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The Impact of Crank-Nicolson Finite Difference Method in Valuation of Options

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Abstract

This paper presents Crank-Nicolson (CN) analysis for valuation of options. This particular method solves Black-Scholes (BS) partial differential equation (PDE) by means of numerical solutions for pricing options. The deviation values were derived from BS analytical solutions, adopting certain criteria using three standard deviations as a measure for pricing effects. Results showed when options are overpriced, underpriced as well as no-mispricing this is in line with theoretical predictions and significant improvement over previous efforts. In the same scenario, a non-parametric test discovered by Kolmogorov-Smirnov (KS) was performed; the test revealed that there exist a statistically difference between distributions of BS and CN. Also, the initial stock prices of no-mispricing were compared and it was seen that initial stock prices of 70 and 40 are the best for call and put options. The work presented here has profound implication for future studies of option prices and may one day help solve the problems of option traders.

Keywords: Crank-Nicolson, Option pricing, Mispricing effects, B-S PDE and Standard deviation.

1. Introduction

In financial modeling, European options have been an issue which almost devastated the efforts of option traders; their valuation became very difficult until 1973. Before then, there was no generally accepted model that could give an answer to option traders the value of an option at expiration. Because of this great challenge in pricing options, [2] came up by solving Black-Scholes (B-S) partial differential equation (PDE) which was used in pricing options. Later, [11], [12],[14] and [15] etc. observed in their different studies that Black-Scholes model misprices options considerably; some of them say it overprices and underprices options which obviously became a burden to option traders again and society at large. That is to say the correctness of the model is still questionable and the major part of the model is how to predict the future volatility of the underlying asset, hence determine a correct option price. In the sequel, mispricing simply means having a price which does not accurately match the intrinsic value of

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the item, [13]. When there's mispricing option traders will not be encouraged to invest in a stock market. With no mispricing option traders, investors, economist, policy makers and Federal Government can use it for decision making and planning.
Motivated by the biasedness in the said B-S PDE in pricing options, the writers of this paper have partitioned mispricing into three interacting components such as overpricing, underpricing and no mispricing. These are subsets of pricing effects; the essence of these partitions is to ensure efficiency in estimating the parameters in pricing options.
Our interest in this paper is to identify the sources of mispricing in order to reduce pricing bias. The bias in valuation can cause unreasonable loss for trader, [3]; as valuation is a key feature of trading system.
However, the advantage of this proposed method over the previous is determining levels of pricing effects, more importantly when there is "no mispricing".

2. Methodology

The Black-Scholes model is based on seven assumptions:

- The asset price follows a Brownian motion with μ and σ as constants.
- There are no transaction costs or taxes. All securities are perfectly divisible.
- There is no dividend during the life of the derivatives.
- There are no riskless arbitrage opportunities.
- The security trading is continuous.
- The option is exercised at the time of maturity for European option.

They gave the formula for the prices of European calls option as

$$C = SN(\mathrm{d}_1) - Ke^{-rt}N(\mathrm{d}_2) \quad (1a)$$

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad (1b)$$

$$d_2 = d_1 - \sigma\sqrt{T} \quad (1c)$$

where C is Price of a call option, S is price of underlying asset, K is the strike price, r is the riskless rate, T is time to maturity, σ^2 is variance of underlying asset, σ is standard deviation of the (generally referred to as volatility) underlying asset, and N is the cumulative normal distribution.
Similarly, the formula for prices of European put option is given as

$$P = SN(d_1) - Ke^{-rt}N(d_2) \quad (2)$$

Where P is the price of a put option and the meaning of other parameters remain the same as in (1) ,[11].

2.1 Derivation of Black-Scholes (BS) Partial Differential Equation (PDE).

According to [15], the derivation of B-S PDE is based on Ito process with an assumption that the stock prices follow a geometric Brownian motion, ie

$$dS = \mu S dt + \sigma S dx \quad (3)$$

Where S is the stock price, μ is the drift, σ is the volatility of underlying asset and dx is a wiener process.
Suppose we have an option whose $V(S,t)$ depends only on S and t. Assuming also that the asset price is perturbed by a small change dS, then the function V will also change. Using Ito's lemma

$$dV = \sigma S\frac{\partial V}{\partial S}dx + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt \tag{4}$$

According to [17], the value of one portfolio having one stock can be expressed with the function $V(S,t)$.

$$\pi = V - \Delta S \tag{5a}$$

The change in the portfolio at time dt in (5a) is given by

$$d\pi = dV - \Delta dS \tag{5b}$$

Putting (3),(4) into (5b),we find that π follows random walk, given by.

$$d\pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dx + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S\right)dt$$

To eliminate the random component in this random walk, let

$$\Delta = \frac{\partial V}{\partial S} \tag{6}$$

Note that Δ is the value of $\frac{\partial V}{\partial S}$ at the start of the time step dt. This results in a portfolio whose increment is wholly deterministic so that

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt \tag{7}$$

Now that the portfolio is riskless it should earn riskless return. The change in the portfolio at time dt becomes (after substituting (3) and (4) into (5) and dividing through by dt)

$$d\pi = r\pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$ and

$$r\left(V-\Delta S\right)dt=\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

Which implies that

$$rV-rS\frac{\partial V}{\partial S}=\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$$

This gives the solution

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \qquad (8)$$

This is the Black-Scholes partial differential equation.

2.2 European Call Option

The B-S PDE for European Call Option with value $C\left(S,t\right)$ is given in the following equation:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0 \qquad (9)$$

The following initial and boundary conditions for Call Option:

$$\begin{aligned}&C\left(0,\mathrm{t}\right)=0\ .\\&C\left(S,t\right)=S\ \text{ when }\ S\to\infty\\&C\left(S,T\right)=\max\left(S-K,0\right)\end{aligned} \qquad (10)$$

2.3 European Put Option

The B-S PDE for European put option with value $P\left(S,t\right)$ is defined as in (11)

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}+rS\frac{\partial P}{\partial S}-rP=0 \qquad (11)$$

With the following initial and boundary conditions:

$$\begin{aligned}&P\left(0,t\right)=Ke^{-rt}\\&P\left(S,t\right)=0\ when\ S\to\infty\\&P\left(S,T\right)=\max\left(K-S,0\right)\end{aligned} \qquad (12)$$

The method for solving (12) is similar to the solution method of (11) but the only difference is the boundary conditions.

2.4 The Crank-Nicolson finite difference Method for an Option pricing Model

The Crank-Nicolson finite difference method is to overcome the stability short-comings by applying the stability and convergence restrictions of the explicit finite difference methods. It is essentially an average of the implicit and explicit methods. However, to carry out a Crank-

Nicolson approximation method on Black-Scholes partial differential equation, there will be a price time mesh ,the vertical axis in the mesh represents the stock prices, while the horizontal axis represents time. Thus, each grid point in the mesh denotes a horizontal index i and a vertical index j such that each point in the mesh is the option price for a definite time and a definite stock price. At all times in the mesh $j\Delta s$ is equal to the stock price, and $i\Delta t$ is equal to the time. There exist boundary conditions which aids in the numerical computations; using the pay-off function. The expiration, $t = T$ and the option are computed for all the different stock prices using boundary conditions. To obtain the prices at $t = 0$, the model solves backwards for each time step from t=T [18].

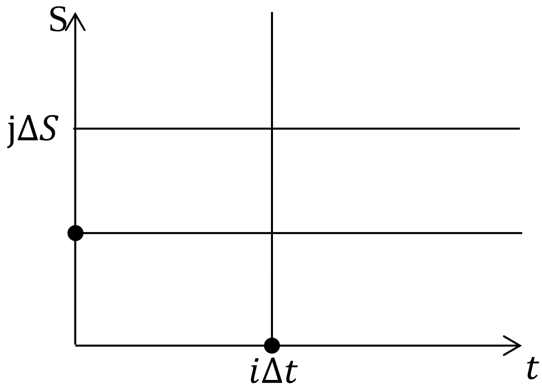


Figure 1a: An illustration of Price time mesh.
Recall that the Black-Scholes partial differential equation (8).

Let a function $V(\mathrm{S},t)$ in two dimensional grid points, that is i and j denote the indices for stock price, S and time, t respectively. The function $V(S,t) = V_i^j$; this can be stated with the following difference scheme [10].

$$Z_i^{\,j} = \frac{1}{2}\sigma^2 S^2 DSS + rS_i DS - rV_i^{\,j} \tag{13}$$

where

$$S = i\Delta s\text{, for } 0 \le i \le m, \quad t = j\Delta t \text{ for } 0 \le j \le i$$

$$DSS = \frac{V^j_{\;i+1} - 2V^j_{\;i} + V^j_{\;i-1}}{\Delta^2} \tag{14}$$

$$DS = \frac{V^j_{\;i+1} - V^j_{\;i-1}}{2\Delta S} \tag{15}$$

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below.

