of the displacement of the point M_0 (abscissa s) of the beam; after deformation, M_0 comes in M, the coordinates of which are $s+w$, u. We suppose that the beam is unextensible, hypothesis that is expressed by

$$\left[\left(1+w'\right)^2+u'^2\right]\mathrm{d}s^2=\mathrm{d}s^2 \quad \left(w'=\frac{\partial w}{\partial s};u'=\frac{\partial u}{\partial s}\right)$$

or

$$w'=-\frac{1}{2}u'^2 \tag{2.1}$$

We are going to study the small oscillations of the system beam-body-liquid about its equilibrium position, obviously in linear theory.

As usual, we are considering that the linearized velocities and accelerations are "true" velocities and accelerations, in order to avoid writing needless formulas in the following calculations.

3 Equations of the motion of the system

3.1 Equations of motion of the beam

The beam, of constant density ρ, is submitted to the gravity, to the reactions in O and to the reactions of the body, the resultant of which being

$mgR_x\vec{\boldsymbol{x}} + mgR_y\vec{\boldsymbol{y}}$ and the moment about O: $mgN\vec{\boldsymbol{z}}$ $(\vec{\boldsymbol{z}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}})$.

g is the acceleration due to gravity and we set

$$m = m_0 + m_\ell$$

m_0 and m_ℓ being respectively the mass of the body and the mass of the liquid.

The boundary conditions in O are

$$u(0,t) = 0\,;\, u'(0,t) = 0\,;\, w(0,t) = 0\,.$$

It is well-known that $\widehat{Ox, AX} = \theta = u'(L,t)$.

In order to obtain the equations of the small oscillations of the beam, it is convenient to use the Hamilton principle.

We write

$$\begin{cases} \int_{t_1}^{t_2} \left\{ \delta T + \delta W + \delta\left(-\dfrac{EI}{2} \int_0^L u''^2 \,\mathrm{d}s \right) + mgR_x \delta w(L,t) + mgR_y \delta u(L,t) + mgN \delta u'(L,t) \right\} \mathrm{d}t \\ = 0 \end{cases}$$

$(t_1 < t_2)$ in the set of the functions $u(s,t)$ that are twice continuously differentiable with respect to t and four times continuously differentiable with respect to s, verifying the boundary conditions in O and taking the same values for $t = t_1$ and $t = t_2$.

T is the kinetic energy of the beam, i.e. $\dfrac{1}{2}\int_0^L \rho \dot{u}^2 \,\mathrm{d}s$; W is the function of the force of gravity, i.e $W = \int_0^L \rho g w \,\mathrm{d}s$, and $\dfrac{EI}{2}\int_0^L u''^2 \,\mathrm{d}s$ is the potential energy of the beam (I moment of inertia of the section, E Young's modulus, considered as constant).

At first, we have, using classically an integration by parts:

$$\int_{t_1}^{t_2} \delta T \,\mathrm{d}t = \int_{t_1}^{t_2} \left[\int_0^L \rho \dot{u} \delta \dot{u} \,\mathrm{d}s \right] \mathrm{d}t = \int_0^L \left[\int_{t_1}^{t_2} \rho \dot{u} \delta \dot{u} \,\mathrm{d}t \right] \mathrm{d}s = -\int_{t_1}^{t_2} \left[\int_0^L \rho \ddot{u} \delta u \,\mathrm{d}s \right] \mathrm{d}t$$

After, we can write, using (2.1):

$$\int_0^L w \,\mathrm{d}s = \int_0^L w \,\mathrm{d}(s-L) = \left[(s-L) w \right]_0^L - \int_0^L (s-L) w' \,\mathrm{d}s = \frac{1}{2}\int_0^L (s-L) u'^2 \,\mathrm{d}s\,,$$

so that we have

$$\delta W = \int_0^L \rho g w \,\mathrm{d}s = -\rho g \int_0^L \frac{\partial}{\partial s}\left[(s-L)u'\right]\delta u \,\mathrm{d}s .$$

Two integration by parts give

$$\begin{cases} \delta\left(\dfrac{EI}{2}\displaystyle\int_0^L u''^2 \,\mathrm{d}s\right) = EI\displaystyle\int_0^L u''^2 \delta u'' \,\mathrm{d}s \\ = EI\left[u''(L,t)\delta u'(L,t) - u'''(L,t)\delta u(L,t) + \displaystyle\int_0^L u^{IV}\delta u \,\mathrm{d}s\right] \end{cases}$$

Finally, we have

$$\delta w(L,t) = \int_0^L w'(s,t)\,\mathrm{d}s = -\int_0^L u'\delta u' \,\mathrm{d}s = -u'(L,t)\delta u'(L,t) + \int_0^L u''\delta u \,\mathrm{d}s$$

The Hamilton principle is expressed by the equation

$$\int_{t_1}^{t_2} \begin{Bmatrix} \displaystyle\int_0^L \left[-\rho\ddot{u} - \rho g \frac{\partial}{\partial s}\left[(s-L)u'\right] - EIu^{IV} + mgR_x u''\right]\delta u \,\mathrm{d}s \\ +\left[EIu'''(L,t) - mgR_x u'(L,t) + mgR_y\right]\delta u(L,t) \\ +\left[-EIu''(L,t) + mgN\right]\delta u'(L,t) \end{Bmatrix} \mathrm{d}t = 0$$

from which we deduce the equations

$$\begin{cases} \rho\ddot{u} + \rho g \dfrac{\partial}{\partial s}\left[(s-L)u'\right] + EIu^{IV} - mgR_x u'' = 0 \\ EIu'''(L,t) - mgR_x u'(L,t) + mgR_y = 0 \\ EIu''(L,t) - mgN = 0 \end{cases} \tag{3.1}$$

to which we must add

$$u(0,t) = 0\,;\;\; u'(0,t) = 0\,. \tag{3.2}$$

(3.1), (3.2) are the equations of motion of the beam.

3.2 Equations of motion of the liquid

Let us consider now the motion of the liquid.

At first, we suppose that the liquid is heterogeneous and we denote by $\rho^*(x,y,t)$ its density and by $p^*(x,y,t)$ the pressure.

Let M_ℓ the particle of the liquid that occupies the position (x,y) at the instant t. If $\vec{V}_r(x,y,t)$ is the velocity of M_ℓ with respect to the body, we set

$$\vec{U}(x,y,t)=\int_{t_e}^{t}\vec{V}_r(x,y,\tau)\,\mathrm{d}\tau$$

where t_e is the date of the equilibrium position.

We have

$$\vec{U}(x,y,t_e)=0\,;\;\dot{\vec{U}}=\frac{\partial\vec{U}}{\partial t}=\vec{V}_r$$

$\vec{U}$ can be considered as a small relative displacement of the particle.

