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Fund Management Strategies for a Defined Contribution (DC) Pension Scheme under the Default Fund Phase IV

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Abstract

This work studied and developed pension fund management strategies in a DC scheme, during the distribution phase. The Pension plan member (PPM) is allowed to invest in a risk-free and a risky asset, under the constant elasticity of variance (CEV) model. The constrained optimization program was developed and transformed into a nonlinear partial differential equation, using the associated Hamilton Jacobi Bellman equation. The explicit solution of the constant relative risk aversion (CRRA) is obtained, using Legendre transform, dual theory, and change of variable methods. It is established herein, with a proposition that the elastic parameter, β, say, must not necessarily be equal to one ($\beta \neq 1$). A theorem is constructed and proved on the pension wealth investment strategy. Through a sensitivity analysis, we exposed the dangers of CRRA utility options during the period after retirement. A numerical simulation was used to buttress our investigation.

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1. Introduction

There are two major designs of pension plan, namely, the defined benefit (DB) pension, and the defined contribution (DC) pension plan. As the names implies, in that of the DB, the benefits of the plan member are defined, and the sponsor bears the financial risk. Whereas, in the DC pension plan, the contributions are defined, the retirement benefits depends on the contributions and the investment returns, and the contributors (the plan members) bears the financial risk. Recently, the DC pension has taken dominance over the DB pension plan in the pension scheme, since DC pension plan is fully funded, which makes it easier for the plan managers (Pension Fund Administrators (PFAs') and the Pension Fund Custodians (PFCs') to invest equitably in the market, and also makes it easier for the plan members to receive their retirement benefit as and when due.

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Investment strategies of the contributions, which in turn is a strong determinant of the investment returns vis-a-vis the benefits of the contributors at retirement must be given optimum attention. Recent publications in economic Journals and other reputable Mathematics and Science Journals have brought to light, variety of methods of optimizing investment strategies and returns. For instance, some researchers have made various contributions in this direction, particularly, in DC Pension Plan. Cairns et al [4], did a work on, "stochastic life styling: optimal dynamic asset allocation for defined contribution pension plans. In their work, various properties and characteristics of the optimal asset allocation strategy, both with and without the presence of non-hedge able salary risk were discussed. The significance of alternative optimal strategy by pension providers was established. Wang, and Chen [15] investigated a defined contribution (DC) pension plan investment problem during the accumulation phase under the multi-period mean-variance criterion. Mwanakatwe et al [12] analysed the optimal investment strategies for a DC pension fund under the Hull-White interest rate model. Under this model, the pension fund manager can invest capital in the bank account, stock index, and real estates. More so, Battocchio et al [2] studied optimal pension management in a stochastic framework, they came out with a significant result.

In order to deal with optimal investment strategy, the need for maximization of the expected utility of the terminal wealth became necessary. Example, the Constant Relative Risk Aversion (CRRA) utility function, and (or) the Constant Absolute Risk Aversion (CARA) utility function were used to maximize the terminal wealth. Cairns et al [4], Gao [8], Boulier, et al [3], Deelstra, et al [7], and Xiao et al [16] used CRRA to maximize terminal wealth. However, Gao [10] used the CRRA and the CARA to maximize terminal wealth, and this triggered our research. Ours is a modification of his work, by considering different categories of contributors, with some other additional assumptions made. Our task in this work is to establish, with a theorem the fact that the elastic parameter $\beta \neq 1$, which is lacking in his work. We used a similar approach in obtaining some of our results.

1.1 Preliminaries

We start with a complete and frictionless financial market that is continuously open over the fixed time interval $[0, T]$, for $T > 0$, representing the retirement time of any plan member.

We assume that the market is composed of the risk-free asset (cash), and risky asset (stock). Let (Ω, F, P) be a complete probability space, where Ω is a real space and P is a probability measure, $\{w_s(t), w_t(t)\}$ are two standard orthogonal Brownian motions, $\{F_t(t), F_s(t)\}$ are right continuous filtrations whose information are generated by the two standard Brownian motions $\{w_s(t), w_t(t)\}$, whose sources of uncertainties are respectively to the stock market and time evolution.

2. Methodology

2.1 Hamilton-Jacobi-Bellman (HJB) equation

Assume we represent $u = u_s$ as the strategy and we define the utility attained by the contributor from a given state y at time t as

$$\phi_u\left(t,r,y\right)=E_u\left[U\left(Y\left(t\right)\right):r\left(t\right)=r,y\left(t\right)=y\right], \tag{2.1.1}$$

Where t is the time, r is the short interest rate and y is the wealth. Our interest here is to find the optimal value function

$$\phi\left(t,r,y\right)=Sup_u\phi_u\left(t,r,y\right) \tag{2.1.2}$$

and the optimal strategy $u^*=u_s^*$ such that

$$\phi_{u^*}\left(t,r,y\right)=\phi\left(t,r,y\right) \tag{2.1.3}$$

2.2. Legendre Transformation

The Legendre transform and dual theory help to transform the nonlinear partial differential equation that is formed due to (2.1), to a linear partial differential equation.

Theorem 2.2 (Jonsson and Sircar [11]): Let $f: R^n \rightarrow R$ be a convex function for $z>0$, then the Legendre transform is defined as;

$$L\left(z\right)=\max_y\left\{f\left(y\right)=zy\right\}, \tag{2.2.1}$$

Where $L(z)$ is the Legendre dual of $f\left(y\right)$.

