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Application of the Bernstein Polynomials for Solving the Nonlinear Fredholm Integro-Differential Equations

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Abstract

In this article an efficient numerical method for finding solution of the nonlinear Fredholm integro-differential equations on base of Bernstein polynomials basis would be presented. For this purpose at the beginning we express briefly some properties of Bernstein polynomials and after that with respect to relation between Bernstein and Legendre polynomials, operational matrices of integration and product of Bernstein Polynomials and also dual operational matrix of Bernstein basis vector, all will be presented. Then with approximate approach the solution of integro-differential equation with $C^T\phi(x)$ form (in which C is the unknown coefficients vector and $\phi(x)$ is the Bernstein basis vector) and it's usage of presented matrices, mentioned equation and it's initial conditions will be converted to an equivalent matrix equation. Coefficients vector C is the solution of this matrix equation. At the end with presentation of five numerical examples the method will be evaluated.

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Article Info: *Revised* : July 18, 2011. *Published online* : November 30, 2011 ational matrix of product, Dual operational matrix, Integro-differential equation, Legendre polynomial

1 Introduction

One of the most important mathematical dialogs which has capture the attention of authors, scientists which is foundation of researches is the fascinating subject of integro-differential equations. These equations in the beginning of nineteen hundred was presented by Volterra [1-3] and these equations were used to provide solutions on the area of engineering, physics, chemistry and biology. In the literature many analytical and numerical methods has been existed for solution of these equations. Since solution of these equations in an analytical form was not easy often time numerical method has been used to solve these equations. In the recent years many different authors provided several numerical methods for solving these equations. In this section some of these solution methods will be presented. Ordokhani [4] has used Walsh functions operational matrix with Newton-Cotes nodes for solving of Fredholm-Hemmerstein integro-differential equations. Authors [5] developed the Sinc method and used it for solving a class of nonlinear Fredholm integro-differential equations. With in [6] semi-orthogonal B-spline scaling functions and wavelets and their dual functions are presented for approximate the solution of linear and nonlinear second order Fredholm integro-differential equations. One dimensional nonlinear integro-differential equations has been solved by the Newton's method and Tau method in [7]. Saberi nadjafi and Ghorbani in [8] have used He's homotopy perturbation method for solving integral and integro-differential equations and then has been compared with these traditional methods, namely the Adomian decomposition method, the direct computation method and the series solution method. For more information the interested reader could refer to [9-14] which uses different numerical method for resolving linear and nonlinear integro-differential equations.

In this article, firstly we present operational matrices of integration and product for the Bernstein polynomials (B-polynomials) and also dual operational matrix of Bernstein basis vector, by the expansion of B-polynomials in terms of Legendre polynomials. Then we utilize them for solving s-th order nonlinear Fredholm integro-differential equation

$$\sum_{j=0}^{s} \rho_j(x)\, y^{(j)}(x) = g(x) + \lambda \int_0^1 k(x,t)\, [y(t)]^p dt, \qquad 0 \le x, t \le 1, \tag{1}$$

with the initial conditions

$$y^{(k)}(0) = b_k, \qquad 0 \le k \le s-1, \tag{2}$$

where $y^j(x)$ is the j-th derivative of the unknown function that will be determined, $k(x,t)$ is the kernel of the integral equation, $g(x)$ and $\rho_j(x), j = 0,1,\ldots,s$ are known analytic functions, p is a positive integer and $\lambda, b_k, k = 0,1,\ldots,s-1$ are suitable constants. The main characteristic of this technique is that it reduces these equations to those of an easily soluble algebraic equation, thus greatly simplifying the equations. This method can be used to solve all types of linear and nonlinear equations such as differential and integral equations, so it is known as a powerful method.

The organization of this article is as follows: in Section 2, we introduce the B-polynomials and their properties. Section 3 is devoted to the function approximation by using B-polynomials basis. Section 4 introduces the expansion of B-polynomial in terms of Legendre basis and vice versa. The operational matrices of integration product and dual operational matrix of Bernstein basis vector will be derived in Section 5. Section 6 is devoted to the solution method of integro- differential equations. In section 7, we provide some numerical examples. And the final Section offers our conclusion.

