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Sequential Estimation of the Mean of a Class of Skewed Distributions

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Abstract

In this paper, we propose a sequential procedure $(t, \hat{\mu}_t)$ for estimating the mean, μ, of a class of skewed probability density functions, subject to the loss function $L_a = a^2(\hat{\mu}_t - \mu)^2 + t,$ where a is a given positive number, t is a stopping time of the type proposed by Robbins (1959) and $\hat{\mu}_t$ is a bias-corrected estimator of μ. We provide a second-order asymptotic expansion, as $a \to \infty$, for the regret with respect to the loss L_a. For the Pareto and Skew-uniform distributions, the proposed sequential procedure $(t, \hat{\mu}_t)$ performs better than the procedure $(t, \overline{X}_t),$ in the sense that it has a lower asymptotic regret. Moreover, the regret is negative for large values of a under the Gamma, Pareto, Rayleigh and Skew-uniform distributions. Using the loss considered by Chow and Yu (1981) and Martinsek (1988), we propose a bias-corrected estimator of μ and provide a second-order asymptotic expansion, as $a \to \infty$, for the incurred regret.

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1 Introduction

Let X_1, X_2, … be independent random variables with common probability density function $f_\theta(x)$, where the value of θ is unknown, but lies in some interval $\Omega \subset (-\infty, \infty)$. Suppose that X_1, X_2, … are to be observed sequentially up to stage n at a cost of one unit per observation and that when observation is terminated, the population mean

$$\mu = \int_{-\infty}^{\infty} x f_\theta(x) dx$$

is estimated by an appropriate estimator, $\hat{\mu}_n$, and the loss incurred is of the form

$$L_a(\hat{\mu}_n, \theta) = a^2(\hat{\mu}_n - \mu)^2 + n, \tag{1}$$

where a is a known positive number, determined by the cost of estimation relative to the cost of a single observation. Robbins (1959) proposed the sequential procedure $(t, \overline{X}_t)$, which stops the sampling process after observing X_1, …, X_t and estimates μ by $\hat{\mu}_t = \overline{X}_t$, where

$$t = \inf\left\{ n \geq m_a : n > a\sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{n}} \right\} \tag{2}$$

with m_a being a positive integer.

Let $\mathscr{C}$ denote the class of skewed probability density functions, $f_\theta(x)$, $\theta \in \Omega$, for which the skewness is independent of θ. This class contains, among others, the density functions of the following distributions:

1- GAMMA(α, θ): the Gamma distribution with known shape parameter α and scale parameter $\beta = \theta$. Its density function is

$f_\theta(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{x/\theta}, x > 0$ and its skewness is $\gamma = 2/\sqrt{\alpha}$.

2- PARETO(α, θ): the Pareto distribution with known shape parameter $\alpha > 0$ and scale parameter $\beta = \theta$. Its density function is $f_\theta(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, x \geq \theta$ and its skewness is $\gamma = \frac{2(1+\alpha)}{\alpha-3}\sqrt{\frac{\alpha-2}{\alpha}}$ for $\alpha > 3$.

3- RAYLEIGH(θ): the Rayleigh distribution with shape parameter $\alpha = \theta$. Its density function is $f_\theta(x) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}, x > 0$ and its skewness is $\gamma = \frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{3/2}}$.

4- SKEWUNIFORM(λ, θ): the Skew-uniform distribution with known λ and unknown θ. Its density function is $f_\theta(x) = \frac{1}{\theta^2}[\max\{\min\{\lambda x, \theta\}, -\theta\} + \theta]$, for $-\theta < x < \theta$ and its skewness is $\gamma = \frac{2\lambda(5\lambda^2 - 9)}{5(3-\lambda^2)^{3/2}}$ for $-\sqrt{3} < \lambda < \sqrt{3}$.