If we use a forward difference approximation to the time partial derivative we obtain explicit scheme

$$\frac{V_i^{j+1} - V_i^{j}}{\Delta t} + Z_i^{j} = 0 \qquad (16)$$

and similarly we obtain the implicit scheme

$$\frac{V_i^{j+1} - V_i^{j}}{\Delta t} + Z_i^{j+1} = 0 \qquad (17)$$

The averages of equations (16) and (17) yields Crank-Nicolson method of approximation

$$\frac{V_i^{j+1} - V_i^{j}}{\Delta t} + \frac{1}{2}\left(Z_i^{j} + Z_j^{j+1}\right) = 0 \qquad (18)$$

From equation(18)

$$V_i^{j} - \frac{\Delta t}{2} Z_i^{j} = V_i^{j+1} + \frac{\Delta t}{2} Z_i^{j+1} \qquad (19)$$

$$\frac{\Delta t}{2} Z_i^{j} = V_i^{j+1} - V_i^{j} + \frac{\Delta t}{2} Z_i^{j+1}$$

$$\therefore Z_i^{j} = \frac{2}{\Delta t}\left(u_i^{j+1} - u_i^{j}\right) - Z_i^{j+1} \qquad (20)$$

Substituting (13) in (20) gives in view of (14) and (15) we obtain after collecting like term in V_{i-1} ,

$$\frac{\sigma^2 S^2}{2}\left[\frac{V_{i+1}^{j} - 2V_i^{j} + V_{i-1}^{j}}{(\Delta S)^2}\right] + rS_i\left[\frac{V_{i+1}^{j} - V_{i-1}^{j}}{2\Delta S}\right] - rV_i^{j} = \frac{2}{\Delta t}\left(V_i^{j+1} - V_i^{j}\right) - Z_i^{j+1}$$

That is

$$\frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}\right] + \frac{rS_i}{2\Delta S}\left[u_{i+1}^{j} - u_{i-1}^{j}\right] - ru_i^{j} = \frac{2}{\Delta t}\left(u_i^{j+1} - u_i^{j}\right) + Z_i^{j+1}$$

Collecting like terms in of V_{i-1}, V_i *and* V_{i+1} and simplifying gives

$$Z_i^{j} = V_{i-1}^{j}\left[\frac{\Delta t \sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t S_i}{2\Delta S}\right] + V_i^{j}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t \sigma^2 S_i^2}{(\Delta S)^2} - r\Delta t\right] + V_{i+1}^{j}\left[\frac{\Delta t \sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t S_i}{2\Delta S}\right]$$

and

$$Z_i^{j+1} = V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t S_i}{2\Delta S}\right] + V_i^{j+1}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t\sigma^2 S_i^2}{(\Delta S)^2} - r\Delta t\right] + V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t S_i}{2\Delta S}\right]$$

Using (13) in (18) solving simultaneously and taking the average of these two equations we obtain

$$V_{i-1}^{j}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta t S_i}{4\Delta S}\right] + V_i^{j}\left[1 - \left(\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right)\right] + V_{i+1}^{j}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta t S_i}{4\Delta S}\right]$$

$$= V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta t S_i}{4\Delta S}\right] + V_i^{j+1}\left[1 - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] + V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta t S_i}{4\Delta S}\right] \quad (21)$$

The expressions inside the square brackets will be replaced with the coefficients a, b, c. The following equations obtained.

$$aV_{i-1}^{j} + bV_i^{j} + cV_{i+1}^{j} = aV_{i-1}^{j+1} + bV_i^{j+1} + cV_{i+1}^{j+1} \quad (22)$$

Equation (22) can now be represented in matrix form as follows

$$XV^j = YV^{j+1},\ j = 0,1,2,\ldots$$

$$\Rightarrow u^j = X^{-1}YV^{j+1}$$

where $V^j = (V_{1,i,}V_{2,i,}V_{3,i,}\ldots,V_{m,i})^T$, ($V^i$ *is an* $m\times n$)

$$\begin{bmatrix} b & c & 0 & \cdots & 0 \\ -a & b & c & \cdots & 0 \\ 0 & -a & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & -a_{M-1} & b_{M-1} \end{bmatrix} \begin{bmatrix} V_1^j \\ V_2^j \\ V_3^j \\ \vdots \\ V_{M-1}^j \end{bmatrix} = \begin{bmatrix} X_1^j \\ X_2^j \\ X_3^j \\ \vdots \\ X_{M-1}^j \end{bmatrix}$$

$$\begin{bmatrix} b & c & 0 & \cdots & 0 \\ a & b & c & \cdots & 0 \\ 0 & a & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} \end{bmatrix} \begin{bmatrix} V_1^{j+1} \\ V_2^{j+1} \\ V_3^{j+1} \\ \vdots \\ V_{M-1}^{j+1} \end{bmatrix} = \begin{bmatrix} Y_1^{j+1} \\ Y_2^{j+1} \\ Y_3^{j+1} \\ \vdots \\ Y_{M-1}^{j+1} \end{bmatrix}$$

3. Modeling Pricing Effects of call and put options

Let $X_i, i = 1,...,T,$ be the Black-Scholes exact values for T trading days and $\bar{X} = \frac{1}{T}\sum X_i$

be the mean of the option value. Define $d_i = X_i - \bar{X}$, as the deviation from the mean. Define also

$$d_i^{\pm} = \begin{cases} X_i > \bar{X},\ i = 1,2,...,m_1 (\mathrm{m}_1\ is\, number\, of\ positive\ d_i) \\ X_i < \bar{X},\ i = 1,2,...,m_2 (m_2\ is\, number\, of\ negative\ d_i) \end{cases}$$

$$The\, mean\, of\ d_i^+\ is\, \overline{d_i}^+ = \frac{1}{m_1}\sum_{i=1}^{m_1} d_i^+,\ the\, mean\, of\ \ d_i^-\ \ is$$

$$\overline{d_i}^- = \frac{1}{m_2}\sum_{i=1}^{m_2} d_i^-,\quad \overline{d_i} = \frac{1}{T}\sum_{i=1}^{T} d_i$$

Here, we use standard deviation in creating strategies for investing and trading because it helps measure market volatility and predict performance trends. Note that $m_1 + m_2 = T$ if and only if $d_i \neq 0,$ otherwise $m_0 + m_1 + m_2 = T$,where m_0 is the number of zero differences $(d_i = 0).$ The standard deviations of the different components $d_i^-, d_i^+\ and\ d_i, \overline{d}^- < \overline{d} < \overline{d}^+\ are$

$$S_{d^+} = \sqrt{\frac{\sum\left(d_i^+\right)^2}{m-1}},\ S_{d^-} = \sqrt{\frac{\sum\left(d_i^-\right)^2}{m-1}},\ S_d = \sqrt{\frac{\sum\left(d_i\right)^2}{T-1}} \tag{23}$$

For simplicity and without loss of generality, we also define the following: $S_1, S_2, S_3,$ representing S_{d^+}, S_{d^-} and S_d respectively and

$$\beta_k = \frac{S_k}{\sqrt{\tau}}, k = 1,2,3 \tag{24}$$

Where τ is $\sqrt{252}$ trading days.

3.1 Criteria for selection

Let the index of price function be $\beta_k X_i$,where β_k is a constant and X_i a vector. $\beta_k X_i = \{A, B, C\}$ where $A, B,$ and C are the products of X_i and β_1, β_2 and β_k respectively. The maximum value r_1 will be referred to as overpricing if r_1 =max $\{A, B, C\}$,is middle value, r_2 as underpricing and r_3 as no mispricing if $r_3 =$ min

$\{A, B, C\}$ in line with [6] and [8].

4. Results

Table 1: Comparing the performance between the Black-Scholes exact values and Crank-Nicolson finite difference method for European Call Option with K = 25, r = 0.2 and T = 1

Sigma	S_o = 40, K = 25			S_o = 50, K = 25, r=0.2,		
	B-S Exact values	C-N	Relative Error	B-S Exact Values	C-N	Relative Error
0.25	19.5398	19.5378	1.0236E-04	29.5321	29.4815	1.7134E-03
0.3	19.5695	19.5564	6.6941E-04	29.5357	29.3841	5.1328E-03
0.35	19.6371	19.5926	2.266E-04	29.5506	29.2378	0.0106
0.4	19.7508	19.6468	5.2656E-03	29.5877	29.0650	0.0177
0.45	19.9117	19.7185	9.7028E-03	29.6565	28.8896	0.0259
0.5	20.1167	19.8070	0.0154	29.7625	28.7302	0.0347
0.55	20.3607	19.9121	0.0220	29.9075	28.5990	0.0438
0.6	20.6383	20.0334	0.0293	30.0906	28.5022	0.0528
0.65	20.9441	20.1704	0.0369	30.3094	28.4420	0.0616
0.7	21.2733	20.3221	0.0447	30.5604	28.4178	0.0701
0.75	21.6219	20.4873	0.0525	30.8401	28.4269	0.0782
0.8	21.9861	20.6641	0.06013	31.1446	28.4655	0.0860
0.85	22.3630	20.8506	0.06763	31.4707	28.5293	0.0935
0.9	22.7497	21.0445	0.07495	31.8150	28.6141	0.1006
0.95	23.1442	21.2436	0.08212	32.1746	28.7154	0.1075
1.0	23.5443	21.4457	0.08913	32.5470	28.8296	0.1142

Values in Tables 1 and 2 were generated by fixing r=0.2,k=25 while allowing S_0 and σ vary in (1) for call options such that $S_0 = (40, 50, 60, 70)$, $\sigma = (0.25, 0.30, ..., 1.00)$ .presented in column 1 are the difference values of σ, columns 2 and 3 are the exact values of BS and that of CN respectively. The 4th column gives the relative error (RE)
of the difference between the estimated prices using BS and CN pricing schemes given by $|BS - CN| / BS$ .RE is the ratio of absolute difference between BS and CN to BS such that when this ratio is very small, the performances of both BS and CN are equivalent, otherwise they are noticeable difference as can be observed in Table 1.The difference in performance increases from near zero with increasing value of σ .Visual inspection of figure 1 show that BS and CN are indistinguishable when $0 < \sigma < 0.5$ and the differences start to when σ is in the range $0.5 \le \sigma \le 1$ .Using same values of σ, r and K as in left panel of Table 1, only S_0 changed from 40 to 50 in the Table1(right panel) panels respectively. From Tables 1 and 2, we observe that besides influence of σ on RE, the initial price S_0 increases both BS exact values and CN approximate values and by implication, increases the RE as between BS and CN with varying values of S_0 and σ.