By using the Coriolis theorem, the Euler's equation takes the form:

$$\rho^*\ddot{\vec{U}}=-\overrightarrow{\mathbf{grad}}p^*+\rho^* g\vec{x}-\rho^*\ddot{u}(L,t)\vec{y}-\rho^*\ddot{u}'(L,t)\left[(x-L)\vec{y}-y\vec{x}\right]\text{ in }\Omega_t \quad (3.3)$$

We must add

$$\mathrm{div}\dot{\vec{U}}=0\;\text{ in }\;\Omega_t\text{ (incompressibility),} \quad (3.4)$$

$$\frac{\partial\rho^*}{\partial t}+\mathrm{div}\left(\rho^*\vec{V}_a\right)=0\text{ in }\;\Omega_t\text{ (continuity equation),} \quad (3.5)$$

where $\vec{V}_a$ is the velocity of the particle with respect to Oxy, and the kinematic condition

$$\dot{\vec{U}}\cdot\vec{n}=0\text{ on }\;S_t\,, \quad (3.6)$$

where $\vec{n}$ is the unit vector of the external normal.

(3.3), (3.4), (3.5), (3.6) are the equations of motion of the liquid.

3.3 Equations of the system body- liquid

Finally, we study the motion of the system rigid body-liquid, which is submitted to the gravity and to the reactions of the beam in A.

The theorem of momentum gives easily the equations

$$\int_{\Omega_t} \rho^* \left(-\ddot{\theta} y + \ddot{U}_x\right) \mathrm{d}\Omega_t = mg\left(1 - R_x\right) \tag{3.7}$$

$$m\ddot{u}\left(L,t\right) + \left[m_0 a + \int_{\Omega_t} \rho^* \left(x - L\right) \mathrm{d}\Omega_t\right] \ddot{u}'\left(L,t\right) + \int_{\Omega_t} \rho^* \ddot{U}_y \, \mathrm{d}\Omega_t = -mgR_y \tag{3.8}$$

The theorem of moment of momentum gives, after a few calculations

$$\begin{cases} I_A \ddot{u}'\left(L,t\right) + \left[m_0 a + \int_{\Omega_t} \rho^* \left(x - L\right) \mathrm{d}\Omega_t\right] \ddot{u}\left(L,t\right) + \int_{\Omega_t} \rho^* \left[\left(x - L\right)\ddot{U}_y - y\ddot{U}_x\right] \mathrm{d}\Omega_t \\ = -\left[m_0 a + \int_{\Omega_t} \rho^* \left(x - L\right) \mathrm{d}\Omega_t\right] g u'\left(L,t\right) - g\int_{\Omega_t} \rho^* Y \, \mathrm{d}\Omega_t - mgN \end{cases} \tag{3.9}$$

where I_A is the moment of inertia of the system about A.

(3.7), (3.8), (3.9) are the equations of motion of the system body-liquid.

3.4 Equilibrium equations

At the equilibrium position, we have $u = 0$, $\vec{\boldsymbol{U}} = \boldsymbol{0}$.

If p_0 and ρ_0 are the values of p^* and ρ^* in this position, we must have

$$\overrightarrow{\mathbf{grad}} p_0 = \rho_0 g \vec{\boldsymbol{x}} ,$$

so that p_0 and ρ_0 are functions of x only, with

$$\frac{\mathrm{d} p_0\left(x\right)}{\mathrm{d}x} = \rho_0\left(x\right) g$$

We suppose classically that the density is an increasing function of the depth, i.e. that $\rho_0'\left(x\right) = \dfrac{\mathrm{d}\rho_0\left(x\right)}{\mathrm{d}x}$ is positive.

The equations (3.1) give

$$R_y = 0 , \quad N = 0 , \text{ for } t = t_e$$

and the condition

$$\int_{\Omega_t} \rho^* Y \, \mathrm{d}\Omega_t \Big|_{t=t_e} = 0 ,$$

deduced from (3.9) is verified by symmetry.

4 Transformation of the equation of motion

4.1 Transformation of Euler's equation

We set

$$\rho^* = \rho_0(x) + \tilde{\rho}(x,y,t) + \cdots; \quad p^* = p_0(x) + p(x,y,t) + \cdots,$$

where $\tilde{\rho}$ and p are of the first order with respect to the amplitude of the oscillations.

The linearized continuity equation (3.5) is

$$\frac{\partial \tilde{\rho}}{\partial t} + \left(-\dot{u}'(L,t)y + \dot{U}_x\right)\rho_0'(x) = 0,$$

and then, integrating between t_e and t:

$$\tilde{\rho} = -\rho_0'(x)\left[-u'(L,t)y + U_x\right]$$

Consequently, the linearized Euler's equation can be written

$$\begin{cases} \rho_0(x)\ddot{\boldsymbol{U}} = -\rho_0'(x)\left[-u'(L,t)y + U_x\right]\vec{\boldsymbol{x}} - \overrightarrow{\mathbf{grad}}\,p - \rho_0(x)\ddot{u}(L,t)\vec{\boldsymbol{y}} \\ \qquad - \rho_0(x)\ddot{u}'(L,t)\left[(x-L)\vec{\boldsymbol{y}} - y\vec{\boldsymbol{x}}\right] \end{cases} \tag{4.1}$$

4.2 Transformation of the equation (3.8)

If b is the distance from A to the centre of mass of the system body-liquid in the equilibrium position, the linearized equation (3.8) can be written

$$m\ddot{u}(L,t) + mb\ddot{u}'(L,t) + \int_\Omega \rho_0(x)\ddot{U}_y \,\mathrm{d}\Omega = -mgR_y \tag{4.2}$$

4.3 Transformation of the equation (3.9)

In order to linearized the equation (3.9), we must calculate at the first order the integral

$$\int_{\Omega_t} \rho^* Y \,\mathrm{d}\Omega_t \ .$$

Since $x = L + X\cos\theta - Y\sin\theta \sim L + X - \theta Y \quad (\theta = u'(L,t))$, we have

$$\int_{\Omega_t} \rho^* Y \,\mathrm{d}\Omega_t = \int_{\Omega_t} (\rho_0 + \tilde{\rho}) Y \,\mathrm{d}\Omega_t = \int_\Omega \rho_0 (L + X - \theta Y) Y \,\mathrm{d}\Omega + \int_\Omega \tilde{\rho} Y \,\mathrm{d}\Omega$$

$$= \int_\Omega \rho_0 (L + X) Y \,\mathrm{d}\Omega - \theta \int_\Omega \rho_0' (L + X) Y^2 \,\mathrm{d}\Omega + \int_\Omega \tilde{\rho} Y \,\mathrm{d}\Omega$$

where Ω is the domain occupied by the liquid at the instant t, geometrically identical to Ω_t.