In this paper, we propose a bias-corrected estimator $\hat{\mu}_t$ of μ. and provide a second-order asymptotic expansion, as $a \to \infty$, for the regret $r_a(t, \hat{\mu}_t)$ with respect to the loss defined by (1). It is seen that the asymptotic regret is negative for the Gamma, Pareto, Rayleigh and Skew-uniform distributions. We also provide second-order asymptotic expansion, as $a \to \infty$, for the regret with respect to the more general loss function considered by Chow and Yu (1981) and Martinsek (1988).

In the Normal case, Starr and Woodroofe (1969) showed that $r_a(t, \overline{X}_t) = O(1)$ as $a \to \infty$. Woodroofe (1977) showed that $r_a(t, \overline{X}_t) = 0.5 + o(1)$ as $a \to \infty$ if $m_a \geq 4$. For the Gamma and Poisson cases, Starr and Woodroofe (1972) and Vardi (1979) obtained bounded regret using stopping times different from the one in (2). For the distribution-free case, Ghosh and Mukhopadhyay (1979) and Chow and Yu (1981) established asymptotic risk efficiency based on (2) under certain conditions. Tahir (1989) proposed a class of bias-reduction estimators of the mean of the one-parameter exponential family and provided a second order approximation for the regret.

2 Preliminary Notes

Let t be as in (2). Martinsek (1988) indicated that

$$E[\overline{X}_t] = \mu - \frac{\gamma}{2a} + o\left(\frac{1}{a}\right) \tag{3}$$

as $a \to \infty$, provided that $E[|X_1|^{8+p}] < \infty$ for some $p > 0$, where γ denotes the population skewness; that is, $\gamma = \sigma^{-3}E[(X_1 - \mu)^3]$, where σ is the population standard deviation. Thus, $\overline{X}_t$ is an asymptotically biased estimator of μ if $f_\theta(x) \in \mathscr{C}$.. Consider the bias-corrected estimator

$$\hat{\mu}_n = \overline{X}_n + \frac{\gamma}{2a} \tag{4}$$

for $n \geq 1$. Then, $E[\hat{\mu}_t] = \mu + o(1)$ as $a \to \infty$, by (3).

In order to define the regret incurred by the sequential procedure $(t, \hat{\mu}_t)$ under the loss (1), we first assume that $X_1, \ldots, X_n$ have been observed sequentially up to a predetermined stage n from a population with density function $f_\theta(x) \in \mathscr{C}$. The risk incurred by estimating μ by (4), subject to the loss (1), is

$$R_a(n,\theta) = E[L_a(n,\hat{\mu}_n)]$$
$$= E[a^2(\overline{X}_n - \mu)^2] + \frac{a^2\gamma}{a^2}E[(\overline{X}_n - \mu)] + \frac{\gamma^2}{4} + n$$
$$= \frac{\mathrm{a}^2\sigma^2}{n} + \frac{\gamma^2}{4} + n,$$

This risk is minimized with respect to n by choosing n as the greatest integer less than or equal to $n_a = a\sigma$. The minimum risk is

$$R_a^*(\theta) = R_a(n_a,\theta) = 2a\sigma + \frac{\gamma^2}{4}$$

for $a > 0$. Since σ is unknown, there is no fixed-sample-size procedure that attains the minimum risk in practice. Therefore, we propose to use the sequential procedure $(t,\hat{\mu}_t)$, where t be as in (2). The performance of this procedure is measured by its regret, which is defined below.

Definition 2.1 *The regret of the procedure* $(t,\hat{\mu}_t)$ *under the loss (2) is defined as*

$$r_a(t,\hat{\mu}_t) = E[L_a(t,\hat{\mu}_t)] - R_a^*(\theta) = E[a^2(\hat{\mu}_t - \mu)^2 + t] - 2a\sigma - \frac{\gamma^2}{4} \quad (5)$$

for $a > 0$.