Since $f\left(y\right)$ is convex, from theorem 2.2 we defined the Legendre transform

$$\hat{\phi}\left(t,r,z\right)=Sup\left\{\phi\left(t,r,y\right)-zy:0<y<\infty\right\},\ 0<t<T \tag{2.2.2}$$

Where $\hat{\phi}$ is the dual of ϕ and $z>0$ is the dual variable of x.
The value of y where this optimum is attained is denoted by $h(t,r,z)$, so that

$$h\left(t,r,z\right)=\inf\left\{y:\phi\left(t,r,y\right)\geq zy+\hat{\phi}\left(t,r,z\right)\right\}\ 0<t<T. \tag{2.2.3}$$

The function h and $\hat{\phi}$ are closely related and can be referred to as the dual of ϕ. These functions are related as follows

$$\hat{\phi}\left(t,r,z\right)=\phi\left(t,r,h\right)-zh \tag{2.2.4}$$

where,

$$h\left(t,r,z\right)=y,\phi_y=z,h=-\hat{\phi}_z. \tag{2.2.5}$$

At terminal time, T, we denote

$$\hat{U}\left(z\right)=Sup\left\{U\left(y\right)-zy:0<y<T\right\},\ \text{and}\ \phi\left(z\right)=Sup\left\{y:\mathrm{U}\left(y\right)\geq zy+\hat{U}\left(z\right)\right\}.$$

As a result

$$\phi(z)=U^{-1}(z) \tag{2.2.6}$$

Where ϕ is the inverse of the marginal utility U and note that $\phi(T,r,y)=U(y)$.

At terminal time T, we can define

$h(T,r,y)=\inf_{y>0}\{y:U(y)\geq zy+\hat{\phi}(\mathrm{t},r,z)\}$ and $\hat{\phi}(\mathrm{t},r,z)=Sup_{y>0}\{U(y)-zy\}$

so that $h(\mathrm{T},r,z)=U^{-1}(z)$.

3. The Model

This session introduces the financial market and proposes the optimization problems in the Default Fund phase IV.

3.1 The Financial Market

Here, we consider a financial market that consists of a risk-free asset (i.e., cash in the bank) and a risky asset (stock).

Let the risk-free asset, C_t, say, at any positive time, t, evolve as;

$$dC_t = rC_t dt \tag{3.1.1}$$

where r represents constant rate of interest.

Next, we denote the price of the risky asset (stock) at any positive time, t, by S_t, as in Gao [10], Akpanibah et al [1], and Njoku et [13] thus;

$$dS_t = \mu S_t dt + kS_t^{\beta+1} dW_t \tag{3.1.2}$$

where $\mu(\mu>r)$ represents the instantaneous rate of return on stock, β (β ≤ 0) is the elastic constant parameter, k is a constant, kS_t^{β} represents the instantaneous volatility.

Let $\{W_t;t\geq 0\}$ denote a standard Brownian motion, defined on a probability space, $(\Omega,\mathrm{F},\mathrm{P})$ where $F=\{F_t\}$ is an augmented filtration generated by the Brownian motion.

3.2 Model Assumption

Consistent with the Nigerian Pension Reform Act of 2004 [14], we make the following assumptions

(a) The Pension Scheme accumulates wealth.
(b) There are different categories of contributions.
(c) The contributors will not willingly withdraw from the scheme.
(d) Payments are made to the retirees.

(e) An accumulated amount is paid to the Next-of-kin of the dead contributors, at the instance of death by any contributor(s).
(f) A certain amount is retained from the payment made to the Next-of-kin of dead contributors, by the Pension managers (i.e., management fee).

3.3 Model Formation (i.e., the Optimization Program)

The fund accruing from the contributors can be invested in both Bank and stock. Particularly, the fund to be invested by the fund manager is the surplus, which is the fund that is available after each period of routine disbursements. That is, let the contribution process be

$$dy = (1+\theta_i) C_{i+1} dt \tag{3.3.1}$$

and the payment process

$$dj = b_{i+1} dt + (a_{i+1} - \eta) dW_s . \tag{3.3.2}$$

Then the surplus

$$dP = dy - dj = (1+\theta_i) C_{i+1} dt - \left[b_{i+1} dt + (a_{i+1} - \eta) dW_s \right]$$

$$= (C_{i+1} + \theta_i C_{i+1} - b_{i+1}) dt - (a_{i+1} - \eta) dW_s . \tag{3.3.3}$$

Therefore our task is to find an optimal investment strategy for the assets for the remaining periods after retirement, to enable us maximize the expected utility at each retirement period.

Without loss of generality, the pension wealth is denoted by $Y(t)$ at any time $0 < t < T < T+N$, and it evolves stochastically, thus:

$$dY(t) = u_s Y(t) \frac{dS_t}{S_t} + (1-u_s) Y(t) \frac{dC_t}{C_t} + (C_{i+1} + \theta_i C_{i+1} - b_{i+1}) dt - (a_{i+1} - \eta) dW_s \tag{3.3.4}$$

$i = 0,1,...,n-1$ and $\theta_0 = 0, \theta_1 = 1, \theta_2 = 2,..,$ θ_i (an integer) = staff loading, where; $a_{i+1} > 0$ represents various amount that is paid to the Next-of-Kin of the dead contributors, $b_{i+1} > 0$ represents various amount paid to retired contributors, $c_{i+1} > 0$ represents various amount contributed, η is the service charge deducted from the a_{i+1}.