The first integral of the right-hand side is equal to zero by symmetry and we have, in linear theory

$$\int_\Omega \tilde{\rho} Y \,\mathrm{d}\Omega = \int_\Omega \rho_0' (L + X)(-\theta Y + U_X) Y \,\mathrm{d}\Omega,$$

so that

$$\int_{\Omega_t} \rho^* Y \,\mathrm{d}\Omega_t = -\int_\Omega \rho_0' (L + X) U_X Y \,\mathrm{d}\Omega,$$

or, Ω being now the domain occupied by the liquid in the equilibrium position

$$\int_{\Omega_t} \rho^* Y \,\mathrm{d}\Omega_t = -\int_\Omega \rho_0'(x) y U_x \,\mathrm{d}\Omega$$

Then, the linearized equation (3.9) becomes

$$\begin{cases} I_A \ddot{u}'(L,t) + mb\ddot{u}(L,t) + \int_\Omega \rho_0(x)\left[(x - L)\ddot{U}_y - y\ddot{U}_x\right]\mathrm{d}\Omega \\ = -mgbu'(L,t) + g\int_\Omega \rho_0'(x) y U_x \,\mathrm{d}\Omega - mgN \end{cases} \tag{4.3}$$

4.4 New equation of the beam

Obviously by virtue of the equation (3.7), in the equations (3.1) of the motion of the beam, we must replace R_x by 1.

5 The particular case of an almost-homogeneous liquid

5.1 Definition of an almost -homogeneous liquid

Let h the vertical diameter of the domain Ω. In Ω, we have, if d is the distance from the center of mass of the liquid to A:

$$|x - L - d| < h$$

We suppose that the density of the liquid in equilibrium position has the form

$$\rho_0(x) = \rho_0\left[1 + \beta(x - L - d)\right] + o(\beta h),$$

where ρ_0 and β are positive constant, β such that $(\beta h)^2$, $(\beta h)^3$,…are negligible with respect to βh.

In this case, the liquid is called *almost-homogeneous in* $\boldsymbol{\Omega}$.

We restrict ourselves to this case. Then, like in the Boussinesq theory of the convection fluid motions [7, p 16], we replace in the equations of motion

$$\rho_0(x) \text{ by } \rho_0, \quad \rho_0^{'}(x) \text{ by } \rho_0\beta .$$

5.2 Elimination of the reactions of the rigid body

At first, the Euler's equation takes the form

$$\ddot{\vec{\boldsymbol{U}}} = -\frac{1}{\rho_0}\overrightarrow{\mathbf{grad}}p - \ddot{u}(L,t)\vec{\boldsymbol{y}} - \ddot{u}'(L,t)\left[(x-L)\vec{\boldsymbol{y}} - y\vec{\boldsymbol{x}}\right] - \beta g\left[-u'(L,t)y + U_x\right]\vec{\boldsymbol{x}} \tag{5.1}$$

On the other hand, since, after integration of (3.4) and (3.6) between t_e and t, $\operatorname{div}\vec{\boldsymbol{U}} = 0$ in Ω, $\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{n}} = 0$ on S, we have

$$\int_\Omega U_y d\Omega = \int_\Omega \vec{\boldsymbol{U}} \cdot \overrightarrow{\mathbf{grad}} y \, d\Omega = 0 .$$

so that the equations (4.2) becomes

$$m\ddot{u}(L,t) + mb\ddot{u}'(L,t) = -mgR_y \tag{5.2}$$

Eliminating R_y and N between (4.3), (5.2) and the equations of motion of the beam, we obtain

$$\rho\ddot{u} + \rho g\frac{\partial}{\partial s}\left[(s-L)u'\right] + EIu^{IV} - mgu'' = 0 \tag{5.3}$$

$$m\ddot{u}(L,t)+mb\ddot{u}'(L,t)-EIu'''(L,t)+mgu'(L,t)=0 \tag{5.4}$$

$$\begin{cases} I_A\ddot{u}'(L,t)+mb\ddot{u}(L,t)+\rho_0\int_\Omega\left[(x-L)\ddot{U}_y-y\ddot{U}_x\right]\mathrm{d}\Omega \\ +mbgu'(L,t)-\rho_0\beta g\int_\Omega yU_x\,\mathrm{d}\Omega+EIu''(L,t)=0 \end{cases} \tag{5.5}$$

5.3 New transformation of Euler’s equation

In the following, we introduce the spaces [12]

$$J_0(\Omega)=\left\{\vec{\boldsymbol{u}}\in\left[L^2(\Omega)\right]^2,\ \mathrm{div}\vec{\boldsymbol{u}}=0,\ \vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}}=0 \text{ on } S\right\}$$

$$G(\Omega)=\left\{\vec{\boldsymbol{u}}=\overrightarrow{\mathbf{grad}}p,\ p\in H^1(\Omega)\right\}$$

We have the well known orthogonal decomposition [12]:

$$\left[L^2(\Omega)\right]^2=J_0(\Omega)\oplus G(\Omega)$$

In order to eliminate the pressure, we project the Euler’s equation (5.1) on $J_0(\Omega)$.

If P_0 is the orthogonal projector of $\left[L^2(\Omega)\right]^2$ on $J_0(\Omega)$, we obtain

$$\ddot{\vec{\boldsymbol{U}}}+\mathrm{P}_0\left[(x-L)\vec{\boldsymbol{y}}-y\vec{\boldsymbol{x}}\right]\ddot{u}'(L,t)+\mathrm{K}\vec{\boldsymbol{U}}-\beta g\,\mathrm{P}_0(y\vec{\boldsymbol{x}})u'(L,t)=0 \tag{5.6}$$

where K is the operator from $J_0(\Omega)$ in $J_0(\Omega)$ defined by

$$\mathrm{K}\vec{\boldsymbol{U}}=\beta g\,\mathrm{P}_0(U_x\vec{\boldsymbol{x}}).$$

5.4 The operator K

The operator K has a basic role in the problem. It was studied by Capodanno [2]. It is self- adjoint and its spectrum is identical to its essential spectrum, denoted by $\sigma_{ess}(\mathrm{K})$, and it is the interval $[0,\beta g]$.

We sketch the proof. By a Weyl’s theorem [12], it is sufficient to prove that, for every $0<\mu<1$, there exists a sequence $\{\vec{\boldsymbol{U}}_k\}$ such that

$$\frac{\left\| \frac{1}{\beta g}\mathrm{K}\vec{\boldsymbol{U}}_k - \mu\vec{\boldsymbol{U}}_k \right\|}{\left\| \vec{\boldsymbol{U}}_k \right\|} \to 0 \text{ when } k \to +\infty$$

We construct a sequence $\{\vec{\boldsymbol{U}}_{n\,m}\}$ such that $\{\vec{\boldsymbol{U}}_{n\,m}\} = \left(\frac{\partial \Delta \mathrm{q}_{n\,m}}{\partial y}, -\frac{\partial \Delta \mathrm{q}_{n\,m}}{\partial x} \right)^{\mathrm{t}}$ and

$\mathrm{q}_{n\,m} = e^{i(nx+my)}\,\mathrm{q}(x,y)$, $q(x,y) \in \mathscr{D}(\Omega)$ and equal to 1 in a circle $|x - x_0| \le r$, contained in Ω.