The stopping time t in (2) can be rewritten as

$$t = \inf\left\{ n \geq m_a : n\left(\frac{V_n}{n}\right)^{-1/2} > a \right\},$$

where

$$V_n = \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \quad (6)$$

for $n \geq 1$. Let $U_a = t(V_t/t)^{-1/2} - a$ denote the excess over the stopping boundary. Chang and Hsiung (1979) showed that U_a converges in distribution to a random variable U as a $\to \infty$.

Lemma 2.2. *Let t be as in (2). Then,* $\frac{t}{a} \to \sigma$ *w.p.1 as* $a \to \infty$*. Moreover, If* $E[|X_1|^{8+p}] < \infty$ *for some* $p > 0$*, then*

$$E[t] = a + \nu - 0.5 - \frac{3}{8}\sigma^4(\kappa - 1) + o(1)$$

as $a \to \infty$*, where* $\nu = E[U]$ *is the asymptotic mean of the excess over the boundary and* $\kappa = \sigma^{-4}E[(X_1 - \mu)^4]$ *is the population kurtosis.*

Proof: The first assertion follows from Lemma 1 of Chow and Robbins (1965). The second assertion is adopted from Chang and Hsiung (1979).

3 Main Results

3.1 Asymptotic regret under the loss (1)

Let $X_1, X_2, \ldots$ be as in Section 1. The following theorem provides a second-order asymptotic expansion for the regret in (5).

Theorem 3.1. Let t be defined by (2) with m_a being such that $\delta\sqrt{a} \le m_a = o(a)$ as $a \to \infty$ for some $\delta > 0$. For any probability density function $f_\theta(x) \in \mathscr{C}$ with respect to which $E[|X_1|^{8+p}] < \infty$ for some $p > 0$,

$$r_a(t, \hat{\mu}_t) = 2.75 - 0.75\kappa + 2\gamma^2 - \frac{\gamma}{2} + o(1)$$

as $a \to \infty$.

Proof: Substituting (4) in (5) yields

$$\begin{aligned} r_a(t, \hat{\mu}_t) &= E[a^2(\overline{X}_t - \mu)^2 + t - 2a\sigma] + a\gamma E[(\overline{X}_t - \mu)] \\ &= r_a(t, \overline{X}_t) + a\gamma E[(\overline{X}_t - \mu)] \end{aligned} \tag{7}$$

for $a > 0$. Moreover,

$$aE[(\overline{X}_t - \mu)] = -\gamma/2 + o(1) \quad \text{and} \quad r_a(t, \overline{X}_t) = 2.75 - 0.75\kappa + 2\gamma^2 + o(1) \tag{8}$$

as $a \to \infty$, by (3) and Martinsek (1983). Take the limit as $a \to \infty$ in (7) and use (8) to complete the proof.

The distributions considered in Tables 1-5 in Section 4 below are positively skewed, except for the Skew-uniform distribution with $-\sqrt{3} < \lambda < -\dfrac{3}{\sqrt{5}}$ and Skew-Laplace distribution with λ = 0.5. For Table 1, the minimum value of ρ^* is 75/28 ≈ 2.68, which is attained when $\alpha = 49$. The tables show that

1- the sequential procedure $(t, \hat{\mu}_t)$ is a clear improvement over the procedure $(t, \overline{X}_t)$ since its asymptotic regret is lower, except for the Skew-uniform distribution with $\lambda = -1.4$.

2- the asymptotic regret of the procedure $(t, \hat{\mu}_t)$ under the PARETO(5, θ) and SKEWUNIFORM(λ, θ) distributions is negative; which means that, for large values of a that the procedure $(t, \hat{\mu}_t)$ performs better for these distributions than the best fixed-sample-size procedure.