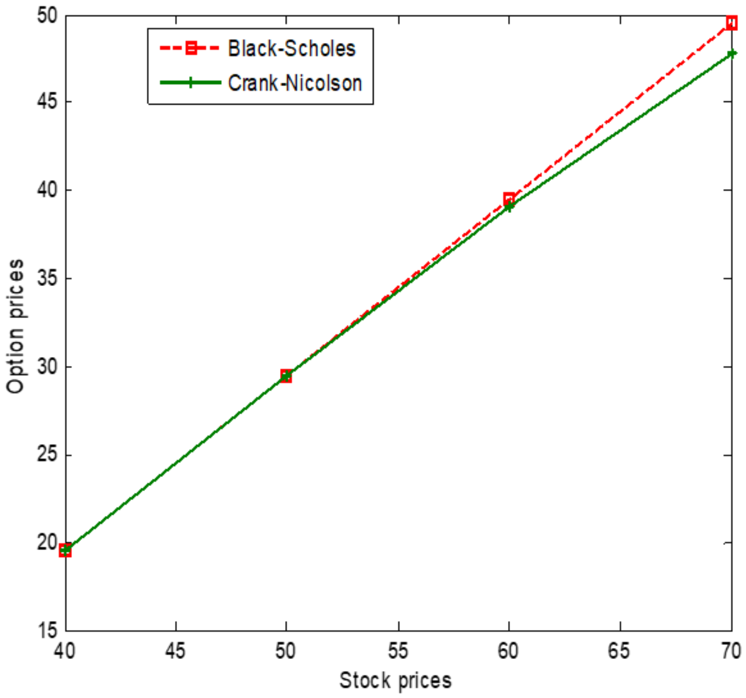


Figure 1: Comparing Black-Scholes with Crank-Nicolson numerical solution under different stock prices for call option.

Table 2: Comparing the performance between the Black-Scholes exact values and Crank-Nicolson finite difference method for European Call Option with K = 25, r = 0.2 and T = 1

Sigma	So = 60, K = 25			So =70, K =25, r=0.2		
	B-S Exact values	C-N	Relative Error	B-S Exact values	C-N	Relative Error
0.25	39.5317	39.1058	0.0108	49.5317	47.7497	0.0360
0.3	39.5322	38.7498	0.0198	49.5318	47.0748	0.0496
0.35	39.5353	38.3467	0.0301	49.5325	46.4363	0.0625
0.4	39.5469	37.9326	0.0408	49.5361	45.8475	0.0745
0.45	39.5750	37.5347	0.0516	49.5476	45.3156	0.0854
0.5	39.6277	37.1722	0.0620	49.5736	44.8451	0.0954
0.55	39.7106	36.8569	0.0719	49.6208	44.4385	0.1044
0.6	39.8273	36.5944	0.0812	49.6946	44.0960	0.1127
0.65	39.9788	36.3858	0.0899	49.7985	43.8150	0.1202
0.7	40.1643	36.2288	0.09799	49.9342	43.5912	0.1270
0.75	40.3820	36.1188	0.10557	50.1020	43.4186	0.1334
0.8	40.6295	36.0504	0.11270	50.3009	43.2908	0.1394
0.85	40.9038	36.0175	0.11946	50.5291	43.3010	0.14305
0.9	41.2022	36.0141	0.1259	50.7846	43.1432	0.1505
0.95	41.5216	36.0350	0.1321	51.0650	43.1115	0.1558
1.0	41.8593	36.0750	0.1382	51.3675	43.1010	0.1609

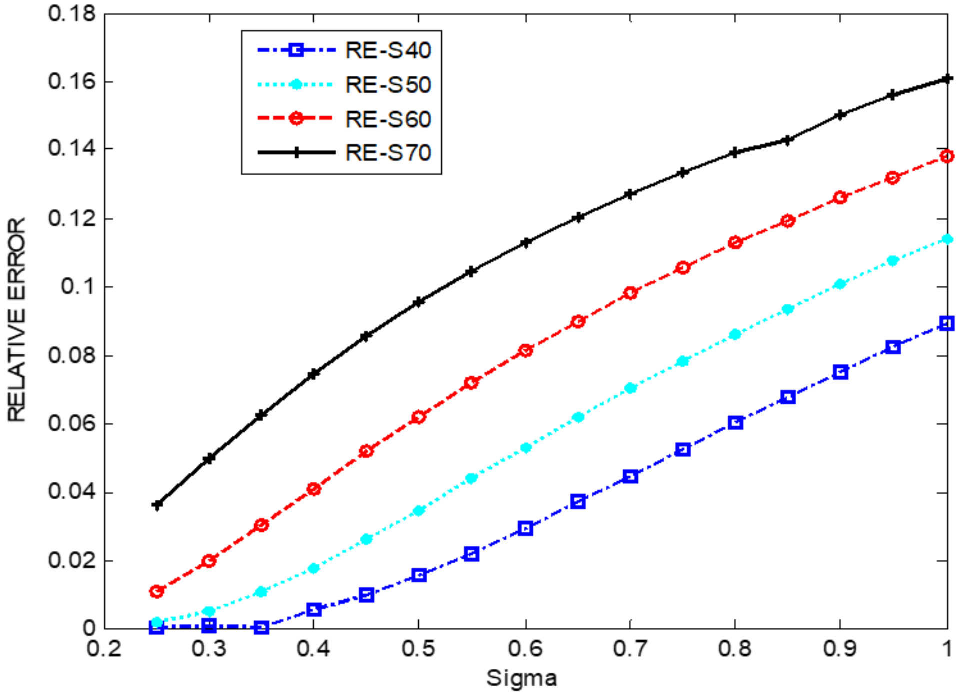


Figure 2: Levels of relative errors of different initial stock prices for call option.

This shows the error differences with different initial stock prices as against different variations of sigma. These also agree with our explanations in Tables 1 and 2 respectively.

Table 3: Levels of pricing effects under Call option when initial stock $S_0 = 40$

	So = 40, K = 25,					
Sigma	B-S Exact values	C-N	Over-pricing	Under-pricing	No Mispricin g	$r_1 - r_2 = r_3^*$
0.25	19.5398	19.5378	1.6648	1.0083	0.6175	0.6565
0.3	19.5695	19.5564	1.6673	1.0098	0.6184	0.6575
0.35	19.6371	19.5926	1.6731	1.01327	0.6205	0.6598
0.4	19.7508	19.6468	1.6828	1.01914	0.6241	0.6637
0.45	19.9117	19.7185	1.6965	1.02744	0.6292	0.6691
0.5	20.1167	19.8070	1.7139	1.03802	0.6357	0.6759
0.55	20.3607	19.9121	1.7347	1.05061	0.6434	0.6841
0.6	20.6383	20.0334	1.7584	1.06494	0.6522	0.6935
0.65	20.9441	20.1704	1.7844	1.08072	0.6618	0.7037
0.7	21.2733	20.3221	1.8125	1.09770	0.6722	0.7148
0.75	21.6219	20.4873	1.8422	1.1157	0.6833	0.7265
0.8	21.9861	20.6641	1.8732	1.1345	0.6948	0.7387
0.85	22.3630	20.8506	1.9053	1.1539	0.7067	0.7514
0.9	22.7497	21.0445	1.9383	1.1788	0.7189	0.7595
0.95	23.1442	21.2436	1.9719	1.1942	0.7314	0.7777
1.0	23.5443	21.4457	2.0060	1.2149	0.7440	0.7911