We can prove that

$$\frac{1}{\beta g}\mathrm{K}\vec{\boldsymbol{U}}_{n\,m} = \left(\frac{\partial^3 \mathrm{q}_{n\,m}}{\partial y^3}, -\frac{\partial^3 \mathrm{q}_{n\,m}}{\partial x \partial y^2} \right)^{\mathrm{t}}$$

and that

$$\frac{1}{\beta g}\mathrm{K}\vec{\boldsymbol{U}}_{n\,m} - \frac{n^2}{n^2+m^2}\vec{\boldsymbol{U}}_{n\,m} = o\left(n^2+m^2\right)$$

where $\frac{o\left(n^2+m^2\right)}{n^2+m^2}$ is uniformly bounded in Ω.

For every $\varepsilon > 0$, it is possible to find a rational number $\frac{\tilde{m}}{\tilde{n}}$ such that

$$\mu < \frac{\tilde{n}^2}{\tilde{n}^2+\tilde{m}^2} < \mu + \varepsilon$$

Choosing $m = k\tilde{m}$, $n = k\tilde{n}$, we can prove that the sequence $\{\vec{\boldsymbol{U}}_{k\tilde{n}\,k\tilde{m}}\}$ satisfies the Weyl's theorem.

5.5 Introduction of new operators

In the following, we introduce the operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{M}}$ from $J_0(\Omega)$ in $\mathbb{C}$ defined by

$$\tilde{\mathrm{L}}\vec{\boldsymbol{U}} = \int_\Omega \left[(x-L)U_y - yU_x \right] \mathrm{d}\Omega \,; \quad \tilde{\mathrm{M}}\vec{\boldsymbol{U}} = \int_\Omega yU_x \,\mathrm{d}\Omega \,.$$

It is easy to see that $\tilde{\mathrm{L}}$ and $\mathrm{P}_0\left[\left(x-L\right)\vec{\boldsymbol{y}}-y\vec{\boldsymbol{x}}\right]$ and $\tilde{\mathrm{M}}$ and $\mathrm{P}_0\left(y\vec{\boldsymbol{x}}\right)$ are mutually adjoint.

6 Operatorial equations of the small motion of the system

6.1 A formal variational equation

The equations of motion are (5.3); (5.4); (5.5); (5.6); (3.2).

We multiply the equation (5.3) by a smooth function $\bar{\tilde{u}}\left(s\right)$ such that $\tilde{u}\left(0\right)=0$, $\tilde{u}'\left(0\right)=0$; we integrate on $\left(0,L\right)$ and we perform integrations by parts. After, we multiply the equation (5.4) by $\bar{\tilde{u}}\left(L\right)$ and equation (5.5) by $\bar{\tilde{u}}'\left(L\right)$. Adding the results, we obtain after little long, by easy calculations:

$$\begin{cases}\int_0^L \rho\ddot{u}\bar{\tilde{u}}\,\mathrm{d}s+m\ddot{u}\left(L,t\right)\bar{\tilde{u}}\left(L\right)+mb\left[\ddot{u}\left(L,t\right)\bar{\tilde{u}}'\left(L\right)+\ddot{u}'\left(L,t\right)\bar{\tilde{u}}\left(L\right)\right]+I_A\ddot{u}'\left(L,t\right)\bar{\tilde{u}}'\left(L\right)\\ +\rho_0\,\tilde{\mathrm{L}}\ddot{\vec{\boldsymbol{U}}}\cdot\bar{\tilde{u}}'\left(L\right)+EI\int_0^L u''\bar{\tilde{u}}''\,\mathrm{d}s+g\int_0^L\left[\rho\left(L-s\right)+m\right]u'\bar{\tilde{u}}'\,\mathrm{d}s+mbgu'\left(L,t\right)\bar{\tilde{u}}'\left(L\right)\\ -\rho_0\beta g\,\tilde{\mathrm{M}}\vec{\boldsymbol{U}}\cdot\bar{\tilde{u}}'\left(L\right)=0\end{cases}\qquad(6.1)$$

6.2 The space V

We introduce the space

$$V=\left\{u\in H^2\left(0,L\right);\ u\left(0\right)=0;\ u'\left(0\right)=0\right\}$$

By virtue of the generalized Poincaré inequality [17], we see that

$$\left[EI\int_0^L\left|u''\right|^2\mathrm{d}s+g\int_0^L\left[\rho\left(L-s\right)+m\right]\left|u'\right|^2\mathrm{d}s\right]^{1/2}$$

defines on V a norm, that is equivalent to the classical norm $\left\|u\right\|_2$ of $H^2\left(0,L\right)$.

Then, by virtue of a trace theorem,

$$\left(u,\tilde{u}\right)_V = EI\int_0^L u''\overline{\tilde{u}}''\mathrm{d}s + g\int_0^L\left[\rho\left(L-s\right)+m\right]u'\overline{\tilde{u}}'\ \mathrm{d}s + mbgu'\left(L\right)\overline{\tilde{u}}'\left(L\right),$$

can be taken as a scalar product in V, the associated norm $\|u\|_V$ being equivalent in V to $\|u\|_2$.

6.3 The space H

Now, we denote by H the completion of V for the norm associated to the scalar product

$$\left(u,\tilde{u}\right)_H = \int_0^L \rho u\overline{\tilde{u}}\,\mathrm{d}s + mu\left(L,t\right)\overline{\tilde{u}}\left(L\right) + mb\left[u\left(L,t\right)\overline{\tilde{u}}'\left(L\right)+u'\left(L,t\right)\overline{\tilde{u}}\left(L\right)\right]$$
$$+ I_A u'\left(L,t\right)\overline{\tilde{u}}'\left(L\right)$$

It is a scalar product, since the quadratic form

$$mu^2\left(L\right)+2mbu\left(L\right)u'\left(L\right)+I_A u'^2\left(L\right)$$

is positive, by virtue of the well-known inequality $I_A > mb^2$.

The imbedding from V in H is obviously dense, it is continuous by virtue of a trace theorem in $H^2\left(0,L\right)$. Finally, it is compact. Indeed, let a sequence $\left\{u_n\right\}\in V$ that converges weakly in V to $u\in V$.

This sequence converges strongly in $L^2\left(0,L\right)$ and the sequence of traces $\left\{u_n\left(L\right)\right\}$, $\left\{u_n'\left(L\right)\right\}$ converge strongly to $u\left(L\right), u'\left(L\right)$ in $\mathbb{C}$.

Remark.