2 B-polynomials and their properties

The B-polynomials of m-th degree are defined on the interval $[0,1]$ as [15]

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \qquad 0 \le i \le m,$$

where

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

There are $m+1$, m-th degree B-polynomials. For mathematical convenience, we usually set, $B_{i,m}(x) = 0$, if $i < 0$ or $i > m$. These polynomials are quite easy to write down: the coefficients can be obtained from Pascal's triangle. It can easily be shown that each of the B-polynomials is positive and also the sum of all the B-polynomials is unity for all real $x \in [0,1]$, i.e,

$$\sum_{i=0}^{m} B_{i,m}(x) = 1, \qquad x \in [0,1].$$

3 Function approximation

B-polynomials defined above form a complete basis [16] over the interval $[0,1]$. It is easy to show that any given polynomial of degree m can be expressed in terms of linear combination of the basis functions. A function $f(x)$ defined over $[0,1]$ may be expanded as

$$f(x) \simeq P_m(x) = \sum_{i=0}^{m} c_i B_{i,m}(x), \qquad m \geq 1. \tag{3}$$

Eq. (3) can be written as

$$P_m(x) = C^T \phi(x),$$

where C and $\phi(x)$ are $(m+1) \times 1$ vectors given by

$$C = [c_0, c_1, \ldots, c_m]^T, \tag{4}$$

and

$$\phi(x) = [B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)]^T. \tag{5}$$

The use of an orthogonal basis on $[0,1]$ allows us to directly obtain the least-squares coefficients of $P_m(x)$ in that basis, and also ensures permanence of these coefficients with respect to the degree m of the approximant, i.e., all the coefficients of P_{m+1} agree with those of $P_m(x)$, except for that of the newly introduced term. The B-polynomials are not orthogonal. But, these can be expressed in terms of some orthogonal polynomials, such as the Legendre polynomials. The Legendre polynomials constitute an orthogonal basis that is well suited [17, 18] to least-squares approximation.

4 Expansion of B-polynomials in terms of Legendre basis and vice versa

To use the Legendre polynomials for our purposes it is preferable to map this to $[0,1]$. A set of shifted Legendre polynomials, denoted by $\{L_k(x)\}$ for $k=0,1,\ldots$, is orthogonal with respect to the weighting function $w(x)=1$ over the interval $[0,1]$. These polynomials satisfy the recurrence relation [19]

$$(k+1)\,L_{k+1}(x)=(2k+1)\,(2x-1)\,L_k(x)-kL_{k-1}(x)\,,\qquad k=1,2,\ldots,$$

with

$$L_0(x)=1,$$
$$L_1(x)=2x-1.$$

The orthogonality of these polynomials is expressed by the relation

$$\int_0^1 L_j(x)\,L_k(x)\,dx=\begin{cases}\frac{1}{2k+1}, & j=k,\\ \\ 0, & j\neq k,\end{cases}\qquad j,k=0,1,2,\quad\ldots. \tag{6}$$

when the approximant (3) is expressed in the Legendre form

$$P_m(x)=\sum_{j=0}^{m} l_j L_j(x)\,,$$

by using Eq. (6) we can obtain the Legendre coefficients as

$$l_j=(2j+1)\int_0^1 L_j(x)\,f(x)\,dx,\qquad j=0,\ldots,m.$$

Now consider a polynomial $P_m(x)$ of degree m, expressed in the m-th degree Bernstein and Legendre bases on $x\in[0,1]$:

$$P_m(x)=\sum_{j=0}^{m} c_j B_{j,m}(x)=\sum_{k=0}^{m} l_k L_k(x)\,. \tag{7}$$

We write the transformation of the Legendre polynomials on $[0,1]$ into the m-th degree Bernstein basis functions as

$$B_{k,m}(x)=\sum_{i=0}^{m} w_{k,i} L_i(x)\,,\qquad k=0,\ldots,m. \tag{8}$$

The elements $w_{k,i}, k, i = 0, 1, \ldots, m$, form a $(m+1) \times (m+1)$ basis conversion matrix W. With respect to [20] the elements of W are as follows:

$$w_{k,j} = \frac{(2j+1)}{m+j+1} \binom{m}{k} \sum_{i=0}^{j} (-1)^{j+i} \frac{\binom{j}{i}\binom{j}{i}}{\binom{m+j}{k+i}}, \qquad k, j = 0, \ldots, m.$$