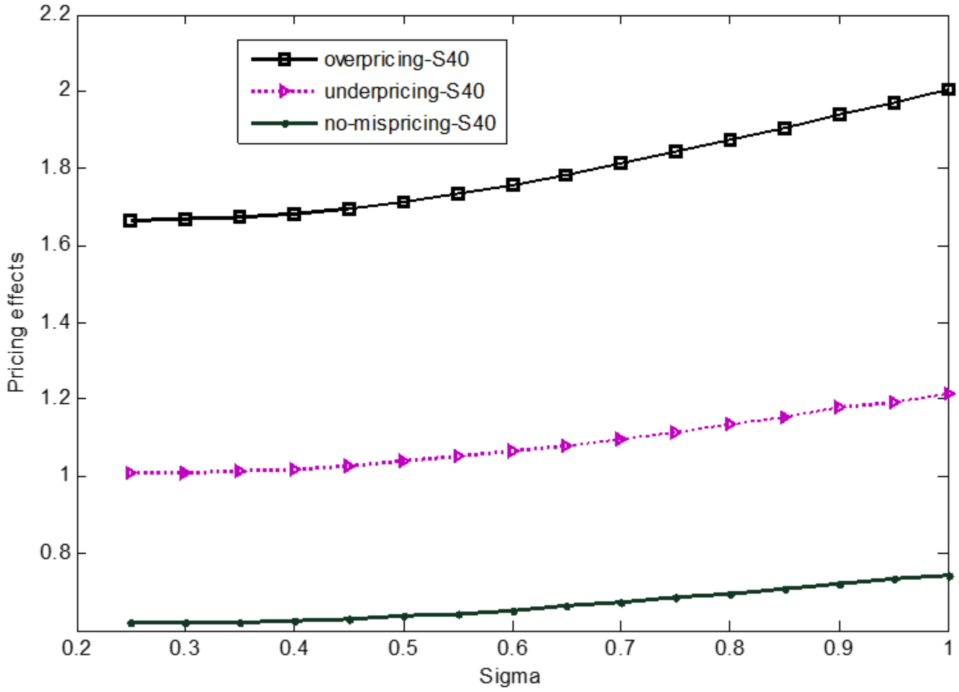


Figure 3: Levels of pricing effects when initial stock price is 40 for call option.

The graph of pricing effects with the same initial stock price of 40 with variations of sigma, overpricing yields high prices. Then the underpricing falls below the mean following the underlying asset. No- mispricing is close to the origin with a little rise at the peak; indicating that option prices match with the intrinsic value of stocks. The three plots, showed that, an increase in the sigma values dominantly indicate an increase in the option prices over the trading period of one year, (see figure 3).

Table 4: Levels of pricing effects under Call option whenthe initial stock $S_0 = 50$

	So = 50, K = 25, r=0.2					
Sigma	B-S Exact values	C-N	Over – pricing	Under – pricing	No Mis - Pricing	$r_1 - r_2 = r_3^*$
0.25	29.5321	29.4815	1.8960	1.3349	0.5168	0.5611
0.3	29.5357	29.3841	1.8962	1.3350	0.5169	0.5612
0.35	29.5506	29.2378	1.8971	1.3357	0.5171	0.5614
0.4	29.5877	29.0650	1.8995	1.3374	0.5178	0.5621
0.45	29.0565	28.8896	1.9040	1.3405	0.5190	0.5635
0.5	29.7625	28.7302	1.9108	1.3453	0.5208	0.5655
0.55	29.9075	28.5990	1.9201	1.3518	0.5234	0.5683
0.6	30.0906	28.5022	1.9318	1.3601	0.5266	0.5717
0.65	30.3094	28.4420	1.9459	1.3699	0.5304	0.576
0.7	30.5604	28.4178	1.9610	1.3813	0.5348	0.5797
0.75	30.8401	28.4269	1.9799	1.3939	0.5370	0.586
0.8	31.1446	28.4655	1.9995	1.4077	0.5450	0.5918
0.85	31.4707	28.5293	2.0204	1.4225	0.5507	0.5979
0.9	31.8150	28.6141	2.0425	1.4380	0.5568	0.6045
0.95	32.1746	28.7154	2.06561	1.4543	0.5631	0.6113
1.0	32.5470	28.8296	2.0895	1.4711	0.5696	0.6184

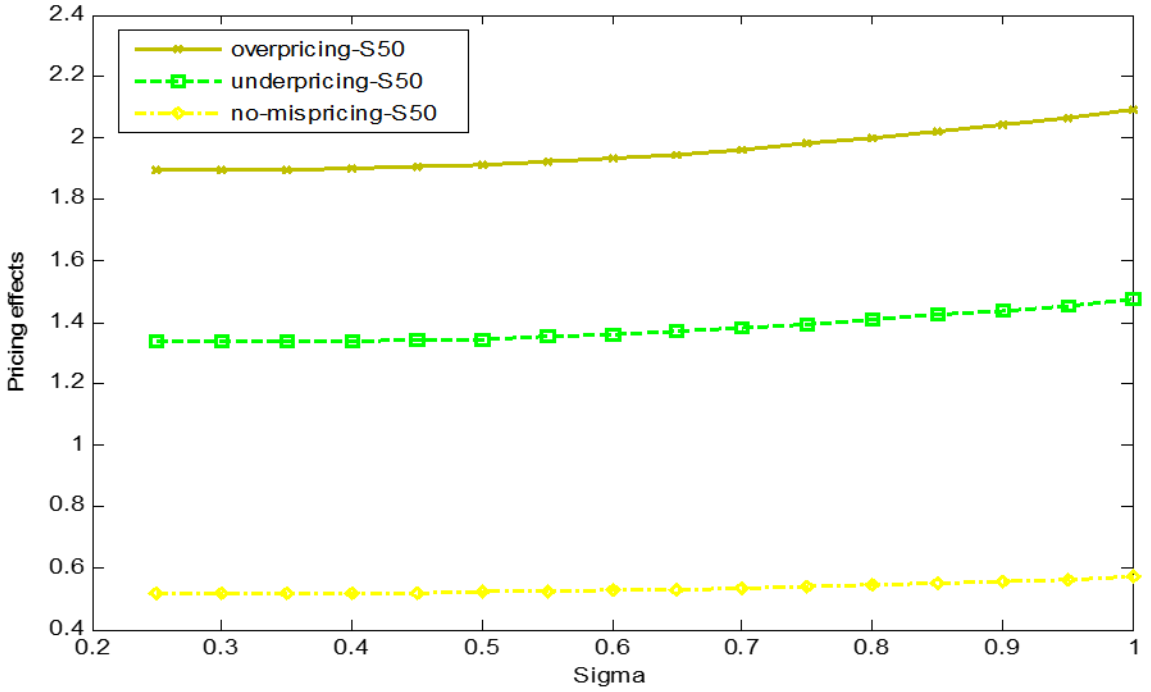


Figure 4: Levels of pricing effects when initial stock is 50 for call option.

In figure 4, we have observed, that the pricing effects increased above the previous in figures 3, due to the increase in the initial stock price of 50. So, overpricing trend increased and move forward which may not interest any option trader in any form. Whereas the underpricing trend is at the mean level; this is not also reliable for pricing call option within the stipulated trading days. Then no-mispricing still lie close to origin along sigma values, showing a good levels of pricing call option.

Table 5: Levels of pricing effects under Call option wheninitial stock $S_0 = 60$

So = 60, K=25, $r = 0.2$							
Sigma	B-S Exact values	C-N	Over-pricing	Under-pricing	No Mis-pricing		$\beta_1 - \beta_2 = \beta^*_3$
0.25	39.5317	39.1058	1.9410	1.3797	0.5495		0.5613
0.3	39.5322	38.7498	1.9410	1.3797	0.5495		0.5613
0.35	39.5353	28.3467	1.9412	1.3798	0.5495		0.5614
0.4	39.5469	37.9326	1.9418	1.3802	0.5497		0.5616
0.45	39.5750	37.5347	1.9431	1.3812	0.5501		0.5619
0.5	39.6277	37.1722	1.9457	1.3830	0.5508		0.5627
0.55	39.7106	36.8569	1.9498	1.3859	0.5520		0.5639
0.6	39.8273	36.5944	1.9555	1.3899	0.5536		0.5656
0.65	39.9788	36.3858	1.9630	1.3953	0.5557		0.5677
0.7	40.1643	36.2288	1.9721	1.4017	0.5583		0.5704
0.75	40.3820	36.1188	1.9828	1.4093	0.5613		0.5735
0.8	40.6295	36.0504	1.9949	1.4180	0.5648		0.5769
0.85	40.9038	36.0175	2.0084	1.4275	0.5686		0.5809
0.9	41.2022	36.0141	2.0230	1.4380	0.5727		0.5850
0.95	41.5216	36.0350	2.0387	1.4491	0.5772		0.5896
1.0	41.8593	36.0750	2.0553	1.4609	0.5818		0.5944