If $u\in H$, there exists a sequence $\left\{u_n\right\}\in V$ such that $\left\|u_n-u\right\|_H\to 0$ and consequentl, $\left\|u_n-u_m\right\|_H\to 0$ when $n,m\to\infty$. Then, we have $\left|u_n\left(L\right)-u_m\left(L\right)\right|\to 0$, is that $u_n\left(L\right)$ has a strong limit in $\mathbb{C}$, that we call naturally $u\left(L\right)$. So, we give a sense to $u\left(L\right)$, and in same manner, to $u'\left(L\right)$, when $u\in H$.

6.4 Transformation of the equation (6.1)

Then, the equation (6.1) is equivalent to the equation

$$(\ddot{u},\tilde{u})_H + \rho_0 \tilde{\mathrm{L}} \ddot{\vec{\boldsymbol{U}}} \cdot \bar{\tilde{u}}'(L) + (u,\tilde{u})_V - \rho_0 \beta g \tilde{\mathrm{M}} \vec{\boldsymbol{U}} \cdot \bar{\tilde{u}}'(L) = 0 \quad \forall \tilde{u} \in V . \tag{6.2}$$

6.5 Reduction of the equation (6.2) to an operatorial equation

From the inequality

$$I_A u'^2(L) + 2mbu(L)u'(L) + mu^2(L) \geq \left(I_A - mb^2\right) u'^2(L),$$

we deduce

$$\left|u'(L)\right| \leq \left(I_A - mb^2\right)^{-1/2} \|u\|_H$$

so that

$$\left|\tilde{\mathrm{L}} \vec{\boldsymbol{U}} \cdot \bar{\tilde{u}}'(L)\right| \leq \left(I_A - mb^2\right)^{-1/2} \left|\tilde{\mathrm{L}} \vec{\boldsymbol{U}}\right| \|\tilde{u}\|_H ,$$

Consequently, there exists a bounded operator $\hat{\mathrm{L}}$ from $J_0(\Omega)$ in H such that

$$\tilde{\mathrm{L}} \vec{\boldsymbol{U}} \cdot \bar{\tilde{u}}'(L) = \left(\hat{\mathrm{L}} \vec{\boldsymbol{U}}, \tilde{u}\right)_H$$

In the same manner, we can write

$$\tilde{\mathrm{M}} \vec{\boldsymbol{U}} \cdot \bar{\tilde{u}}'(L) = \left(\hat{\mathrm{M}} \vec{\boldsymbol{U}}, \tilde{u}\right)_H$$

The equation (6.2) becomes

$$(\ddot{u},\tilde{u})_H + \rho_0 \left(\hat{\mathrm{L}} \ddot{\vec{\boldsymbol{U}}}, \tilde{u}\right)_H + (u,\tilde{u})_V - \rho_0 \beta g \left(\hat{\mathrm{M}} \vec{\boldsymbol{U}}, \tilde{u}\right)_H = 0 \quad \forall \tilde{u} \in V .$$

Classically [14], if A_0 is the unbounded operator of H, which is associated to the sesquilinear form $(u,\tilde{u})_V$ and to the pair (V,H), the precedent equation is equivalent to the operatorial equation

$$\ddot{u} + \rho_0 \hat{\mathrm{L}} \ddot{\vec{\boldsymbol{U}}} + \mathrm{A}_0 u - \rho_0 \beta g \hat{\mathrm{M}} \vec{\boldsymbol{U}} = 0, \qquad u \in D(\mathrm{A}_0) \subset V \subset H. \tag{6.3}$$

6.6 Transformation of Euler's equation in operatorial equation

We have, for each $\tilde{\vec{U}} \in J_0(\Omega)$, and with obvious notations for the scalar products:

$$\left(\mathrm{P}_0\left[(x-L)\vec{y}-y\vec{x}\right]\ddot{u}'(L,t),\tilde{\vec{U}}\right)_{J_0(\Omega)} = \left(\ddot{u}'(L,t),\tilde{\mathrm{L}}\tilde{\vec{U}}\right)_{\mathbb{C}} = \overline{\left(\tilde{\mathrm{L}}\tilde{\vec{U}},\ddot{u}'(L,t)\right)}_{\mathbb{C}}$$
$$= \overline{\left(\hat{\mathrm{L}}\tilde{\vec{U}},\ddot{u}\right)}_{H} = \overline{\left(\tilde{\vec{U}},\hat{\mathrm{L}}^*\ddot{u}\right)}_{J_0(\Omega)}$$
$$= \left(\hat{\mathrm{L}}^*\ddot{u},\tilde{\vec{U}}\right)_{J_0(\Omega)}$$

where we denote by $\hat{\mathrm{L}}^*$ the adjoint of $\hat{\mathrm{L}}$.
Therefore, we can write

$$\mathrm{P}_0\left[(x-L)\vec{y}-y\vec{x}\right]\ddot{u}'(L,t) = \hat{\mathrm{L}}^*\ddot{u}$$

and, in the same manner

$$\mathrm{P}_0(y\vec{x})u'(L,t) = \hat{\mathrm{M}}^* u$$

Taking the scalar product by $\tilde{\vec{U}}$ in $J_0(\Omega)$ of the members of the Euler's equation (5.6), we obtain, using the precedent results

$$\left(\ddot{\vec{U}} + \hat{\mathrm{L}}^*\ddot{u} + \mathrm{K}\vec{U} - \beta g\,\hat{\mathrm{M}}^* u,\tilde{\vec{U}}\right)_{J_0(\Omega)} = 0, \qquad \forall\tilde{\vec{U}} \in J_0(\Omega)$$

and finally the operatorial equation

$$\ddot{\vec{U}} + \hat{\mathrm{L}}^*\ddot{u} + \mathrm{K}\vec{U} - \beta g\,\hat{\mathrm{M}}^* u = 0. \qquad (6.4)$$

(6.3) and (6.4) are the operatorial equations of the problem, for the unknowns $u \in V$, $\vec{U} \in J_0(\Omega)$.

6.7 Operatorial equations with bounded operators

In order to eliminate the unbounded operator A_0, we set

$$\hat{u} = \mathrm{A}_0^{1/2}\, u \in H$$

The equation (6.3) and (6.4)become

$$\mathrm{A}_0^{-1}\,\ddot{\hat{u}} + \rho_0\,\mathrm{A}_0^{-1/2}\,\hat{\mathrm{L}}\,\ddot{\vec{\boldsymbol{U}}} + \hat{u} - \rho_0 \beta g\,\mathrm{A}_0^{-1/2}\,\hat{\mathrm{M}}\vec{\boldsymbol{U}} = 0\,, \tag{6.5}$$

$$\hat{\mathrm{L}}^*\,\mathrm{A}_0^{-1/2}\,\ddot{\hat{u}} + \ddot{\vec{\boldsymbol{U}}} - \beta g\,\hat{\mathrm{M}}^*\,\mathrm{A}_0^{-1/2}\,\hat{u} + \mathrm{K}\vec{\boldsymbol{U}} = 0\,, \tag{6.6}$$

where all the operators are bounded.