Similarly, we write the transformation of the B-polynomials on $[0, 1]$ into m-th degree Legendre basis functions as

$$L_k(x) = \sum_{j=0}^{m} \Lambda_{k,j} B_{j,m}(x), \qquad k = 0, \ldots, m, \tag{9}$$

The elements $\Lambda_{k,j}, k, j = 0, 1, \ldots, m$ form a $(m+1) \times (m+1)$ basis conversion matrix Λ. Replacing Eq.(9) into Eq.(7) and re-arranging the order of summation, we obtain

$$c_j = \sum_{k=0}^{m} l_k \Lambda_{k,j}, \qquad j = 0, \ldots, m. \tag{10}$$

With respect to [20] the basis transformation (9) is defined by the elements

$$\Lambda_{k,j} = \frac{1}{\binom{m}{j}} \sum_{i=r}^{\min\{j,k\}} (-1)^{k+i} \binom{k}{i} \binom{k}{i} \binom{m-k}{j-i},$$

$$r = \max\{0, j+k-m\},$$

of the matrix Λ for $k, j = 0, \ldots, m$. If we denote the Legendre basis vector as

$$L(x) = [L_0(x), L_1(x), \ldots, L_m(x)]^T, \tag{11}$$

using Eqs. (5, 8, 9) and (11) we have

$$\phi(x) = WL(x), \tag{12}$$

and

$$L(x) = \Lambda \phi(x).$$

5 Operational matrices of integration, product and dual of B-polynomials

5.1 B-polynomials operational matrix of integration

Let P_b be an $(m+1)\times(m+1)$ operational matrix of integration, then

$$\int_0^x \phi(t)\,dt \simeq P_b\,\phi(x)\,, \qquad 0 \le x \le 1.$$

As we did in [21], this matrix is given by

$$P_b = WP\Lambda,$$

where the $(m+1)\times(m+1)$ matrix P is the operational matrix of integration of the shifted Legendre polynomials on the interval $[0,1]$ and can be obtained as [22]

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2m-1} & 0 & \frac{1}{2m-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2m+1} & 0 \end{bmatrix}.$$

5.2 B-polynomials operational matrix of product

In this Subsection, we present a general formula for finding the operational matrix of product of m-th degree B-polynomials. Suppose that C is an arbitrary $(m+1)\times 1$ vector, then $\hat{C}$ is an $(m+1)\times(m+1)$ operational matrix of product whenever

$$C^T\phi(x)\phi^T(x) \simeq \phi^T(x)\hat{C}. \tag{13}$$

Using Eq. (12) and since $C^T\phi(x) = \sum\limits_{i=0}^{m} c_i B_{i,m}$, we have

$$
\begin{aligned}
&C^T\phi(x)\phi^T(x)\\
&\quad= \left(C^T\phi(x)\right)\left(L^T(x)W^T\right)\\
&\quad= \left[L_0\left(x\right)\left(C^T\phi\left(x\right)\right),\, L_1\left(x\right)\left(C^T\phi\left(x\right)\right),\, \ldots,\, L_m(x)\left(C^T\phi\left(x\right)\right)\right] W^T\\
&\quad= \left[\sum_{i=0}^{m} c_i\left(L_0\left(x\right)B_{i,m}\left(x\right)\right), \sum_{i=0}^{m} c_i\left(L_1\left(x\right)B_{i,m}(x)\right),\, \ldots,\, \sum_{i=0}^{m} c_i\left(L_m(x)B_{i,m}(x)\right)\right] W^T
\end{aligned} \tag{14}
$$

Now, the functions of $L_k(x)B_{i,m}(x)$ is being approximate by the B-polynomials in the form of bellow functions,

$$
L_k\left(x\right)B_{i,m}\left(x\right) \simeq \eta_{k,i}^T\phi\left(x\right), \qquad i,k=0,1,\ldots,m.
$$