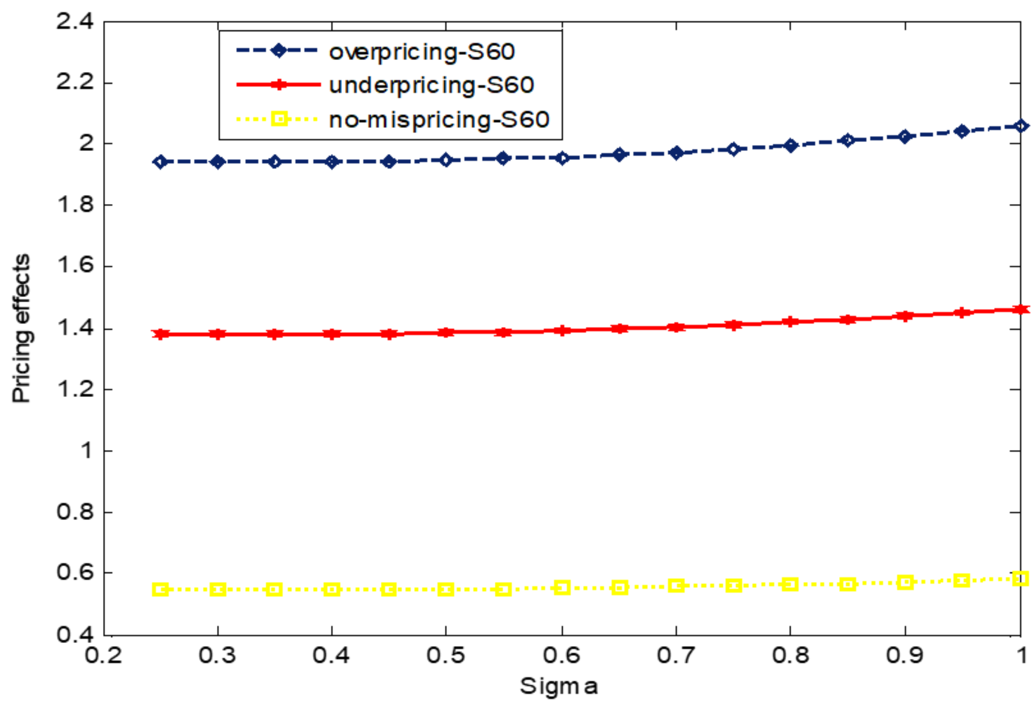


Figure 5: Levels of pricing effects when initial stock is 60 for call option.

It is clear that the plot of figure 5 is implemented following the initial stock price of 60. Overpricing yields a very high trend followed by underpricing which is a bit above the mean level. In this circumstance, profit making may not be possible due to inaccuracies in the valuation. Then no-mispricing effects revolves close to the origin ; indicating no pricing error, with this, decision making is possible.

Table 6: Levels of pricing effects under Call option when initial stock $S_0 = 70$.

So = 70, K = 25, $r = 0.2$						
Sigma	B-S Exact values	C-N	Over-pricing	Under-pricing	No Mis-pricing	$r_1 - r_2 = r_3^*$
0.25	49.5317	47.7497	1.8921	1.4860	0.4309	0.4061
0.3	49.5318	47.0748	1.8921	1.4860	0.4309	0.4061
0.35	49.5325	46.4363	1.8921	1.4860	0.4309	0.4061
0.4	49.5361	45.8475	1.8923	1.4861	0.4310	0.4062
0.45	49.5476	45.3156	1.8927	1.4864	0.4311	0.4063
0.5	49.5736	44.8451	1.8937	1.4872	0.4313	0.4065
0.55	49.6208	44.4385	1.8955	1.4886	0.4317	0.4069
0.6	49.6946	44.0960	1.8983	1.4908	0.4323	0.4075
0.65	49.7985	43.8150	1.9023	1.4940	0.4332	0.4083
0.7	49.9342	43.5912	1.9075	1.4980	0.4344	0.4095
0.75	50.1020	43.4186	1.9139	1.5031	0.4359	0.4108
0.8	50.3009	43.2908	1.9215	1.5090	0.4376	0.4125
0.85	50.5291	43.3010	1.9302	1.5159	0.4396	0.4143
0.9	50.7846	43.1432	1.9399	1.5235	0.4418	0.4164
0.95	51.0650	43.1115	1.9507	1.5320	0.4443	0.4187
1.0	51.3675	43.1010	1.9622	1.5410	0.4469	0.4212

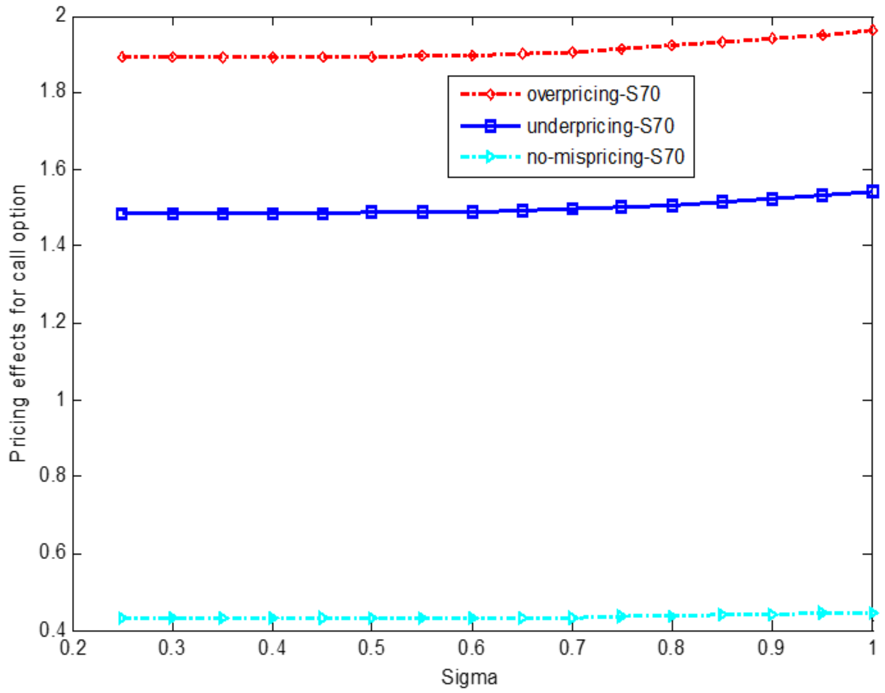


Figure 6: Levels of pricing effects when initial stock is 70 for call option

In figure 6, when the stock price changed to 70, the overpricing trend increased far more than any other plots, and then the underpricing trend rose above the mean level of the plot. This can lead to some levels of loss in the hand of any option trader for call option. In the contrary, the no-mispricing trend is stable, consistent and lies parallel along sigma starting from origin, implying no type 1 error due to chance. This situation can interest investors to be encouraged to invest in a stock exchange business which is index in millions of naira.

However, Tables 3 through 6 was generated following Section 2.2 and subsection 2.2.1 using (23) and (24) respectively withvarious BS exact values for call option. Therefore, column 4 were subtracted from column 5 to generate column 7 from number of Tables mentioned above .This is a show of comparison for model adequacy.

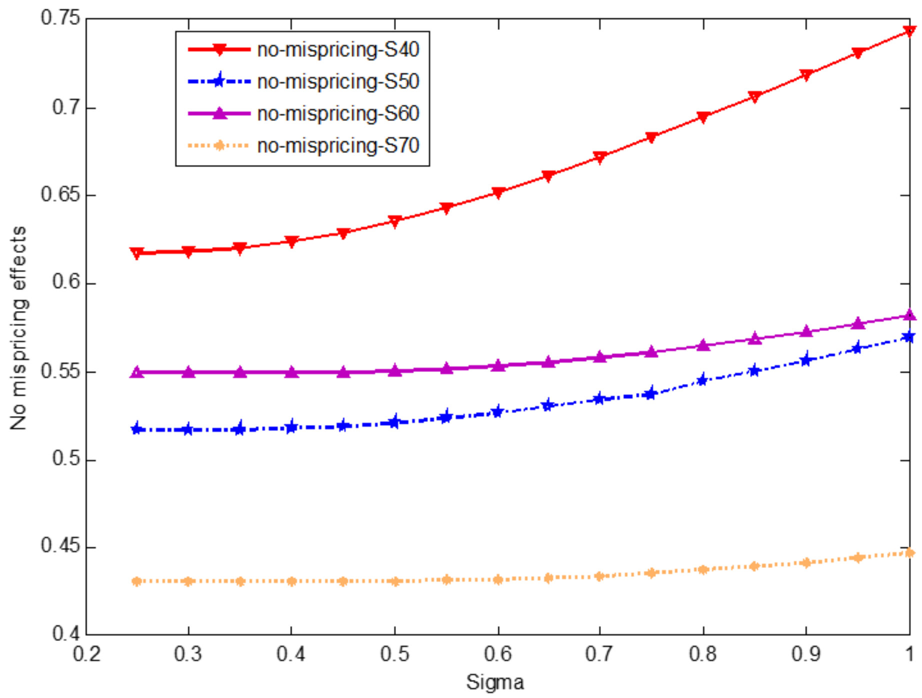


Figure 7: The summary results of no-mispricing effects with different initial stock prices.

In scenario Figure 7, the plot reveals that the initial stock price of 70 remains the best for call option, as you see that, it lies along the sigma axis below the mean level of the plot. This observation leads to a beneficial profit margin within 252 trading days and beyond.