However, relevant to the provisions of the Nigerian Pension Reform Act of 2004 [14], on the eligibility condition for signing up on the Pension Scheme, by both government and private sectors, we have;

$$dY(t) = u_s Y(t) \frac{dS_t}{S_t} + (1-u_s) Y(t) \frac{dC_t}{C_t} + (C_{i+1} + \theta_i C_{i+1} - b_{i+1}) dt - (a_{i+1} - \eta) dW_s ,$$

(3.3.5)

$i = 4, 5.., n-1,$ and $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, ..., \theta_n = n$ (a positive integer)= staff loading.

Assuming, $b_{i+1} = C_{i+1} + r_d C_{i+1}$; $r < r_d$, and r_d represents discounted interest, then

$$dY(t) = u_s Y(t)\frac{dS_t}{S_t} + (1-u_s)Y(t)\frac{dC_t}{C_t} + (\theta_i - r_d)C_{i+1}dt - (a_{i+1} - \eta)dW_s, \tag{3.3.6}$$

$i = 4, 5, .., n-1,$ and $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, ..., \theta_n = n, \theta_i > 0$ (integer)= staff loading.

Taking into (3.1.1), (3.1.2) and (3.3.6), one obtains the wealth process

$$\begin{aligned} dY(t) &= u_s Y(t)\left[\mu dt + kS^\beta dW_t\right] + (1-u_s)Y(t)rdt + (\theta_i - r_d)C_{i+1}dt - (a_{i+1} - \eta)dW_s \\ &= u_s Y(t)\mu dt + u_s Y(t)kS^\beta dW_t + Y(t)rdt + u_s Y(t)rdt + (\theta_i - r_d)C_{i+1}dt - (a_{i+1} - \eta)dW_s \\ &= \left(u_s Y(t)\mu + Y(t)r - u_s Y(t)r + (\theta_i - r_d)C_{i+1}\right)dt + u_s Y(t)kS^\beta dW_t - (a_{i+1} - \eta)dW_s \end{aligned} \tag{3.3.7}$$

$i = 4, 5, .., n-1,$ and $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, ..., \theta_n = n, \theta_i > 0$ (integer)= staff loading.

Based on the wealth process in (3.3.7), the Pension manager seeks a strategy, u_s^*, which maximizes the utility function, such that $u_s^* = \max E\left(U\left(Y(\mathrm{T})\right)\right), \forall u(\mathrm{t})$. Where u(•) is an increasing concave utility function, which satisfies the Inada conditions;

$U'(+\infty) = 0,$ and $U'(0) + \infty$ (cf. Gao [10])

4. Applying the Associated HJB Equation to maximize equation (3.3.7)

Let the maximizing equation be,

$$dY(t) = \left(u_s Y(t)\mu + Y(t)r - u_s Y(t)r + (\theta_i - r_d)C_{i+1}\right)dt + u_s Y(t)kS^\beta dW_t - (a_{i+1} - \eta)dW_s,$$
$$u_s \geq 0, 1 - u_s \geq 0$$

Then, applying the associated HJB to the maximizing equation above, one obtains

$$\begin{aligned} &\phi_t + \mu S\phi_s + Y(\mathrm{t})\left[u_s(\mu - r) + r\right]\phi_y + (\theta_i - r_d)C_{i+1}\phi_y + \frac{1}{2}k^2 S^{2\beta+2}\phi_{ss} + \frac{1}{2}u_s^2 Y^2 k^2 S^{2\beta}\phi_{yy} - \frac{1}{2}\alpha\gamma u_s Y(\mathrm{t})kS^\beta a_{i+1}\phi_{yy} \\ &+ \frac{1}{2}\alpha\gamma u_s Y(\mathrm{t})kS^\beta \eta\phi_{yy} + \frac{1}{2}(a_{i+1} - \eta)^2\phi_{yy} + \frac{1}{2}\alpha\gamma k^2 S^{\beta+2}Y(\mathrm{t})\mathrm{u}_s\phi_{sy} - \frac{1}{2}\alpha\gamma kS^{\beta+1}a_{i+1}\phi_{sy} + \frac{1}{2}\alpha\gamma kS^{\beta+1}\eta\phi_{sy} = 0 \end{aligned}$$
(4.1)

To obtain the optimal value u_s^*, we differentiate (4.1) with respect to u_s

$$\varphi_t + \mu s\varphi_s + (\theta_i - r_d)C_{i+1}\varphi_y + \frac{1}{2}k^2 s^{2\beta+2}\varphi_{ss} + \frac{1}{2}(a_{i+1}-\eta)^2\varphi_{yy} - \frac{1}{2}\alpha\gamma k s^{\beta+1}a_{i+1}\varphi_{sy} + \frac{1}{2}\alpha\gamma k s^{\beta+1}\eta\varphi_{sy}$$
$$+\sup_{u_s}\left\{\begin{array}{l} Y(\mathrm{t})\left(u_s(\mu-r)+r\right)\varphi_y + \frac{1}{2}u_s^2 Y(\mathrm{t})^2 k^2 S^{2\beta}\varphi_{yy} - \frac{1}{2}\alpha\gamma u_s Y(\mathrm{t})kS^{\beta}a_{i+1}\varphi_{yy} + \frac{1}{2}\alpha\gamma u_s kS^{\beta}\eta\varphi_{yy} \\ +\frac{1}{2}\alpha\gamma k^2 s^{\beta+2}Y(\mathrm{t})u_s\varphi_{sy}\end{array}\right\} = 0, \tag{4.2}$$