Using Eq. (10) we can obtain the elements of vector $\eta_{k,i}$, for $i,k=0,1,\ldots,m$. As [21] we have

$$
\sum_{i=0}^{m} c_i\left(L_k\left(x\right)B_{i,m}(x)\right) \simeq \phi^T(x)\tilde{C}_k, \tag{15}
$$

where

$$
\tilde{C}_k = \left[\eta_{k,0}, \eta_{k,1}, \ldots, \eta_{k,m}\right] C, \qquad k=0,1,\ldots,m.
$$

If we define a $(m+1)\times(m+1)$ matrix $\tilde{C}=[\tilde{C}_0,\tilde{C}_1,\ldots,\tilde{C}_m]$, then by using Eqs. (14) and (15) we have,

$$
\begin{aligned}
C^T\phi\left(x\right)\phi^T(x) &\simeq \phi^T(x)\left[\tilde{C}_0,\, \tilde{C}_1,\, \ldots,\tilde{C}_m\right] W^T\\
&= \phi^T(x)\,\tilde{C}\,W^T,
\end{aligned}
$$

and so we have the operational matrix of product as

$$
\hat{C} = \tilde{C}W^T.
$$

5.3 Dual operational matrix

In this Subsection, we want to present dual operational matrix of $\phi(x)$. With taking integration of cross product of two Bernstein basis vectors, a matrix of $(m+1)\times(m+1)$ dimensional will be resulted which will be indicated as:

$$
H = \int_0^1 \phi\left(x\right)\phi^T\left(x\right)dx. \tag{16}
$$

This matrix is known by dual operational matrix of $\phi(x)$ [23] and will be calculated as follow:
Since the integrals of the products of Bernstein basis functions by using [23]

$$\int_0^1 (1-x)^r x^i dx = \frac{1}{(r+i+1)\binom{r+i}{i}}, \qquad i, r \in N \cup \{0\},$$

is as follows:

$$\int_0^1 B_{k,m}(x)\, B_{i,j}(x)\, dx = \binom{m}{k}\binom{j}{i} \int_0^1 x^{k+i}(1-x)^{m+j-k-i} dx = \frac{\binom{m}{k}\binom{j}{i}}{(m+j+1)\binom{m+j}{k+i}}.$$

Therefore we have

$$H = \frac{1}{2m+1}\begin{bmatrix} \frac{\binom{m}{0}\binom{m}{0}}{\binom{2m}{0}} & \frac{\binom{m}{0}\binom{m}{1}}{\binom{2m}{1}} & \cdots & \frac{\binom{m}{0}\binom{m}{m}}{\binom{2m}{m}} \\ \frac{\binom{m}{1}\binom{m}{0}}{\binom{2m}{1}} & \frac{\binom{m}{1}\binom{m}{1}}{\binom{2m}{2}} & \cdots & \frac{\binom{m}{1}\binom{m}{m}}{\binom{2m}{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\binom{m}{m}\binom{m}{0}}{\binom{2m}{m}} & \frac{\binom{m}{m}\binom{m}{1}}{\binom{2m}{m+1}} & \cdots & \frac{\binom{m}{m}\binom{m}{m}}{\binom{2m}{2m}} \end{bmatrix}.$$

Also by using Eq. (12), we have

$$H = \int_0^1 \phi(x)\, \phi^T(x)\, dx = \int_0^1 (W L(x))\, (W L(x))^T dx$$

$$= W\left[\int_0^1 L(x)\, L^T(x) dx\right] W^T = W D W^T,$$

where D is a $(m+1)\times(m+1)$ matrix and by using Eq. (6) is defined as,

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{5} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \frac{1}{(2m+1)} \end{bmatrix}.$$

6 Method of Solution

Consider the s-th order nonlinear Fredholm integro-differential equation (1) with the initial conditions (2).