Table 8: Comparing the performance between the Black-Scholes exact values and Crank-Nicolson finite difference method for European Put Option with K = 100, r = 0.2 and T = 1

Sigma Sigma	S_o = 40, K = 100			S_o = 50, K = 100,		
	B-S Exact values	C-N	Relative Error	B-S Exact Values	C-N	Relative Error
0.25	41.8817	41.8778	9.3119E-05	32.0183	31.9605	1.8052E-03
0.3	41.9211	41.8977	5.5819E-04	32.2717	32.1011	5.2864E-03
0.35	42.0213	41.9458	1.7967E-03	32.6694	32.3076	0.01107
0.4	42.2025	42.0283	4.1277E-03	33.1966	32.5633	0.01908
0.45	42.4721	42.1446	7.7109E-03	33.8322	32.8510	0.029002
0.5	42.8277	42.2898	0.01256	34.5553	33.1562	0.04049
0.55	43.2622	42.4578	0.01859	35.3479	33.4685	0.05317
0.6	43.7659	42.6422	0.02568	36.1951	33.7805	0.06671
0.65	44.3292	42.8374	0.03365	37.0850	34.0870	0.08084
0.7	44.9429	43.0389	0.04236	38.0079	34.3849	0.09532
0.75	45.5988	43.2432	0.05166	38.9559	34.6722	0.10996
0.8	46.2893	43.4474	0.06139	39.9226	34.9476	0.1246
0.85	47.0083	43.6494	0.07145	40.9028	35.2108	0.1392
0.9	47.7501	43.8478	0.08172	41.8921	35.4616	0.1535
0.95	48.5099	44.0415	0.09211	42.8869	35.7001	0.1676
1.0	49.2837	44.2297	0.10255	43.8839	35.9265	0.1813

Similarly, Values in Tables 8 and 9 were generated by fixing r=0.2, k=100 while allowing S_0 and σ vary in (2) for put options such that $S_0 = (40, 50, 60, 70), \sigma = (0.25, 0.30, ..., 1.00)$. The same interpretations for Tables 1,2 and figures 1,3 of call options applies to Tables 8,9 and figures 8,9 of put options; this particular plot shows a downward trend indicating put options which is direct opposite of call options(buying/selling).

Table 9: Comparing the performance between the Black-Scholes exact values and Crank-Nicolson finite difference method for European Put Option with K = 100, r = 0.2 and T = 1

Sigma	S_o = 60, K = 100			S_o = 70, K = 100		
	B-S Exact values	C-N	Relative Error	B-S Exact Values	C-N	Relative Error
0.25	22.7672	22.4676	0.01316	14.9160	14.0823	0.0559
0.3	23.4960	22.8954	0.02556	16.1862	14.8246	0.0841
0.35	24.3677	23.3644	0.04117	17.5121	15.5168	0.1139
0.4	25.3408	23.8438	0.05907	18.8713	16.1517	0.1441
0.45	26.3860	24.3156	0.07847	20.2504	16.7300	0.1738
0.5	27.4826	24.7698	0.0987	21.6405	17.2554	0.2026
0.55	28.6158	25.2013	0.1193	23.0357	17.7326	0.2302
0.6	29.7745	25.6080	0.1399	24.4317	18.1664	0.2564
0.65	30.9508	25.9896	0.1603	25.8252	18.5617	0.2813
0.7	32.1384	26.3466	0.1802	27.2137	18.9225	0.3047
0.75	33.3325	26.6801	0.1996	28.5950	19.2526	0.3267
0.8	34.5291	26.9914	0.2183	29.9675	19.5554	0.3474
0.85	35.7250	27.2822	0.2363	31.3296	19.8338	0.3669
0.9	36.9177	27.5537	0.2536	32.6800	20.0903	0.3852
0.95	38.1049	27.8076	0.2621	34.0175	20.3272	0.4024
1.0	39.2848	28.0450	0.2861	35.3412	20.5466	0.4186

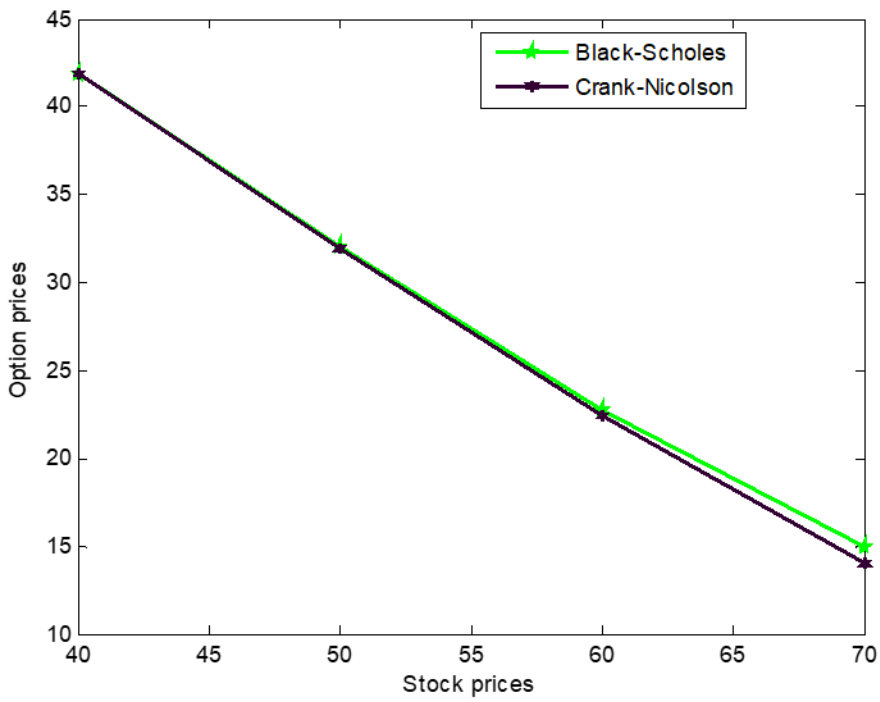


Figure 8: Comparing Black-Scholes with Crank-Nicolson numerical solution under different stock prices for put option.

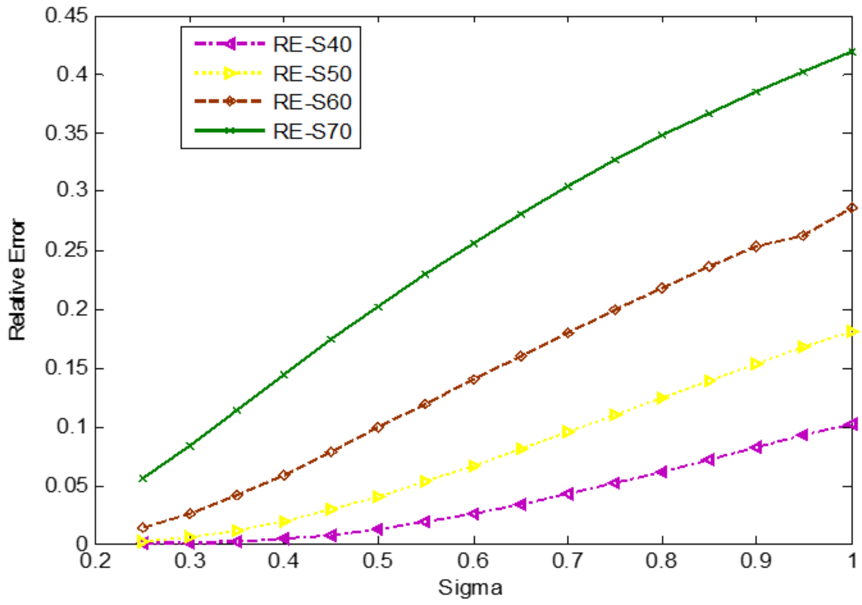


Figure 9: The Levels of relative errors under different initial stock prices for put option

Table 10: Levels of pricing effects under Put option when initial stock $S_0 = 40$

	So = 40, K = 100, $r = 0.2$					
Sigma	B-S Exact values	C-N	Over-pricing	Under-pricing	No Mispricing	$r_1 - r_2 = r_3^*$
0.25	41.8817	41.8778	6.6089	4.1337	2.3119	2.4752
0.3	41.9211	41.8977	6.6151	4.1376	2.3140	2.4775
0.35	42.0213	41.9458	6.6310	4.1475	2.3196	2.4835
0.4	42.2025	42.0283	6.6596	4.1654	2.3296	2.4942
0.45	42.4721	42.1446	6.7021	4.1920	2.3445	2.5101
0.5	42.8277	42.2898	6.7582	4.2271	2.3641	2.5311
0.55	43.2622	42.4578	6.8268	4.2699	2.3881	2.5569
0.6	43.7659	42.6422	6.9063	4.3197	2.4159	2.5866
0.65	44.3292	42.8374	6.9951	4.3753	2.4470	2.6198
0.7	44.9429	43.0389	7.0920	4.4359	2.4808	2.6561
0.75	45.5988	43.2432	7.1955	4.5006	2.5171	2.6949
0.8	46.2893	43.4474	7.3045	4.5688	2.5552	2.7357
0.85	47.0083	43.6494	7.4179	4.6397	2.5949	2.7782
0.9	47.7501	43.8478	7.5350	4.7129	2.6358	2.8221
0.95	48.5099	44.0415	7.6549	4.7879	2.6777	2.867
1.0	49.2837	44.2297	7.7770	4.8643	2.7205	2.9127