7 Study of the spectrum of the problem

We will prove, at the end of the paper, that the spectrum of the problem exists and lies on the positive real semi-axis.

Then, we seek the solutions of the form

$$u\left(s,t\right) = \mathrm{e}^{\mathrm{i}\omega t} u\left(s\right);\ \vec{\boldsymbol{U}}\left(x,y,t\right) = \mathrm{e}^{\mathrm{i}\omega t}\vec{\boldsymbol{U}}\left(x,y\right)\ (\omega\,\text{real})$$

The precedent equations give

$$\omega^2\left(\mathrm{A}_0^{-1}\,\hat{u} + \rho_0\,\mathrm{A}_0^{-1/2}\,\hat{\mathrm{L}}\,\vec{\boldsymbol{U}}\right) = \hat{u} - \rho_0\beta g\,\mathrm{A}_0^{-1/2}\,\hat{\mathrm{M}}\vec{\boldsymbol{U}}\,, \tag{7.1}$$

$$\omega^2\left(\hat{\mathrm{L}}^*\,\mathrm{A}_0^{-1/2}\,\hat{u} + \vec{\boldsymbol{U}}\right) = -\beta g\,\hat{\mathrm{M}}^*\,\mathrm{A}_0^{-1/2}\,\hat{u} + \mathrm{K}\vec{\boldsymbol{U}} \tag{7.2}$$

7.1 The spectrum in the interval $\omega^2 > \beta g$

We set $\mu = \omega^{-2}$, so that $|\mu| < \left(\beta g\right)^{-1}$.

The equation (7.2) can be written

$$\left(\mathrm{I}_{J_0(\Omega)} - \mu\,\mathrm{K}\right)\vec{\boldsymbol{U}} = -\left(\hat{\mathrm{L}}^* + \mu\beta g\,\hat{\mathrm{M}}^*\right)\mathrm{A}_0^{-1/2}\,\hat{u}$$

Since $\|\mathrm{K}\| = \beta g$, $\mathrm{I}_{J_0(\Omega)} - \mu\,\mathrm{K}$ has an inverse $\mathrm{R}\left(\mu\right)$, which is a self-adjoint and holomorphic operatorial function in $|\mu| < \left(\beta g\right)^{-1}$ and we have

$$\vec{\boldsymbol{U}} = -\mathrm{R}\left(\mu\right)\left(\hat{\mathrm{L}}^* + \mu\beta g\,\hat{\mathrm{M}}^*\right)\mathrm{A}_0^{-1/2}\,\hat{u}$$

Substituting in the first equation, we obtain

$$\mathrm{L}_0(\mu)\hat{u} = \left[\mu\,\mathrm{I}_H - \mathrm{A}_0^{-1} + \rho_0\,\mathrm{A}_0^{-1/2}\left(\hat{\mathrm{L}} + \mu\beta g\hat{\mathrm{M}}\right)\mathrm{R}(\mu)\left(\hat{\mathrm{L}}^* + \mu\beta g\,\hat{\mathrm{M}}^*\right)\mathrm{A}_0^{-1/2}\right]\hat{u} = 0 \qquad (7.3)$$

$\mathrm{L}_0(\mu)$ is a self-adjoint and holomorphic operatorial function in $|\mu| < (\beta g)^{-1}$.

We have

$$\mathrm{L}_0(0) = -\mathrm{A}_0^{-1} + \rho_0\,\mathrm{A}_0^{-1/2}\,\hat{\mathrm{L}}\,\hat{\mathrm{L}}^*\,\mathrm{A}_0^{-1/2}$$

$$\mathrm{L}_0'(0) = \mathrm{I}_H + \rho_0\beta g\,\mathrm{A}_0^{-1/2}\left(\hat{\mathrm{M}}\,\hat{\mathrm{L}}^* + \hat{\mathrm{L}}\,\hat{\mathrm{M}}^*\right)\mathrm{A}_0^{-1/2} + \rho_0\,\mathrm{A}_0^{-1/2}\,\hat{\mathrm{L}}\,\mathrm{K}\,\hat{\mathrm{L}}^*\,\mathrm{A}_0^{-1/2}$$

$\mathrm{L}_0(0)$ is compact, like A_0^{-1} and $\mathrm{A}_0^{-1/2}$. $\mathrm{L}_0'(0)$ is strongly positive like I_H, if βg is sufficiently small.

Therefore [12, p 74], for every ε such that $0 < \varepsilon < (\beta g)^{-1}$, there is, in the interval $]0,\varepsilon[$, a countable set of positive real eigenvalues μ_k, which tend to zero when $k \to +\infty$.

The eigenelements form a Riesz basis in a subspace of H, which has a finite defect.

For our problem, there is a countable set of positive real eigenvalues $\omega_k^2 = \mu_k^{-1}$, which tend to infinity, when $k \to +\infty$.

7.2 The spectrum in the interval $0 \le \omega^2 \le \beta g$

The equation (7.1) can be written

$$\left(\mathrm{I}_H - \omega^2\,\mathrm{A}_0^{-1}\right)\hat{u} = \rho_0\,\mathrm{A}_0^{-1/2}\left(\beta g\,\hat{\mathrm{M}} + \hat{\mathrm{L}}\right)\vec{\boldsymbol{U}}$$

Since $\omega^2 \le \beta g$, $\mathrm{I}_H - \omega^2\,\mathrm{A}_0^{-1}$ has an inverse if βg is sufficiently small and we have

$$\hat{u} = \rho_0\left(\mathrm{I}_H - \omega^2\,\mathrm{A}_0^{-1}\right)^{-1}\mathrm{A}_0^{-1/2}\left(\beta g\,\hat{\mathrm{M}} + \hat{\mathrm{L}}\right)\vec{\boldsymbol{U}}$$

Substituting in (7.2), we obtain

$$\mathrm{K}\vec{\boldsymbol{U}} - \mathrm{W}_0\left(\omega^2\right)\vec{\boldsymbol{U}} = \omega^2\vec{\boldsymbol{U}}\,,\ \ \vec{\boldsymbol{U}} \in J_0\left(\Omega\right),$$

with

$$\mathrm{W}_0\left(\omega^2\right) = \rho_0\left(\omega^2\,\hat{\mathrm{L}}^* + \beta g\,\hat{\mathrm{M}}^*\right)\mathrm{A}_0^{-1/2}\left(\mathrm{I}_H - \omega^2\,\mathrm{A}_0^{-1}\right)^{-1}\mathrm{A}_0^{-1/2}\left(\omega^2\,\hat{\mathrm{L}} + \beta g\,\hat{\mathrm{M}}\right)$$