so that

$$\begin{aligned} \therefore u_S^* &= \frac{\alpha\gamma a_{i+1}}{2Y(\mathrm{t})kS^{\beta}} - \frac{\alpha\gamma\eta}{2Y(\mathrm{t})kS^{\beta}} - \frac{(\mu-r)}{Y(\mathrm{t})k^2S^{2\beta}}\frac{\varphi_y}{\varphi_{yy}} - \frac{\alpha\gamma}{2Y(\mathrm{t})S^{\beta-2}}\cdot\frac{\varphi_{sy}}{\varphi_{yy}} \\ \Rightarrow u_S^* &= \frac{\alpha\gamma(a_{i+1}-\eta)}{2Y(\mathrm{t})kS^{\beta}} - \frac{(\mu-r)}{Y(\mathrm{t})k^2S^{2\beta}}\frac{\varphi_y}{\varphi_{yy}} - \frac{\alpha\gamma}{2Y(\mathrm{t})S^{\beta-2}}\cdot\frac{\varphi_{sy}}{\varphi_{yy}} \end{aligned}. \tag{4.3}$$

Putting back (4.3) into (4.2) gives;

$$\begin{aligned} &\varphi_t + \mu s\varphi_s + \left[Y(\mathrm{t})r + (\theta_i - r_d)C_{i+1} + \frac{\alpha\gamma(\mu-r)(a_{i+1}-\eta)}{2kS^{\beta}}\right]\varphi_y + \frac{1}{2}k^2S^{2\beta+2}\varphi_{ss} + c_1\varphi_{yy} \\ &+\left[\frac{(\alpha\gamma)^2 kS^2(a_{i+1}-\eta)}{4} - \frac{\alpha\gamma kS^{\beta+!}(a_{i+1}-\eta)}{2}\right]\varphi_{sy} - \frac{(\mu-r)^2\varphi_y^2}{2k^2S^{2\beta}\varphi_{yy}} - \frac{\alpha\gamma(\mu-r)\varphi_y\varphi_{sy}}{2S^{\beta-2}\varphi_{yy}} - \frac{k^2S^4\varphi_{sy}^2(\alpha\gamma)^2}{8\varphi_{yy}} = 0 \end{aligned} \tag{4.4}$$

$i = 4,5,..,n-1, \mathrm{r} < r_d$ and $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, ..., \theta_n = n, \theta_{n-1} = n-1, \theta_i > 0$ (integer)= staff loading, and

$$c_1 = \frac{1}{2}(a_{i+1}-\eta)^2 + \frac{3}{4}(\alpha\gamma)^2(a_{i+1}-\eta)^2. \tag{4.5}$$

Having seen that stochastic control problem described in the previous session has been converted to a nonlinear PDE, our next tax is to solve for φ in (4.4)and subsequently substitute it into (4.3), to enable us obtain the optimal wealth investment strategy (i.e., the control strategy).In order to achieve this, we use employ the services of the Dual theory and Legendre transformation.

5. Transforming (4.4) into its Dual and applying Legendre Transformation

Here, we transform the nonlinear second order partial differential equation (4.4) into a linear PDE, using the Dual theory and Legendre transformations in Gao [10], that is;

$$\phi_y = z \text{ and } \phi_t = \hat{\phi}_t, \phi_s = \hat{\phi}_s, \phi_{ss} = \hat{\phi}_{ss} - \frac{\hat{\phi}_{sz}^2}{\hat{\phi}_{zz}}, \phi_{yy} = \frac{-1}{\hat{\phi}_{zz}}, \phi_{ys} = \frac{-\hat{\phi}_{sz}}{\hat{\phi}_{zz}}. \tag{5.1}$$

Putting (5.1) into (4.4)

$$\hat{\varphi}_t + \mu S \hat{\varphi}_s + \left[Y(\mathrm{t}) r + (\theta_i - r_d) C_{i+1} + \frac{\alpha\gamma(\mu - r)}{2kS^{\beta}} (a_{i+1} - \eta) \right] z + \frac{1}{2} k^2 S^{2\beta+2} \hat{\varphi}_{ss} - \frac{c_1}{\hat{\varphi}_{zz}}$$
$$- \left[\frac{(\alpha\gamma)^2 kS^2 (a_{i+1} - \eta)}{4} - \frac{\alpha\gamma k S^{\beta+1} (a_{i+1} - \eta)}{2} \right] \frac{\hat{\varphi}_{sz}}{\hat{\varphi}_{zz}} + \frac{(\mu - r)^2}{2k^2 S^{2\beta}} z^2 \hat{\varphi}_{zz} - \frac{\alpha\gamma(\mu - r)}{2s^{\beta-2}} z \hat{\varphi}_{sz} + \frac{(\alpha\gamma)^2 k^2 S^4}{8} \frac{\hat{\varphi}_{sz}^2}{\hat{\varphi}_{zz}} = 0 \tag{5.2}$$

$i = 4, 5, .., n-1, \mathrm{r} < r_d$ and $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, ..., \theta_n = n, \theta_{n-1} = n-1, \theta_i > 0$ (integer)= staff loading, and $c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{4}(\alpha\gamma)^2 (a_{i+1} - \eta)^2$.