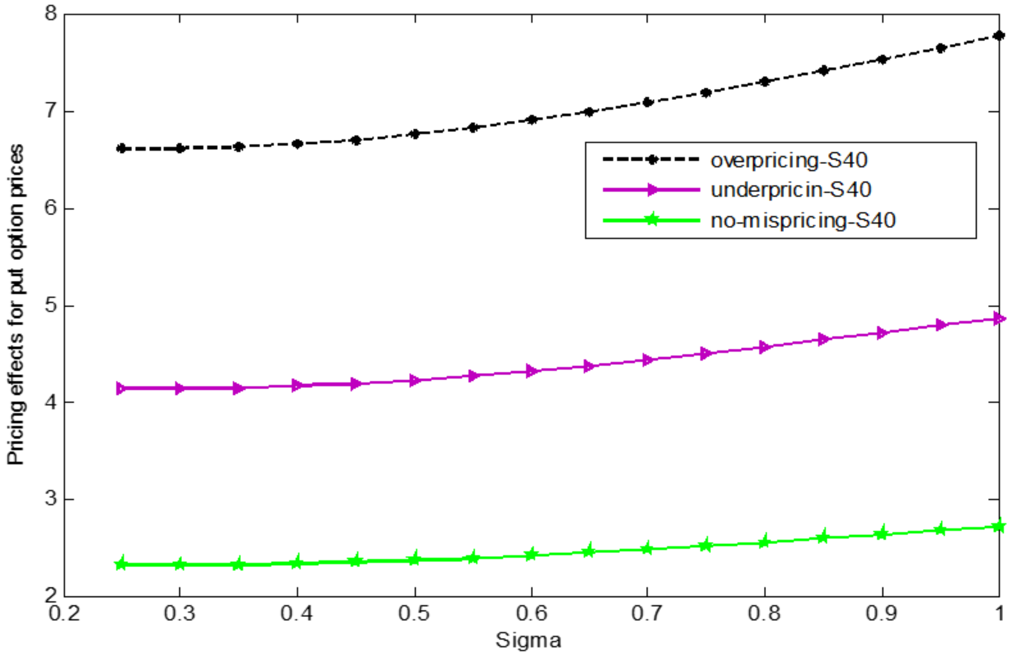


Figure 10: Levels of pricing effects when initial stock is 40 for put option.

In figure 10, overpricing yields an upward trend. Then the underpricing falls in between the two plots which is in line with our criteria for selections of pricing effects. No-mispricing lie along sigma axis indicating no pricing error during the trading period.

Table 11: Levels of pricing effects under Put option wheninitial stock $S_0 = 50$

So = 50, K = 100, $r = 0.2$						
Sigma	B-S Exact values	C-N	Over – pricing	Under – pricing	No Mis - Pricing	$r_1 - r_2 = r_3^*$
0.25	32.0183	31.9605	7.9694	4.2744	3.6117	3.695
0.3	32.2717	32.1011	8.0324	4.3083	3.6402	3.7241
0.35	32.6694	32.3076	8.1314	4.3614	3.6851	3.77
0.4	33.1966	32.5633	8.2626	4.4317	3.7445	3.8309
0.45	33.8322	32.8510	8.4208	4.5166	3.8163	3.9042
0.5	34.5553	33.1562	8.6008	4.6131	3.8998	3.9877
0.55	35.3479	33.4685	8.7981	4.7189	3.9872	4.0792
0.6	36.1951	33.7805	9.0090	4.8320	4.0828	4.177
0.65	37.0850	34.0870	9.2305	4.9508	4.1832	4.2797
0.7	38.0079	34.3849	9.4602	5.0741	4.2873	4.3861
0.75	38.9559	34.6722	9.6961	5.2006	4.3942	4.4955
0.8	39.9226	34.9476	9.9367	5.3297	4.5033	4.607
0.85	40.9028	35.2108	10.1807	5.4605	4.6138	4.7202
0.9	41.8921	35.4616	10.4269	5.5926	4.7254	4.8343
0.95	42.8869	35.7001	10.6745	5.7254	4.8376	4.9491
1.0	43.8839	35.9265	10.9227	5.8585	4.9501	5.0642

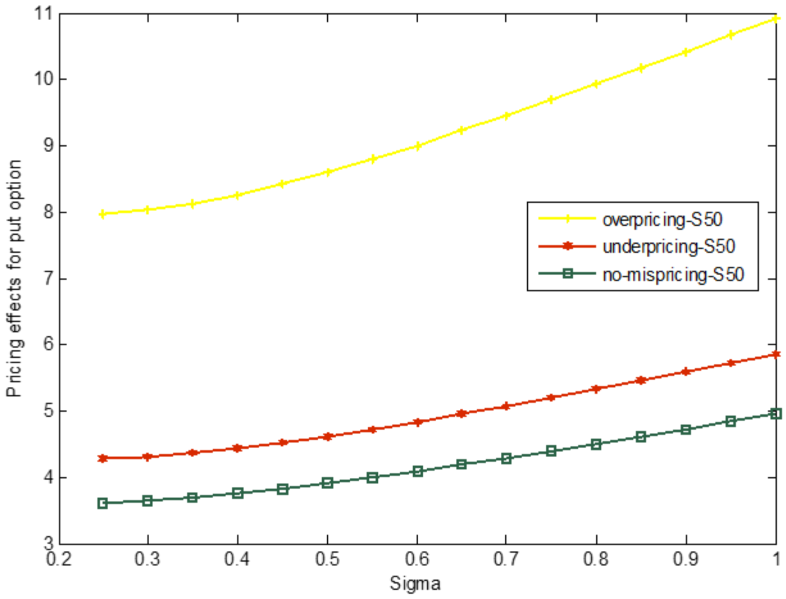


Figure 11: Levels of pricing effects when initial stock is 50 for put option.

It is observed in figure 11 that the three pricing effects moved upwardly with the same initial stock price of 50. Overpricing plot is an indication of high prices of put options which is on disadvantage to an option trader. Underpricing plots comes very close to no-mispricing which cannot be used for decision making; but for no-mispricing the trend lies along sigma axis but starting from point of origin.

Table 12: Levels of pricing effects under Put option wheninitial stock $S_0 = 60$

So = 60, K = 100, $r = 0.2$						
Sigma	B-S Exact values	C-N	Over-pricing	Under-pricing	No Mis-pricing	$r_1 - r_2 = r_3^*$
0.25	22.7672	22.4676	7.7113	4.1869	3.5653	3.5244
0.3	23.4960	22.8954	7.9581	4.3209	3.6795	3.6372
0.35	24.3677	23.3644	8.2533	4.4812	3.8160	3.7721
0.4	25.3408	23.8438	8.5829	4.6602	3.9684	3.9227
0.45	26.3860	24.3156	8.9369	4.8524	4.1320	4.0845
0.5	27.4826	24.7698	9.3084	5.0541	4.3038	4.2543
0.55	28.6158	25.2013	9.6922	5.2624	4.4812	4.4298
0.6	29.7745	25.6080	10.0846	5.4755	4.6627	4.6091
0.65	30.9508	25.9896	10.4830	5.6919	4.8469	4.7911
0.7	32.1384	26.3466	10.8853	5.9103	5.0329	4.975
0.75	33.3325	26.6801	11.2897	6.1298	5.2199	5.1599
0.8	34.5291	26.9914	11.6950	6.3499	5.4073	5.3451
0.85	35.7250	27.2822	12.1001	6.5698	5.5945	5.5308
0.9	36.9177	27.5537	12.5040	6.7892	5.7813	5.7148
0.95	38.1049	27.8076	12.9061	7.0075	5.9672	5.8986
1.0	39.2848	28.0450	13.3058	7.2245	6.1520	6.0813