$\mathrm{W}_0\left(\omega^2\right)$ is an analytical function in $\left[0,\beta g\right]$ and, for each ω^2, $\mathrm{W}_0\left(\omega^2\right)$ is a compact self-adjoint operator, since $\mathrm{A}_0^{-1/2}$ is compact from H onto H.
Setting

$$\mathrm{Z}\left(\omega^2\right) = \mathrm{K} - \mathrm{W}_0\left(\omega^2\right),$$

we obtain the equation

$$\left(\mathrm{Z}\left(\omega^2\right) - \omega^2\,\mathrm{I}_{J_0(\Omega)}\right)\vec{\boldsymbol{U}} = 0\,,\ \ \vec{\boldsymbol{U}} \in J_0\left(\Omega\right) \tag{7.4}$$

Let $\omega_1^2 \in \sigma_{ess}\left(\mathrm{K}\right) = \left[0,\beta g\right]$. By a classical Weyl's theorem [Kopachevskii 2001], the operator $\mathrm{Z}\left(\omega_1^2\right)$ verifies

$$\sigma_{ess}\left[\mathrm{Z}\left(\omega_1^2\right)\right] = \sigma_{ess}\left(\mathrm{K}\right) = \left[0,\beta g\right].$$

For each $\omega_2^2 \in \sigma_{ess}\left[\mathrm{Z}\left(\omega_1^2\right)\right]$, there exists a "Weyl's sequence" [12] $\left\{\tilde{\vec{\boldsymbol{U}}}_{\boldsymbol{n}}\right\} \in J_0\left(\Omega\right)$, that depends on ω_1^2 and ω_2^2, such that

$$\tilde{\vec{\boldsymbol{U}}}_{\boldsymbol{n}} \to 0 \text{ weakly};\ \ \inf\left\|\tilde{\vec{\boldsymbol{U}}}_{\boldsymbol{n}}\right\| > 0\,;\ \ \left(\mathrm{Z}\left(\omega_1^2\right) - \omega_2^2\,\mathrm{I}_{J_0(\Omega)}\right)\tilde{\vec{\boldsymbol{U}}}_{\boldsymbol{n}} \to 0 \ \text{ in }\ J_0\left(\Omega\right).$$

Choosing $\omega_2^2 = \omega_1^2$, the corresponding Weyl's sequence $\left\{\tilde{\tilde{\vec{\boldsymbol{U}}}}_{\boldsymbol{n}}\right\}$ depends on ω_1^2 only and verifies $\left(\mathrm{Z}\left(\omega_1^2\right) - \omega_1^2\,\mathrm{I}_{J_0(\Omega)}\right)\tilde{\tilde{\vec{\boldsymbol{U}}}}_{\boldsymbol{n}} \to 0$ in $J_0\left(\Omega\right)$, so that ω_1^2 belongs to the spectrum of the problem (7.4). ω_1^2 being arbitrary in $\left[0,\beta g\right]$, the spectre of the problem in this interval coincides with its essential spectrum $\left[0,\beta g\right]$.

7.3 Conclusion

The spectrum of the problem is composed by an essential part, which fills the closed interval $[0,\beta g]$, and a discrete part that lies outside this interval and is comprised of a countable set of positive real eigenvalues, whose accumulation point is the infinity.

Physically, the eigenvalues of the point spectrum are a denumerable set of values of resonance, where as the interval $[0,\beta g]$ is a domain of resonance.

8 Existence and uniqueness theorem

The equation (6.3) and (6.4) can be written, setting $\zeta=\left(u,\vec{\boldsymbol{U}}\right)^{\mathrm{t}}\in\chi=H\oplus J_0(\Omega)$:

$$\mathrm{B}\ddot{\zeta}+\mathrm{Q}\zeta=0,\ \ \zeta\in\chi \tag{8.1}$$

with

$$\mathrm{B}=\begin{pmatrix} \mathrm{I}_H & \rho_0\,\hat{\mathrm{L}} \\ \rho_0\,\hat{\mathrm{L}}^* & \rho_0\,\mathrm{I}_{J_0(\Omega)} \end{pmatrix};\qquad \mathrm{Q}=\begin{pmatrix} \mathrm{A}_0 & -\rho_0\beta g\hat{\mathrm{M}} \\ -\rho_0\beta g\hat{\mathrm{M}}^* & \rho_0\,\mathrm{K} \end{pmatrix}$$

8.1 Properties of the operator B

B is obviously bounded from χ into χ and self-adjoint. It is easy to see, that B is strongly positive if the body is preponderant.

Indeed, by direct calculation, we have

$$\left(\mathrm{B}\zeta,\zeta\right)_\chi\geq\|u\|_H^2+\rho_0\left\|\vec{\boldsymbol{U}}\right\|_{J_0(\Omega)}^2-2\sqrt{\frac{\rho_0 I_{A_\ell}}{I_A-mb^2}}\left\|\vec{\boldsymbol{U}}\right\|_{J_0(\Omega)}\|u\|_H,$$

where I_{A_ℓ} is the moment of inertia of the liquid about A.

The right-hand side is a positive quadratic form with respect to $\|u\|_H$, $\|\vec{U}\|_{J_0(\Omega)}$ if

$$I_A - mb^2 > I_{A_\ell},$$

that is verified if the body is preponderant.

8.2 The space ν and properties of the operator Q

We introduce the space $\nu = V \oplus J_0(\Omega)$ equipped with the norm

$$\|\zeta\|_\nu = \left(\|u\|_V^2 + \|\vec{U}\|_{J_0(\Omega)}^2\right)^{1/2}.$$

The imbedding from ν into χ is obviously dense and continuous but it is not compact, because the identical operator $\mathrm{I}_{J_0(\Omega)}$ is not compact.

Q is unbounded and self-adjoint operator of χ.

By direct calculations, we obtain, λ being a positive real number:

$$\begin{cases} (\mathrm{Q}\zeta,\zeta)_\chi + \lambda\|\zeta\|_\chi^2 \geq \|u\|_V^2 + \dfrac{\lambda}{2}\|\vec{U}\|_{J_0(\Omega)}^2 \\ +\dfrac{\lambda}{2}\|\vec{U}\|_{J_0(\Omega)}^2 + \lambda\|u\|_H^2 + \rho_0\beta g\|U_x\|_{L^2(\Omega)}^2 - \beta g\sqrt{\dfrac{\rho_0 I_{A_\ell}}{I_A - mb^2}}\|\zeta\|_\chi^2 \end{cases}$$

so that , if βg sufficiently small

$$(\mathrm{Q}\zeta,\zeta)_\chi + \lambda\|\zeta\|_\chi^2 \geq \min\left(1,\frac{\lambda}{2}\right)\left(\|u\|_V^2 + \|\vec{U}\|_{J_0(\Omega)}^2\right) = \min\left(1,\frac{\lambda}{2}\right)\|\zeta\|_\chi^2$$

Consequently, the form $a(\zeta,\tilde{\zeta}) = (\mathrm{Q}\zeta,\tilde{\zeta})_\chi$ is a sesquilinear, hermitian, continuous and ν coercive with respect to χ form.