3.2 Asymptotic regret under a more general loss function

Let X_1, X_2, … be as in Section 1 and suppose that the loss function for estimating μ is of the form considered by Chow and Yu (1981) and Martinsek (1988); that is,

$$L_a(\mu_n^*, \theta) = a^2 \sigma^{2\beta-2} (\mu_n^* - \mu)^2 + n \tag{9}$$

for $a > 0$, where β is a given positive number and μ_n^* is an estimator of μ. If θ is estimated by $\mu_n^* = \overline{X}_n$, Martinsek (1988) proposed to use the stopping time

$$T = \inf\left\{ n \ge m_a : n > a\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2\right)^{\beta/2}\right\} \tag{10}$$

and showed that the regret of the procedure $(T, \overline{X}_T)$ under the loss (9) is

$$\begin{aligned} r_a^*(T, \overline{X}_T) &= E[a^2\sigma^{2\beta-2}(\overline{X}_T - \mu)^2 + T] - 2a\sigma^\beta \\ &= 3\beta - \frac{\beta^2}{4} + \left(\frac{\beta^2}{4} - \beta\right)\kappa + (\beta^2 + \beta)\gamma^2 + o(1) \end{aligned} \tag{11}$$

as $a \to \infty$, provided that $E[|X_1|^{8+p}] < \infty$ for some $p > 0$. Straightforward calculations yield that, for large values of a,

1) $r_a^*(T, \overline{X}_T)$ is negative under the Gamma distribution with $\alpha = 0.5$ if $0 < \beta < 0.1$.

2) $r_a^*(T, \overline{X}_T)$ is negative under the Pareto distribution with $\alpha = 5$ if $0 < \beta < 1.24$.

Martinsek (1988) also indicated that

$$E[\overline{X}_T] = \mu - \frac{\beta\gamma}{2a\sigma^{\beta-1}} + o\left(\frac{1}{a}\right) \tag{12}$$

as $a \to \infty$. Thus, if the distribution of X_1 is not symmetric, then $\overline{X}_T$ is biased for large values of a.

Proposition 3.2: *Suppose that γ does not depend on θ and let*

$$\mu_n^* = \overline{X}_n + \frac{\beta\gamma}{2a^{1/\beta}n^{1-1/\beta}}$$

for $n \ge 1$, where $\beta > 1$. Let T be defined by (10) with m_a being such that $\delta\sqrt{a} \le m_a = o(a)$ as $a \to \infty$ for some $\delta > 0$. For any probability density function $f_\theta(x) \in \mathscr{C}$ with respect to which $E[|X_1|^{8+p}] < \infty$ for some $p > 0$,

$E[\mu_T^*] = \mu + o(1)$ as $a \to \infty$.

Proof: For $a > 0$,

$$aE[\mu_T^* - \mu] = aE[\overline{X}_T - \mu] + \frac{\beta\gamma}{2}E\left[\left(\frac{T}{a}\right)^{-(1-1/\beta)}\right]. \tag{13}$$

Next, $E[(T/a)^{-(1-1/\beta)}] \to \sigma^{1-\beta}$ as $a \to \infty$ if $\beta > 1$, by the fact that $T/a \to \sigma^\beta$

w.p.1 as $a \to \infty$ and (2.2) of Martinsek (1983). Taking the limit as $a \to \infty$ in (13), using this fact and (12) yields the desired result.

Let $r_a^*(T, \mu_T^*)$ denote the regret of the biased-corrected procedure (T, μ_T^*) under the loss (9). Then,

$$r_a^*(T,\mu_T^*) = E[a^2\sigma^{2\beta-2}(\bar{X}_T-\mu)^2 + T - 2a\sigma^\beta] + \beta\gamma\sigma^{2\beta-2}a^{2-1/\beta}E\left[\frac{1}{T^{1-1/\beta}}\left(\bar{X}_T-\mu\right)\right]$$
$$+\frac{\gamma^2\sigma^{2\beta-2}}{4}E\left[\frac{a^{2-2/\beta}}{T^{2-2/\beta}}\right] \tag{14}$$
$$= r_a^*(T,\bar{X}_T) + \beta\gamma\sigma^{2\beta-2}E\left[\frac{a^{1-1/\beta}}{T^{1-1/\beta}}a\left(\bar{X}_T-\mu\right)\right] + \frac{\gamma^2\sigma^{2\beta-2}}{4}E\left[\frac{a^{2-2/\beta}}{T^{2-2/\beta}}\right]$$