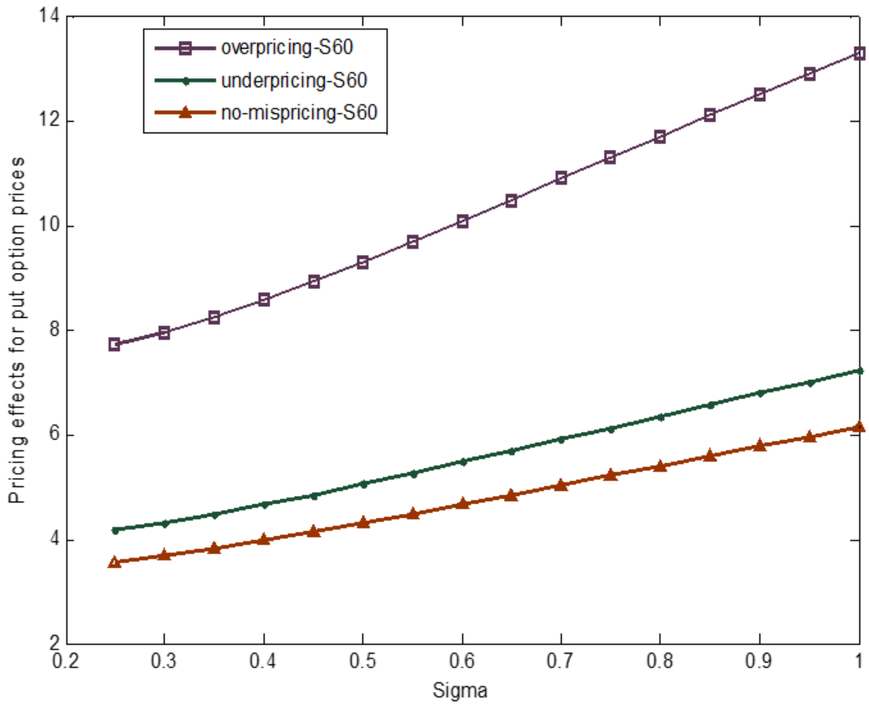


Figure 12 Levels of pricing effects when initial stock is 60 for put option

As shown in figure 12, the overpricing effect increased on steady trend. This is true because by definition. It means to charge too high a price for product. The underpricing effect moved below the mean level of the entire plot which is an indication for underpricing of option values. So no-mispricing still falls within the region starting from the origin lied along sigma axis; this is an impression of goods and services of the option trader which matches the intrinsic value of the item during the trading days of one year.

Table 13: Levels of pricing effects under Put option when initial stock $S_0 = 70$

So = 70, K = 100, $r = 0.2$						
Sigma	B-S Exact values	C-N	Over-pricing	Under-pricing	No Mis-pricing	$r_1 - r_2 = r_3^*$
0.25	14.9160	14.0823	6.1439	3.1413	3.1309	3.013
0.3	16.1862	14.8246	6.6671	3.4088	3.3975	3.2583
0.35	17.5121	15.5168	7.2132	3.6880	3.6758	3.5252
0.4	18.8713	16.1517	7.7731	3.9743	3.9611	3.7988
0.45	20.2504	16.7300	8.3411	4.2647	4.2506	4.0764
0.5	21.6405	17.2554	8.9137	4.5575	4.5423	4.3562
0.55	23.0357	17.7326	9.4884	4.8513	4.8352	4.6371
0.6	24.4317	18.1664	10.0634	5.1453	5.1282	4.9181
0.65	25.8252	18.5617	10.6374	5.4388	5.4207	5.1986
0.7	27.2137	18.9225	11.2093	5.7312	5.7122	5.4781
0.75	28.5950	19.2526	11.7783	6.0221	6.0021	5.7562
0.8	29.9675	19.5554	12.3436	6.3112	6.2902	6.0324
0.85	31.3296	19.8338	12.9047	6.5980	6.5761	6.3067
0.9	32.6800	20.0903	13.4609	6.8824	6.8595	6.5785
0.95	34.0175	20.3272	14.0118	7.1641	7.1403	6.8477
1.0	35.3412	20.5466	14.5570	7.4429	7.4181	7.1141

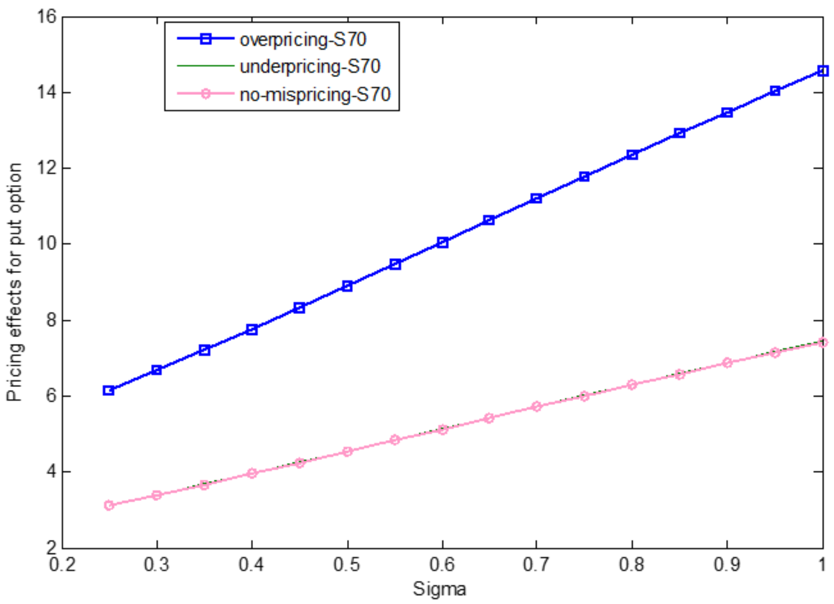
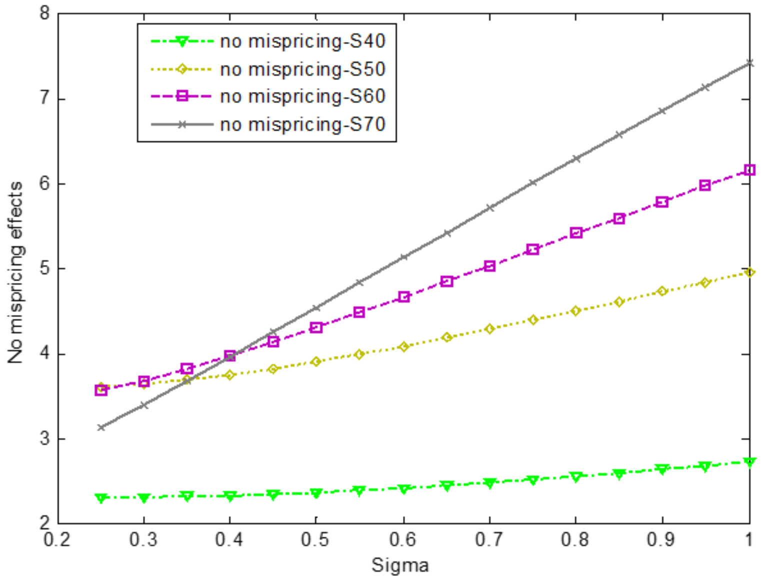


Figure 13: Levels of pricing effects when initial stock is 70 for put option.

In figure 13 scenario, we observed that overpricing effect showed an upward trend. Underpricing merged with no-mispricing because of price differences. With these two pricing effects profit making issue.



In Figure 14, the summary results of no-mispricing effects with different initial stock prices

In Figure 14, the plot reveals that the initial stock price of 40 remains the best for put options, which also have same meaning from figure 7 of call options.

H_0 : The BS call option and CN numerical solutions are from the same distribution.

H_1 : They are not from the same distribution.

Table15: Evaluation of Kolmogorov-Simirnov(KS) test for BS and CN

	S_0			
α	40	50	60	70
0.01	Reject	Accept	Accept	Accept
0.05	Reject	Accept	Accept	Accept

In Table 15, we observed that at $\alpha =$ 0.01 and 0.05 with $S_0 =$ 40, H_0 was rejected. But when initial stock prices increased to 50, 60 and 70 respectively with the same levels of significance H_1 was accepted. We therefore conclude that there is significant difference between BS and CN; it is statistically significant.

H_0 : The BS put option and CN numerical solutions are from the same distribution.

H_1 : They are not from the same distribution.

Table16: Kolmogorov-Simirnov test for BS and CN values come from a common distribution for Put option

	S_0			
α	40	50	60	70
0.01	Reject	Accept	Accept	Accept
0.05	Accept	Accept	Accept	Accept

In Table 16, we noticed that at $\alpha = 0.01$ with S_0 40, H_0 was rejected but when S_0 = 50, 60 and 70, H_1 was accepted. So, when level of significance increased to 0.05 using various initial stock prices of 40, 50, 60 and 70 H_1 was accepted. We can conclude by saying there is significant difference between BS and CN.

5. Conclusion

This paper adopted a method of Crank-Nicolson finite difference method; which is the best finite difference scheme in solving heat equation and similar partial differential equations. It gives better approximations than other schemes. In this work, the analytical formula for valuing European Call and Put options has been valued as well as Black-Scholes partial differential equation using Crank-Nicolson numerical solution for different stock prices; where there was no much difference in both solutions.

However, because of biasedness of BS PDE in mispricing options we formulated a new method where mispricing was divided into three interacting components namely: Overpricing, Underpricing and No mispricing on the premise to reduce pricing bias. In the same vain, deviation values were derived from BS analytical values where we placed certain criteria using statistical tools to determine some levels of pricing effects .Also all the initial stock prices of "no mispricing" were compared in figures 7 and 8; results showed that initial stock prices of 70 and 40 are the best for call and put options respectively. This is as a result of strike price K been lesser than stock prices(S>K) for call option. This type of option is said to be in the money. Whereas put option can only be exercised when the actual price of the asset is less than the strike price of the asset at expiration date. This option is said to be in the money. In another scenario, the model was subjected to goodness of test using KS, the test revealed that both options were statistically significant. Therefore, our novel contribution is unique, efficient and reliable in pricing option in this area of financial mathematics.

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