8.3 Existence and uniqueness theorem

Then, we can apply to the (8.1) a known theorem [8, p. 667-670].

If the initial data verify

$$\zeta(0)=\begin{pmatrix} u(0) \\ \vec{\boldsymbol{U}}(0) \end{pmatrix} \in \nu \; ; \;\; \dot{\zeta}(0)=\begin{pmatrix} \dot{u}(0) \\ \dot{\vec{\boldsymbol{U}}}(0) \end{pmatrix} \in \nu \, ,$$

the problem has one and only one solution such that

$$\zeta(\cdot) \in L^2(0,\boldsymbol{T};\nu) \; ; \;\; \dot{\zeta}(\cdot) \in L^2(0,\boldsymbol{T};\nu) \, ,$$

where $\boldsymbol{T}$ is a positive constant.

8.4 Existence of the spectrum

Setting $\mathrm{B}^{1/2}\zeta = \eta \in \chi$, we replace the equation (8.1) by

$$\ddot{\eta} + \mathrm{C}\eta = 0 \, , \tag{8.2}$$

where $\mathrm{C} = \mathrm{B}^{-1/2}\,\mathrm{Q}\,\mathrm{B}^{-1/2}$ is a self-adjoint unbounded operator of χ.

We are going to prove that C is positive definite, so Q is positive definite.

A_0 being strongly positive in H, we have

$$\left(\mathrm{A}_0\, u, u\right)_H \geq k \left\| u \right\|_H^2 \, , \;\; k > 0 \, .$$

By direct calculations, we obtain easily

$$\left(\mathrm{Q}\zeta,\zeta\right)_\chi \geq \left(k - \rho_0 \beta g \nu_0^2\right) \left\| u \right\|_H^2 + \rho_0 \beta g \left(\left\| U_x \right\|_{L^2(\Omega)} - \nu_0 \left\| u \right\|_H \right)^2$$

with

$$\nu_0 = \frac{\left(\int_\Omega y^2 \, \mathrm{d}\Omega \right)^{1/2}}{\sqrt{I_A - mb^2}}$$

$k - \rho_0 \beta g \nu_0^2$ being positive if βg is sufficiently small, we have

$$\left(\mathrm{Q}\zeta,\zeta\right)_\chi \geq 0 \, , \quad \text{equal to zero only for } \; u = 0 \, , \; U_x = 0 \, .$$

But since $\vec{\boldsymbol{U}} \in J_0(\Omega)$, we have $\dfrac{\partial U_y}{\partial y} = 0$ in the sense of distributions and it is known [16, p. 57] that U_y is absolutely continuous function on each parallel to

Ox limited to S, that has almost everywhere a derivative equal to zero. Since $\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{n}} = 0$ on S, we have $U_y = 0$ and then $\vec{\boldsymbol{U}} = 0$.

Q, and therefore C, is positive definite.

Seeking the solution of (8.2) in the form

$$\eta(.,t) = \mathrm{e}^{\Lambda t} \eta(.),$$

we obtain

$$\mathrm{C}\eta = -\Lambda^2 \eta$$

C, being self-adjoint, has a real spectrum. Since it is positive definite, this spectrum lies on the positive real semi-axis. Therefore, $-\Lambda^2$ is real positive and we can set $\Lambda = \mathrm{i}\omega$, ω real, according to the calculations in the paragraph 7.

References

[1] P. Capodanno, Un exemple simple de problème non standard de vibration: oscillations d'un liquide hétérogène pesant dans un container, *Mechanics Research Communications(MRC)*, **20**(3), (1993), 257-262.

[2] P. Capodanno, Piccole oscillazioni piane di un liquido perfetto incompressibile pesante eterogeneo in un recipiente, *Lecture in the Rome University TR- CTIT*, (2001), 10-16.

[3] P. Capodanno, Sur les petites oscillations d'un pendule élastique, *Rendiconti del Sem. Mat. dell'Università e del Politecnico di Torino*, **51**(1), (1993), 53-64.

[4] P. Capodanno, Petites oscillations planes d'un liquide hétérogène dans un container fermé par un couvercle élastique, *Bulletin de l'Académie Polonaise des Sciences, Série des sciences techniques*, **44**(4), 1ère partie 347-356, 2ème partie, (1996), 357-366.

[5] P. Capodanno and D.Vivona, Mathematical study of the small oscillations of a pendulum filled by an inviscid, incompressible, almost-homogeneous liquid,

Proceedings 14th Conference on Waves And Stability In Continuous Media WASCOM'05, Catania, (June 2005), 71-76.

[6] P. Capodanno and D. Vivona, Mathematical study of the planar oscillations of a heavy almost-homogeneous liquid in a container, *Proceedings 13th Conference on Waves And Stability In Continuous Media WASCOM'07*, Sicily, (30 June –7 July, 2007), 90-95.

[7] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic stability*, Clarendon Press, Oxford, 1961.

[8] R. Dautray and J.L. Lions, *Analyse Mathématique et calcul numérique*, Vol. **8**, Masson, Paris, 1988.

[9] H. Essaouini, *Etude mathématique des petites oscillations de certains systèmes matériels plans comprenant un liquide parfait incompressible quasi-homogène*, Thèse de doctorat en sciences, Tétouan : Université Abdelmalek Essaâdi / Maroc), (2010), 144 pages.

[10] H. Essaouini, L. Elbakkali and P. Capodanno, Analysis of the small oscillations of a heavy almost-homogeneous liquid-gas system, *Mechanics Research Communications(MRC)*, **37**, (2010), 257-262.

[11] H. Essaouini, L. Elbakkali and P. Capodanno, Mathematical study of the small oscillations of a pendulum containing an almost-homogeneous liquid and a barotropic gas, *Zeitschrift fur Angewandte Matematik und Physik (ZAMP)*, **62**, **(**2011), 849-868.

[12] N.D. Kopachevskii and S.G. Krein, *Operator approach to linear problems of hydrodynamics*, Vol. **1**, Birkhauser, Basel, 2001.

[13] H. Lamb, *Hydrodynamics*, Cambridge at the University Press, Cambridge, 1932.

[14] J.L. Lions, *Equations différentielles opérationnelles et problèmes aux limites*, Springer, Berlin, 1961.

[15] N.N. Moiseyev and V.V. Rumyantsev, *Dynamic Stability of Bodies Containing Fluid*, Springer, Berlin, 1968.

[16] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[17] W. Velte, *Direkte Methoden der Variationsrechnung*, B.G., Teubner, Stuttgart, 1976.