Lemma 3.3: *Let T be as in (3.2) with $\beta > 1$. If $E[|X_1|^{8+p}] < \infty$ for some $p > 0$, then*

$$E\left[\frac{a^{1-1/\beta}}{T^{1-1/\beta}}a\left(\bar{X}_T-\mu\right)\right] = \frac{2(\beta-1)}{\sigma^{2\beta+1}} - \frac{\beta\gamma}{2\sigma^{2(\beta-1)}}\,\mathrm{o}(1)$$

as $a \to \infty$.

Proof: First, observe that

$$E\left[\frac{a^{1-1/\beta}}{T^{1-1/\beta}}a\left(\bar{X}_T-\mu\right)\right] = E\left[\left(\frac{a^{1-1/\beta}}{T^{1-1/\beta}} - \frac{1}{\sigma^{\beta-1}}\right)a\left(\bar{X}_T-\mu\right)\right] + \frac{1}{\sigma^{\beta-1}}aE[\bar{X}_T-\mu] \tag{15}$$

for $a > 0$. Moreover,

$$aE[\bar{X}_T-\mu] = -\frac{\beta\gamma}{2\sigma^{\beta-1}} + o(1) \tag{16}$$

as $a \to \infty$, by (12). Next, expand $g(y) = 1/y^{1-1/\beta}$ at $y = \sigma^\beta$, substitute $y = a/T$ and multiply by $a\left(\bar{X}_T-\mu\right)$ to obtain

$$\left(\frac{a^{1-1/\beta}}{T^{1-1/\beta}} - \frac{1}{\sigma^{\beta-1}}\right)a(\bar{X}_T-\mu) = \left(\frac{1}{\beta}-1\right)T_*^{1/\beta-2}\left(\frac{T}{a}-\sigma^\beta\right)a(\bar{X}_T-\mu), \tag{17}$$

where T_* is a random variable such that $|T_* - \sigma^\beta| \le |T/a - \sigma^\beta|$. Next, rewrite T in as $T = \inf\{ n \ge m_a: n(V_n/n)^{-\beta/2} > a\}$, where V_n is as in (6), and let

$$U_a^* = T\left(\frac{V_T}{T}\right)^{-\beta/2} - a$$

denote the excess over the stopping boundary. Expanding $h(y) = y^{-\beta/2}$ at $y = \sigma^2$, substituting $y = V_T/T$ and multiplying by T yields

$$T\left(\frac{V_T}{T}\right)^{-\beta/2} = \frac{T}{\sigma^\beta} - \frac{\beta}{2\sigma^{\beta+2}}(V_T - T\sigma^2) + \frac{\beta(\beta+2)}{8\lambda_T^{\beta/2+2}}\frac{(V_T - T\sigma^2)^2}{T}$$

for $a > 0$, where λ_T is a random variable between V_T/T and σ^2. Furthermore, write $V_T = \sum_{i=1}^{T}(X_i - \mu)^2 - T(\overline{X}_T - \mu)^2$ to obtain

$$U_a^* = \frac{T}{\sigma^\beta} - a - \frac{\beta}{2\sigma^{\beta+2}}(W_T - T\sigma^2) + \frac{\beta}{2\sigma^{\beta+2}}T(\overline{X}_T - \mu)^2 + \frac{\beta(\beta+2)}{8\lambda_T^{\beta/2+2}}\frac{(V_T - T\sigma^2)^2}{T}$$

for $a > 0$, where $W_T = \sum_{i=1}^{T}(X_i - \mu)^2$. It follows from easily that

$$\frac{T}{a} - \sigma^\beta = \frac{\sigma^\beta}{a}(U_a^* - \xi_T) + \frac{\beta}{2a\sigma^2}(W_T - T\sigma^2) \tag{18}$$

for $a > 0$, where

$$\xi_t = \frac{\beta}{2\sigma^{\beta+2}}T(\overline{X}_T - \mu)^2 + \frac{\beta(\beta+2)}{8\lambda_T^{\beta/2+2}}\frac{(V_T - T\sigma^2)^2}{T}.$$