Taking $y = h = -\hat{\phi}_z$ into (5.2), and differentiating with respect to z , we obtain

$$h_t + \mu S h_s - r z h_z - r h - (\theta_i - r_d) C_{i+1} - \frac{\alpha\gamma(\mu - r)(a_{i+1} - \eta)}{2kS^{\beta}} + \frac{1}{2} k^2 S^{2\beta+2} h_{ss} + \frac{c_1 h_{zz}}{h_z^2} - + \left[\frac{(\alpha\gamma)^2 kS^2 (a_{i+1} - \eta)}{4} - \frac{\alpha\gamma k S^{\beta+1} (a_{i+1} - \eta)}{2} \right] \frac{h_s}{h_z} - \frac{\alpha\gamma(\mu - r)}{2S^{\beta-2}} h_s$$
$$- \frac{\alpha\gamma(\mu - r)}{2S^{\beta-2}} z h_{sz} + \frac{(\mu - r)^2}{k^2 S^{2\beta}} z h_z + \frac{(\mu - r)^2}{2k^2 S^{2\beta}} z^2 h_{zz} + \frac{(\alpha\gamma)^2 k^2 S^4}{8} \frac{h_{sz} h_s}{h_z} - \frac{(\alpha\gamma)^2 k^2 S^4 h_s^2 h_{zz}}{8 h_z^2} = 0 \tag{5.3}$$

where,

$$c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{8}(\alpha\gamma)^2 (a_{i+1} - \eta)^2, \tag{5.4}$$

with,

$$u_s^* = \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(\mathrm{t}) k S^{\beta}} + (\mu - r) \frac{z}{\hat{\varphi}_{zz}} - \frac{\alpha\gamma}{2Y(\mathrm{t}) S^{\beta-2}} . \hat{\varphi}_{sz}$$

$$\Rightarrow u_s^* = \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(\mathrm{t}) k S^{\beta}} - (\mu - r) \frac{z}{h_z} - \frac{\alpha\gamma}{2Y(\mathrm{t}) S^{\beta-2}} . h_s \tag{5.5}$$

6. Test for Some Utility Functions

Here, we seek to obtain the explicit solution for the CRRA utility functions, using change of variable method.

6.1 Explicit solution to the CRRA utility

Following Gao [10] and Zhang et al [5], we use

$$h(t, s, z) = z^{\frac{1}{p-1}} \quad , p < 1, p \neq o . \tag{6.1.1}$$

Let us conjecture a solution to (5.3), thus

$$h(t,s,z) = z^{\frac{1}{p-1}} g(t,s) + a(t); a(T) = 0; g(T,s) = 1 \tag{6.1.2}$$

Now, obtaining the various first and second partial derivatives with respect to t, s, z, we have

$$h_t = g_t z^{\frac{1}{p-1}} + a'(t); h_s = g_s z^{\frac{1}{p-1}}; h_z = \frac{-g}{1-p} z^{\frac{1}{p-1}-1}; h_{ss} = g_{ss} z^{\frac{1}{p-1}}; h_{sz} = \frac{-g}{1-p} z^{\frac{1}{p-1}-1}; h_{zz} = \frac{(2-p)g}{(1-p)^2} z^{\frac{1}{p-1}-2} = 0 \tag{6.1.3}$$

Putting (6.1.2) and (6.1.3) into (5.3), we obtain

$$\begin{aligned}
&g_t z^{\frac{1}{p-1}} + a'(t) + \mu s g_s z^{\frac{1}{p-1}} + \frac{rgz^{\frac{1}{p-1}}}{1-p} - rgz^{\frac{1}{p-1}} - ra(t) - (\theta_i - r_d) C_{i+1} - \frac{\alpha\gamma(\mu - r)(a_{i+1} - \eta)}{2ks^{\beta}} \\
&+ \frac{1}{2} k^2 s^{2\beta+2} g_{ss} z^{\frac{1}{p-1}} + \frac{c_1(2-p)}{g} \frac{1}{z^{\frac{1}{p-1}}} - \left[\frac{(\alpha\gamma)^2 ks^2 (a_{i+1} - \eta)}{4} - \frac{\alpha\gamma ks^{\beta+1}(a_{i+1} - \eta)}{2} \right] \frac{g_s(1-p)z}{g} \\
&- \frac{\alpha\gamma(\mu - r)}{2s^{\beta-2}} g_s z^{\frac{1}{p-1}} + \frac{\alpha\gamma(\mu - r) g_s z^{\frac{1}{p-1}}}{2s^{\beta-2}(1-p)} - \frac{(\mu - r)^2 g}{k^2 s^{2\beta}(1-p)} z^{\frac{1}{p-1}} + \frac{(\mu - r)^2 (2-p) g}{2k^2 s^{2\beta}(1-p)^2} z^{\frac{1}{p-1}} + \frac{(\alpha\gamma)^2 k^2 s^4 g_s^2 z^{\frac{1}{p-1}}}{8} \\
&- \frac{k^2 s^4 g_s^2 z^{\frac{1}{p-1}} (2-p)(\alpha\gamma)^2}{8g} = 0
\end{aligned} \tag{6.1.4}$$