Step 1: The functions of $y^j(x), j = 0, 1, \ldots, s$ is being approximate by the B-polynomials. Therefore with approximation $y^s(x)$ in the form of

$$y^{(s)}(x) = C^T \phi(x), \tag{17}$$

where C is defined similar to Eq. (4), we have

$$y^{(j)}(x) = Q_j^T \phi(x), \qquad j = 0,\ 1,\ \ldots\ ,\ s, \tag{18}$$

where Q_j's are $(m+1) \times 1$ vectors and details of obtaining these vectors is given in [21].

Step 2: The function of $k(x,t)$ is being approximate by the B-polynomials in the form of bellow function,

$$k(x,t) = \phi^T(x)\, K_b\, \phi(t), \tag{19}$$

where K_b is a matrix of $(m+1) \times (m+1)$ dimensional and will be calculated as follow:
If we approximate $k(x,t)$ with the shifted Legendre polynomials on the interval $[0,1]$ as

$$k(x,t) = L^T(x)\, K_l\, L(t), \tag{20}$$

in which K_l is a $(m+1) \times (m+1)$ matrix and the entries can be calculated as

$$K_{l i,j} = \frac{(L_i(x), (k(x,t), L_j(t)))}{(L_i(x), L_i(x))\,(L_j(t), L_j(t))}, \qquad for \quad i,\ j = 0,\ 1,\ \ldots\ ,m,$$

where $(.,.)$ denotes the inner product. Then by using Eqs. (12, 19-20), K_b will be as:

$$K_b = (W^T)^{-1} K_l\, W^{-1}.$$

Step 3: In this step, we present a general formula for approximate $y^p(t)$ with the B-polynomials. To do so, by using Eqs. (18, 13) we have

$$y^2(t) = Q_0^T \phi(t)\, \phi^T(t)\, Q_0 = \phi^T(t)\, \hat{Q}_0 Q_0,$$

$$y^3(t) = Q_0^T \phi(t)\, \phi^T(t)\, \hat{Q}_0 Q_0 = \phi^T(t)\, (\hat{Q}_0)^2 Q_0,$$

and so by use of induction $y^p(t)$ will be approximated as

$$y^p(t) = Q_0^T \phi(t)\, \phi^T(t)\, (\hat{Q}_0)^{p-2} Q_0 = \phi^T(t)\, (\hat{Q}_0)^{p-1} Q_0. \tag{21}$$

Now, with approximate $g(x)$ and $\rho_j(x), j = 0, 1, \ldots, s$ in the forms of $G^T \phi(x)$ and $\rho_j(x) = {P_j}^T \phi(x), j = 0, 1, \ldots, s$, respectively in which G and $P_j, j = 0, 1, \ldots, s$, are the coefficients which are defined similarly to Eq. (4) and using Eqs. (18-19, 21) into Eq. (1) we have

$$\sum_{j=0}^{s} P_j^T \phi(x)\, \phi^T(x)\, Q_j = G^T \phi(x) + \lambda \int_0^1 \phi^T(x)\, K_b\, \phi(t)\, \phi^T(t)\, (\hat{Q}_0)^{p-1} Q_0\, dt,$$

using Eqs. (13, 16) we obtain

$$\sum_{j=0}^{s} \phi^T(x)\, \hat{P}_j\, Q_j = \phi^T(x)\, G + \lambda\, \phi^T(x)\, K_b\, H\, (\hat{Q}_0)^{p-1} Q_0,$$

and therefore we get

$$\sum_{j=0}^{s} \hat{P}_j\, Q_j = G + \lambda K_b\, H\, (\hat{Q}_0)^{p-1} Q_0, \tag{22}$$

The matrix equation (22) gives a system of $m+1$ nonlinear algebraic equation which can be solved for the elements of C in Eq. (17). Once C is known, $y(x)$ can be calculated from Eq. (18).

7 Illustrative Examples

In this section, we apply the method presented in this article and solve five examples. The computations associated with the examples were performed using Matlab 7.1.