Substituting (18) in (17) yields

$$\begin{aligned}\left(\frac{a^{1-1/\beta}}{T^{1-1/\beta}} - \frac{1}{\sigma^{\beta-1}}\right)a(\overline{X}_T - \mu) &= \left(\frac{1}{\beta} - 1\right)\sigma^\beta T_*^{1/\beta-2}(U_a - \xi_T)(\overline{X}_T - \mu) \\ &\quad + \left(\frac{1}{\beta} - 1\right)\frac{\beta}{2\sigma^2}T_*^{1/\beta-2}(W_T - T\sigma^2)(\overline{X}_T - \mu) \\ &= \left(\frac{1}{\beta} - 1\right)\sigma^\beta I_1(a) + \frac{1-\beta}{2\sigma^2}I_2(a),\end{aligned} \tag{19}$$

say. Let $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Then,

$$E[|I_1(a)|] = E\left[\left|\frac{T_*^{1/\beta-2}}{T}(U_a - \xi_T)(S_T - \mu T)\right|\right] = \frac{\sigma^\beta}{\sqrt{a\sigma^\beta}} E\left[\left|(U_a - \xi_T)\frac{a}{T}T_*^{1/\beta-2}\frac{(S_T - \mu T)}{\sqrt{a\sigma^\beta}}\right|\right]$$

$$\le \frac{\sqrt{\sigma^\beta}}{\sqrt{a}}\sqrt{E[(U_a - \xi_T)^2]}\sqrt{E\left[T_*^{2/\beta-4}\left(\frac{a}{T}\right)^2\left(\frac{S_T - \mu T}{\sqrt{a\sigma^\beta}}\right)^2\right]}$$

$$\le \frac{1}{\sqrt{a}}\sqrt{2\sigma^\beta E[U_a^2] + 2\sigma^\beta E[\xi_T^2]}\sqrt{E\left[T_*^{2/\beta-4}\left(\frac{a}{T}\right)^2\left(\frac{S_T - \mu T}{\sqrt{a\sigma^\beta}}\right)^2\right]}$$

$$\to 0 \qquad (20)$$

as a $\to \infty$, by Hölder's inequality, the fact that $T_* \to \sigma^\beta$ ($|T_* - \sigma^\beta| \le |T/a - \sigma^\beta| \to 0$ w.p.1 since $T/a \to \sigma^\beta$, as in the first assertion of Lemma 1), $\frac{S_T - \mu T}{\sqrt{a\sigma^\beta}}$ converges in distribution to a Standard Normal random variable by Anscombe's theorem, the facts that $E[U_a^2] \to E[U^2] < \infty$ and $E[\xi_T^2] = O(1)$ as a $\to \infty$ and (2.3), (2.8) and (2.9) of Martinsek (1983). To evaluate $E[I_2(a)]$, observe that

$$I_2(a) = \frac{2a\sigma^\beta}{T}T_*^{1/\beta-2}\frac{(W_T - T\sigma^2)(S_T - \mu T)}{a\sigma^\beta} = 2\sigma^\beta\frac{a}{T}T_*^{1/\beta-2}\left(\frac{W_T - \sigma^2 T}{\sqrt{a\sigma^\beta}} + \frac{S_T - \mu T}{\sqrt{a\sigma^\beta}}\right)^2$$