Factoring out terms that depends on $z^{\frac{1}{p-1}}$, and z, and the ones that is independent of either of the two mentioned, we split (6.1.4) into three, thus

$$a'(t) - ra(t) - (\theta_i - r_d) C_{i+1} - \frac{\alpha\gamma(\mu - r)(a_{i+1} - \eta)}{2ks^{\beta}} = 0, \mathrm{r} < r_d, \tag{6.1.5}$$

$$\left[\frac{(\alpha\gamma)^2 ks^2 (a_{i+1} - \eta)}{4} - \frac{\alpha rks^{\beta+1}(\mu - r)(a_{i+1} - \eta)}{2} \right] \frac{g_s(1-p)}{g} = 0, g \neq 0 \tag{6.1.6}$$

and

$$\begin{aligned}
&g_t + \mu s g_s + \frac{rg}{1-p} - rg + \frac{1}{2} k^2 s^{2\beta+2} g_{ss} + \frac{g}{c_1(2-p)} - \frac{\alpha r(\mu - r) g_s}{2s^{\beta-2}} + \frac{\alpha\gamma(\mu - r) g_s}{2s^{2\beta-2}(1-p)} - \frac{(\mu - r)^2 g}{k^2 s^{2\beta}(1-p)} \\
&+ \frac{(\mu - r)^2 (2-p) g}{2k^2 s^{2\beta}(1-p)^2} - \frac{(\alpha\gamma)^2 k^2 s^4 g_s^2}{8} + \frac{k^2 s^4 g_s^2}{8g}(2-p)(\alpha\gamma)^2 = 0, g \neq 0
\end{aligned} \tag{6.1.7}$$

Solving (6.1.5) at the boundary condition, $a(T) = 0$, we obtain the continuous annuity of duration, $T - t$, yields

$$a(t) = -[T - t] \ell^{-rt} \left[(\theta_i - r_d) C_{i+1} 2k + \alpha\gamma u_s (a_{i+1} - \eta) \right], \tag{6.1.8}$$

where, $a(t) = -[T-t]\left[(\theta_i - r_d)C_{i+1}2k + \alpha\gamma u_s(a_{i+1}-\eta)\right]$, and ℓ^{-rt} are the so called continuous annuity of duration $T-t$, and the continuous technical rate, respectively.

From (6.1.6), we propose the following

Proposition 6.1.1: Let, (a) $dw_s dw_t = \frac{1}{2}\alpha\gamma dt; \frac{\alpha\gamma}{2} \neq 1$ ***(b)*** $\beta \neq 1$ ***, then***

$$\left[\frac{(\alpha\gamma)^2 ks^2(a_{i+1}-\eta)^2}{4} - \frac{\alpha\gamma ks^{\beta+1}(a_{i+1}-\eta)}{2}\right]\frac{g_s(1-p)}{g} = 0; p < 1. \qquad (6.1.9)$$

Proof: Suppose, for contradiction, $\beta = 1, \frac{\alpha\gamma}{2} = 1$.We observe that either of the following two cases arises

Case 1: $\frac{g_s(1-p)}{g} = 0 \Rightarrow p = 1 \Rightarrow\Leftarrow p < 1$, by definition of p in (6.1.1).

Case 2: $\frac{(\alpha\gamma)^2 ks^2(a_{i+1}-\eta)^2}{4} = \frac{\alpha\gamma ks^{\beta+1}(a_{i+1}-\eta)}{2} \Rightarrow \frac{\alpha\gamma s^2}{2} = s^{\beta+1}$.

But $s^2 = s^{\beta+1}$, whenever $\frac{\alpha\gamma}{2} = 1$, satisfied by the supposition above $\Rightarrow \beta = 1$ whenever $\frac{\alpha\gamma}{2} = 1$, by law of indices. But by proposition 1(a), $dw_s dw_t = \frac{1}{2}\alpha\gamma dt \Rightarrow \frac{\alpha\gamma}{2} = dw_s dw_t \neq 1 \Rightarrow\Leftarrow$, contradicting the assumption that $\frac{\alpha\gamma}{2} = 1$.

Hence, the proposition holds, and this completes the proof.

Corollary 6.1.1: The correlation between stock and time, represented, $dw_s dw_t$ is equal to 1, thereby making $\frac{\alpha\gamma}{2} = 1$, holds only if stock attains its maximum yield at every increase in time, which is not realistic.

Lastly, solving (6.1.7), observe firstly that the equation contains some variable coefficients, $s, s^{2\beta+2}, s^{-2\beta}, s^{-(\beta-2)}$, and this makes obtaining solution somewhat difficult. However, in order to overcome this difficulty, we employ the services of power transformation and change of variable technique as in Cox [6].

Assuming,

$$g(t,s) = f(t,\mathrm{j}), j = s^{-2\beta} \qquad (6.1.10)$$

Such that,

$$g_t = j_t ; g_s = -2\beta s^{-(2\beta+1)} f_j ; g_{ss} = 4\beta^2 s^{-2(2\beta+1)} f_{jj} + 2\beta(2\beta+1) s^{-2(\beta+1)} f_j , \tag{6.1.11}$$

then putting (6.1.10) and (6.1.11) into (6.1.7), gives

$$\begin{aligned} & f_t - 2\mu\beta j f_t + \frac{r}{1-p} f - rf + 2k^2\beta^2 j f_{jj} + k^2\beta(2\beta+1) f_j + \frac{1}{c_1(2-p)}.f + \frac{4(\mu-r)\beta j}{\alpha\gamma s^2}.f_t - \frac{4(\mu-r)\beta j}{\alpha\gamma s^2(1-p)}.f_t \\ & -\frac{(\mu-r)^2}{k^2(1-p)}.f + \frac{(\mu-r)^2(2-p)j}{2k^2(1-p)^2}.f - k^2\beta^2 j^2 f_t^2 + \frac{2k^2\beta^2 j^2}{f}.f_t^2 = 0 \end{aligned} \tag{6.1.12}$$

where, $s^{\beta} = \frac{\alpha\gamma}{2}.s$.