Example 1. Consider the first-order nonlinear Fredholm integro-differential equation [5]

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 x\, y^2(t)\, dt, \qquad 0 \le x \le 1,$$

with the initial condition $y(0) = 0$. In this example we have

$$\rho_0(x) = 0, \quad \rho_1(x) = 1, \quad g(x) = 1 - \tfrac{1}{3}x, \quad k(x,t) = x, \quad p = 2.$$

By applying the method in Section 6, the expression of equation matrix will be as follow,

$$Q_1 - G - K_b\, H\, \hat{Q}_0 Q_0 = 0,$$

where for $m = 1$ we have

$$Q_1 = C = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}, \quad K_b = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$\mathrm{H} = \tfrac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q_0 = \tfrac{1}{12}\begin{bmatrix} c_0 - c_1 \\ 7c_0 + 5c_1 \end{bmatrix}, \quad \hat{Q}_0 = \tfrac{1}{12}\begin{bmatrix} 2c_0 & -(c_0 + c_1) \\ c_0 + c_1 & 6c_0 + 4c_1 \end{bmatrix}.$$

Therefore this algebraic equation system

$$\begin{cases} c_0 - 1 = 0, \\ c_1 - \frac{2}{3} - \frac{19\,c_0^2 + 22\,c_0\,c_1 + 7c_1^2}{144} = 0, \end{cases}$$

will be resulted. In which by solving above expression $c_0 = c_1 = 1$, will be obtained. Then by substituting values of c_0, c_1 by $y(x) = {Q_0}^T \phi(x)$, the result will be as $y(x) = x$, that is the exact solution. It is noted that with $N = 5$, the maximum absolute error on the grid points Sinc [5] in the Sinc method [5], is 1.52165×10^{-3}; but in the present method with $m = 1$ (namely only with 2 basis function) the maximum absolute error on the grid points Sinc is equal to zero.

Example 2. Consider the first-order nonlinear Fredholm integro-differential equation [4]

$$x\,y'(x) - y(x) = \frac{4}{5}x^2 - \frac{1}{6} + \int_0^1 (x^2 + t)y^2(t)\,dt, \qquad 0 \le x \le 1,$$

with the initial condition $y(0) = 0$. In this example we have

$$\rho_0(x) = -1, \quad \rho_1(x) = x, \quad g(x) = \tfrac{4}{5}x^2 - \tfrac{1}{6}, \quad k(x,t) = x^2 + t, \quad p = 2.$$

By applying the method in Section 6, the expression of equation matrix will be as follow,

$$\hat{P}_1\, Q_1 - Q_0 - G - K_b\, H\, \hat{Q}_0 Q_0 = 0,$$

where for $m = 2$ we have

$$\hat{P}_1 = \begin{bmatrix} \frac{1}{20} & \frac{-1}{10} & \frac{1}{20} \\ \frac{1}{4} & \frac{1}{2} & \frac{-1}{4} \\ \frac{-1}{20} & \frac{1}{10} & \frac{57}{60} \end{bmatrix}, Q_1 = C = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}, Q_0 = \begin{bmatrix} \frac{1}{60}c_0 - \frac{1}{30}c_1 + \frac{1}{60}c_2 \\ \frac{5}{12}c_0 + \frac{1}{6}c_1 - \frac{1}{12}c_2 \\ \frac{19}{60}c_0 + \frac{11}{30}c_1 + \frac{19}{60}c_2 \end{bmatrix},$$

$$G = \begin{bmatrix} \frac{-1}{6} \\ \frac{-1}{6} \\ \frac{19}{30} \end{bmatrix}, K_b = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & \frac{3}{2} & 2 \end{bmatrix}, H = \frac{1}{5}\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} & 1 \end{bmatrix},$$

$$\hat{Q}_0 = \begin{bmatrix} \frac{67}{1050}c_0 - \frac{1}{75}c_1 - \frac{1}{2100}c_2 & -\frac{93}{2100}c_0 - \frac{1}{25}c_1 - \frac{33}{2100}c_2 & -\frac{3}{1050}c_0 + \frac{1}{50}c_1 + \frac{69}{2100}c_2 \\ \frac{57}{420}c_0 + \frac{1}{10}c_1 - \frac{1}{70}c_2 & \frac{31}{105}c_0 + \frac{1}{6}c_1 + \frac{4}{105}c_2 & -\frac{1}{70}c_0 - \frac{1}{10}c_1 - \frac{57}{420}c_2 \\ -\frac{69}{2100}c_0 - \frac{1}{50}c_1 + \frac{3}{1050}c_2 & \frac{33}{2100}c_0 + \frac{1}{25}c_1 + \frac{93}{2100}c_2 & \frac{701}{2100}c_0 + \frac{182}{525}c_1 + \frac{283}{1050}c_2 \end{bmatrix}.$$