$$-2\sigma^\beta\frac{a}{T}T_*^{1/\beta-2}\left(\frac{W_T - \sigma^2 T}{\sqrt{a\sigma^\beta}}\right)^2 - 2\sigma^\beta\frac{a}{T}T_*^{1/\beta-2}\left(\frac{S_T - \mu T}{\sqrt{a\sigma^\beta}}\right)^2$$

$$\xrightarrow{\text{in distribution}} 2\sigma^{1-2\beta}(2Z)^2 - 2\sigma^{1-2\beta}Z^2 - 2\sigma^{1-2\beta}Z^2 = 4\sigma^{1-2\beta}Z^2 \qquad (21)$$

as $a \to \infty$, by Anscombe's theorem and the fact that $T_* \to \sigma^\beta$ w.p.1 as $a \to \infty$, where Z is a random variable having the Standard Normal distribution. Thus,

$$E[I_2(a)] = 4\sigma^{1-2\beta} + o(1) \qquad (22)$$

as $a \to \infty$, by (21) and (2.3) and (2.4) of Martinsek (1983). Taking expectation in (19) and using (20) and (22) yields

$$E\left[\left(\frac{a^{1-1/\beta}}{T^{1-1/\beta}} - \frac{1}{\sigma^{\beta-1}}\right)a\left(\bar{X}_T - \mu\right)\right] = \frac{2(1-\beta)}{\sigma^{2\beta+1}} + o(1) \qquad (23)$$

as $a \to \infty$. The lemma follows by taking the limit, as $a \to \infty$, in (15) and using (23) and (16).

Theorem 3.4: *Let T be defined by (3.2) with m_a being such that $\delta\sqrt{a} \le m_a = o(a)$ as $a \to \infty$ for some $\delta > 0$ and $\beta > 1$. Then, for any probability density function $f_\theta(x) \in C$ with respect to which $E[|X_1|^{8+p}] < \infty$ for some $p > 0$,*

$$r_a^*(T, \mu_T^*) = 3\beta - \frac{\beta^2}{4} + \left(\frac{\beta^2}{4} - \beta\right)\kappa + (\beta^2 + \beta)\gamma^2 + \frac{2\beta(\beta-1)\gamma}{\sigma^3} - \frac{\beta^2\gamma^2}{2} + \frac{\gamma^2}{4} + o(1)$$

as $a \to \infty$.

Proof: The theorem follows by taking the limit, as $a \to \infty$, in (14) and using (11), Lemma 3.3 and the fact that

$$E\left[\frac{a^{2-2/\beta}}{T^{2-2/\beta}}\right] = \frac{1}{\sigma^{2\beta-2}} + o(1)$$

as $a \to \infty$ if $\beta > 1$, by the fact that $T/a \to \sigma^\beta$ w.p.1 as $a \to \infty$ (see the first assertion of Lemma 2.2) and (2.2) of Martinsek (1983) .

4 Tables

The tables below show the values of ρ and ρ^* for certain skewed distributions, where $\rho^* = \rho - \frac{\gamma}{2}$ is the asymptotic regret incurred by the procedure $(t, \hat{\mu}_t)$ and $\rho = 2.75 - 0.75\kappa + 2\gamma^2$ represents the asymptotic regret incurred by the procedure $(t, \overline{X}_t)$.

Table 1: GAMMA(α, θ) with known α

γ	κ	ρ	ρ*
$\frac{2}{\sqrt{\alpha}}$	$\frac{6}{\alpha}$	$2.75 + \frac{3.5}{\alpha}$	$2.75 + \frac{3.5}{\alpha} - \frac{1}{\sqrt{\alpha}}$

Table 2: PARETO(5, θ)

γ	κ	ρ	ρ*
$\frac{2(1+\alpha)}{\alpha-3}\sqrt{\frac{\alpha-2}{\alpha}} = 4.6476$	$\frac{6(\alpha^3+\alpha^2-6\alpha-2)}{\alpha(\alpha-3)(\alpha-4)} + 3 = 73.8$	**-9.4**	**-11.7238**