Following Proposition 6.1.1, we observe that some cases may arise in (6.1.12), $\beta = 0, \beta < 1$, provided $p \neq 0, p < 1$

Case 1: $\beta = 0$ (i.e., the GBM case), yields

$$f_t + \left[\frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{(\mu-r)^2}{k^2(1-p)} + \frac{2(\mu-r)^2(2-p)}{k^2(1-p)^2(\alpha\gamma)^2 s^2} \right].f = 0, \beta < -1, p \neq 0 \tag{6.1.13}$$

Next, we find the solution of (6.1.13), using the structure below

$$f(t,j) = A(t)\ell^{B(t)j} ; A(T) = 1, B(T) = 0 \Rightarrow f_t = A_t \ell^{B(t)j} + A(t) B_t j \ell^{B(t)j} \tag{6.1.14}$$

Putting (6.1.14) into (6.1.13) yields

$$\frac{A_t}{A(t)} + B_t j + \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{(\mu-r)^2}{k^2(1-p)} + \frac{(\mu-r)^2(2-p)j}{2k^2(1-p)^2} = 0 , \tag{6.1.15}$$

where $s^{\beta} = \frac{\alpha\gamma}{2}.s$ is re substituted and $j = s^{-2\beta}$

Observe that we can further split (6.1.15) by separating the terms that depends on j , thus

$$B_t + \frac{(\mu-r)^2(2-p)}{2k^2(1-p)^2} = 0 \tag{6.1.16}$$

and

$$\frac{A_t}{A(t)}+\frac{r}{1-p}-r+\frac{1}{c_1(2-p)}-\frac{(\mu-r)^2}{k^2(1-p)}=0. \tag{6.1.17}$$

Solving (6.1.16) at the initial condition $B(T)=0$, yields

$$B(t)=[T-t]\left(\frac{u_s^2(2-p)}{2k^2(1-p)}\right), \tag{6.1.18}$$

where $\mu=u_s+r$.

Next, we solve (6.1.17) at initial condition $A(T)=1$, yields

$$A(t)=\ell^{\left[\frac{r}{1-p}-r+\frac{1}{c_1(2-p)}-\frac{u_s^2}{k^2(1-p)}\right][T-t]},\ p\neq 0, p<-1, \tag{6.1.19}$$

where, $\mu=u_s+r$.

Taking into (6.1.18), (6.1.19) and (6.1.14), we have

$$f(t,j)=\ell^{\left[\frac{r}{1-p}-r+\frac{1}{c_1(2-p)}-\frac{u_s^2}{k^2(1-p)}+\frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]}. \tag{6.1.20}$$

In this sequel, we state;

Theorem 6.1: Let equations (6.1.2), (6.1.8), (6.1.10) and (6.1.20) hold. Then the optimal wealth investment made in stock is given as;

$$u_s^*=\frac{\alpha\gamma(a_{i+1}-\eta)}{2Y(t)k}-u_s(p-1)z^{\frac{2p-3}{p-1}}\ell^{-\left[\frac{rp}{1-p}-r+\frac{1}{c_1(2-p)}-\frac{u_s^2}{k^2(1-p)}+\frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]}$$

Proof.

Taking into (6.1.8), (6.1.10), (6.1.20) and (6.1.2), we have

$$h(t,s,z)=z^{\frac{1}{p-1}}\ell^{\left[\frac{r}{1-p}-r+\frac{1}{c_1(2-p)}-\frac{u_s^2}{k^2(1-p)}+\frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]}-\ell^{-rt}\left[2k(\theta_i-r_d)C_{i+1}+\alpha\gamma u_s(a_{i+1}-\eta)\right][T-t] \tag{6.1.21}$$

Differentiating (6.1.21) w r t s and z, yields

$$h_s=0 \tag{6.1.22}$$

and

$$h_z = \frac{1}{p-1} z^{\frac{1}{p-1}-1} \ell^{\left[\frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]}. \tag{6.1.23}$$

Therefore, taking into (6.22), (6.23) and (5.5), thus

$$u_s^* = \frac{\alpha\gamma\left(a_{i+1}-\eta\right)}{2Y(t)k} - u_s\left(p-1\right) z^{\frac{2p-3}{p-1}} \ell^{-\left[\frac{rp}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]}, \tag{6.1.24}$$

where,

$c_1 = \frac{1}{2}\left(a_{i+1}-\eta\right)^2 + \frac{3}{8}\left(\alpha\gamma\right)^2\left(a_{i+1}-\eta\right)^2$, as required.