Table 3: RAYLEIGH(θ)

γ	κ	ρ	ρ*
$\frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{3/2}} = 0.6311$	$3 - \frac{6\pi^2-24\pi+16}{(4-\pi)^2} = 3.2451$	**1.11245**	**0.7969**

Table 4: SKEW-UNIFORM(λ, θ) with λ = -1.4 and λ = 1.35

$\gamma = \frac{2\lambda(5\lambda^2-9)}{5(3-\lambda^2)^{3/2}}$	$\kappa = \frac{2\lambda(5\lambda^2-9)}{5(3-\lambda^2)^{3/2}}$	ρ	ρ*
$\gamma < 0$ if $\lambda \in \left(-\sqrt{3}, -\frac{3}{\sqrt{5}}\right) \cup \left(0, \frac{3}{\sqrt{5}}\right)$ $\gamma > 0$ if $-\frac{3}{\sqrt{5}} < \lambda < 0$	$\kappa > 0$ if $-\sqrt{3} < \lambda < \sqrt{3}$	**-29.9109** (λ = -1.4) **-0.7671** (λ = 1.35)	**-29.6997** (λ = -1.4) **-0.7909** (λ = 1.35)

5 Conclusion

We have proposed a bias-corrected estimator of the mean of a class of skewed probability density functions and provided a second-order asymptotic expansion for the regret under the squared error loss. The results indicate that the proposed procedure performs better than the best fixed-sample-size procedure when the observations are taken from the Gamma, Pareto, Rayleigh or Skew-uniform distribution. For a more general loss function, we have proposed bias-corrected estimator of the mean and provided a second-order asymptotic expansion for the incurred regret.

References

[1] F. Anscombe, Large sample theory of sequential estimation, *Proceedings Cambridge Philos. Soc.*, **48**, (1952), 600 – 607, 1952.

[2] C. Chang, A.K. Gupta and W.J. Huang, Some skew-symmetric models, *Random Operators and Stochastic Equations*, **10**, (2002), 133 – 140.

[3] I.S. Chang and C.A. Hsiung, Approximations to the expected sample size of certain sequential procedures, *Proceedings of the Conference on Recent Developments in Statistical Methods and Applications,* Taipei, December 1979, Inst. Of Math. Acad. Sinica, (1979), 71 – 82.

[4] Y.S. Chow and K.F.Yu, The performance of a sequential procedure for the estimation of the mean, *Annals of Statistics*, **9**, (1981), 189 – 198.

[5] M. Ghosh and N. Mukhopadhyay, Sequential point estimation of the mean when the distribution is unspecified, *Commununications in Statistics -Theory & Methods*, **A8**, (1979), 637 – 652.

[6] A.T. Martinsek, Second order approximation to the risk of a sequential procedure, *Annals of Statistics*, **11**, (1983), 827 – 836.

[7] A.T. Martinsek. Negative regret, optional stopping and the elimination of

outliers. *J. A. S. A.*, **83**, (1988), 160 – 163.

[8] H. Robbins, Sequential estimation of the mean of a normal population, *Probability and Statistics*, The Harold Cramer Volume, (1959), 235 – 245.

[9] N. Starr and M. Woodroofe, Remarks on sequential point estimation. *Proceedings Nat. Acad. Sci.*, **63**, (1969), 285 – 288.

[10] M. Tahir, An asymptotic lower bound for the local minimax regret in sequential point estimation, *Annals of Statistics*, **17**, (1989), 1335 – 1346.

[11] Y. Vardi, Asymptotic optimal sequential estimation: the Poisson case. *Annals of Statistics*, **7**, (1979), 1040 – 1051.

[12] M. Woodroofe, Second order approximations for sequential point and interval estimation, *Annals of Statistics*, **5**, (1977), 984 – 995.

[13] M. Woodroofe, *Non Linear Renewal Theory in Sequential Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 1982.