7. Sensitivity analysis

From (6.1.21) and (6.1.24), if we set $\alpha\gamma = 0$, that is, saying that stock and time have orthogonal relationship, then the satisfaction of the contributors will reduce to (7.6), but has significant negative effect on the optimal investment made in stock, and we have

$$h\left(t,s,z\right) = z^{\frac{1}{p-1}} \ell^{\left[\frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]} - \ell^{-rt}\left[2k\left(\theta_i - r_d\right)C_{i+1}\right]\left[T-t\right] \tag{7.6}$$

and,

$$u_s^* = -u_s\left(p-1\right) z^{\frac{2p-3}{p-1}} \ell^{-\left[\frac{rp}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]}. \tag{7.7}$$

Again, suppose, no money is paid to the Next-of-kin of the dead contributors, that is, setting $a_{i+1} - \eta = 0$, and this yields, respectively

$$h\left(t,s,z\right) = z^{\frac{1}{p-1}} \ell^{\left[\frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)}\right][T-t]} - \ell^{-rt}\left[2k\left(\theta_i - r_d\right)C_{i+1}\right]\left[T-t\right] \tag{7.8}$$

And

$$u_s^* = -u_s\left(p-1\right) z^{\frac{2p-3}{p-1}} \ell^{-\left[\frac{rp}{1-p}-r+\frac{1}{c_1(2-p)}-\frac{u_S^2}{k^2(1-p)}+\frac{u_S^2(2-p)j}{2k^2(1-p)}\right][T-t]} \tag{7.9}$$

Result 1

Then, from (7.6) and (7.8), we see that when stock and time have orthogonal relationship and when no money was paid to the Next-of-kin of the dead contributors, we have the same level of satisfaction on the contributors. However, both have significant effect on the satisfaction of the contributors when there is orthogonal relationship between stock and time, and also when payments are made to the Next-of-kin of the dead contributors.

Result 2

From (7.7) and (7.9), we observe that the introduction of the orthogonal relationship between stock and time, and the nonpayment of benefits to the Next-of-kin of the dead contributors have a negative significant effect on the money invested in stock. However, both introductions have the same negative effect (i.e., a decline in stock investment) on the investment made in stock.

8. Numerical illustration

A numerical example of the proposed model was given to demonstrate the dynamic behaviour of a DC pension fund and optimal investment strategy. Nigeria-National PensionFund Administration (NNPFA) real data was used to illustrate the efficiency of the proposed model. The parameters used are summarized in Table I below, for $T = 35$. $t = 0, 5, 10, 15, 20, 25, 30, 35$ with $i = 4, 5, \dots, n-1$ and $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, \dots, \theta_n = n$.

Table 1: Parameters and their respective values

Name of parameter	**Symbol used**	**Values**
constant rate of interest	r	0.02
Expected stock returns	μ	0.10
Instantaneous stock returns	$u_S = u_L$	0.07
stock volatility	k	0.55
Risk aversion	$p = q$	0.50
Rate of contribution	r_d	0.075
Management fee	η	0.025

Results

Figure 1 describes the investment strategy for the CRRA utility function under the the constant elasticity of variance (CEV) model using (6.1.21) when $c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{8}(\alpha\gamma)^2(a_{i+1} - \eta)^2, j = \ell^{-2\beta}, \beta = 0 \Rightarrow j = 1$.It shows that the stock price has significant effect on the optimal investment made in stock.

Figure 2 shows the influence instantaneous stock returns on the optimal investment strategies using (6.1.24) with $c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{8}(\alpha\gamma)^2(a_{i+1} - \eta)^2, Y(t) = 1000$. It reveals that the optimal investment policies increase with time. That is, as the time passes on, investment in riskless asset decreases. Results suggest that the pension fund manager maintains diversifying the portfolio by investing more in stock since the optimal investment strategies in risky assets increase with time.

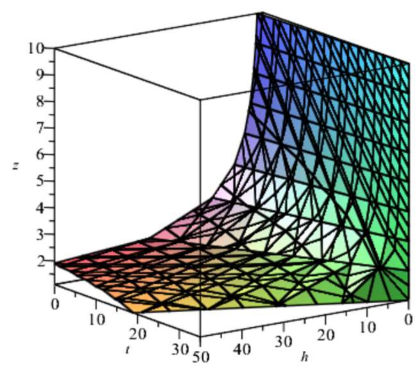

Figure 1: Investment strategy under CEV using (6.1.21)

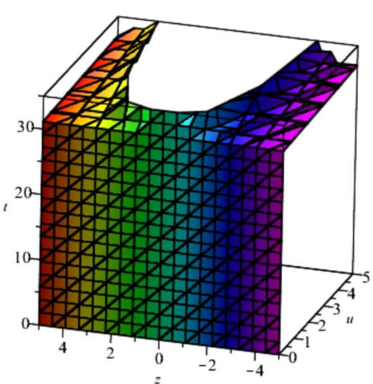

Figure 2: influence of instantaneous stock returns on the optimal investment strategies

8. Conclusion

We studied and established the investment strategy for contracts under the constant elasticity of variance (CEV) as in literature to show that the elastic parameter takes values other than unity, using proposition 6.1.1. We also constructed the Pension wealth investment strategy during the decumulation phase, in a defined contribution (DC) Pension scheme and obtained the explicit solution of the constant relative risk aversion (CRRA) utility functions. From our overall results, we state that is not advisable to use CRRA utility option in averting investment risk, after retirement from service. In view of this, we suggest the option of CARA utility function.

9. Recommendations

In order to obtain a somewhat optimal result, we recommend the fund investment strategy, whose utility option is CARA.

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