That by solving above matrix equation $c_0 = 0, c_1 = 1, c_2 = 2$ will be obtained. Then by substituting values of c_0, c_1, c_2 by $y(x) = Q_0^T \phi(x)$, the result will be as $y(x) = x^2$, that is the exact solution.

Example 3. Consider the second-order nonlinear Fredholm integro-differential equation [7]

$$y''(x) + x\, y'(x) - x\, y(x) = e^x - \sin(x) + \int_0^1 \sin(x).\, e^{-2t} y^2\,(t)\, dt, \quad 0 \le x \le 1, \tag{23}$$

with the initial conditions $y(0) = y^{'}(0) = 1$ and the exact solution $y(x) = e^{x}$. We solve Eq. (23) by using the method in Section 6 with $m = 5$. The comparison among the present method and method in [7] is shown in Table 1. As we see from this Table, it is clear that the result obtained by the present method is very superior to that by the method in [7]. Thus the result for $m = 7$, in this Table will be presented. As we observed in this Table with increasing the value of m, the resulted accuracy increased as well. Figure 1(a, b).

Table 1: Numerical results for Example 3

x	presented method for m=5	method of [7] for m=5, n=5	presented method for m=7
0.0	$2.5065E-006$	0	$3.2038E-009$
0.2	$3.8332E-007$	$4.0000E-007$	$7.1841E-010$
0.4	$1.4274E-007$	$8.1000E-006$	$1.4151E-010$
0.6	$2.5813E-007$	$7.7300E-005$	$4.0671E-011$
0.8	$4.7475E-007$	$4.2480E-004$	$9.1044E-010$
1.0	$2.5068E-006$	$1.6413E-003$	$3.7002E-009$

Example 4. Consider the first-order nonlinear Fredholm integro-differential equation [6]

$$y'(x) + y(x) = \frac{1}{2}(e^{(-2)} - 1) + \int_0^1 y^2(t)\,dt, \qquad 0 \le x \le 1,$$

with the initial condition $y(0) = 1$. The exact solution of this example is $y(x) = e^{-x}$. The absolute difference error for $m = 3, 5, 7$ in Table 2 is being observed. In addition the last column of this Table indicates the existed result in [6] for 34 basis function. As you can observe in the presented method for the less basis function the more accuracy with respect method [6], can be seen. Figure 1(c, d).

Table 2: Numerical results for Example 4

x	presented method			method of [6]
	m=3	m=5	m=7	for m=4
0.125	$1.3471E-004$	$1.7053E-007$	$2.4509E-010$	$9.4E-006$
0.250	$9.4839E-005$	$4.4145E-007$	$1.0202E-010$	$5.1E-006$
0.375	$7.0662E-005$	$1.9861E-007$	$1.6139E-010$	$3.0E-005$
0.500	$1.3596E-004$	$3.7164E-008$	$3.2362E-010$	$4.9E-005$
0.625	$4.3458E-005$	$3.3857E-007$	$1.9197E-010$	$5.5E-005$
0.750	$1.1598E-004$	$6.2605E-007$	$6.6120E-011$	$4.5E-005$
0.875	$1.2047E-004$	$9.7392E-008$	$2.2417E-010$	$2.1E-005$

Table 3: Numerical results for Example 5

x	presented method		
	m=3	m=5	m=9
0.0	$9.9522E-004$	$2.5022E-006$	$2.4740E-012$
0.2	$4.2859E-004$	$3.8321E-007$	$1.9780E-012$
0.4	$1.7370E-004$	$1.4740E-007$	$2.5981E-012$
0.6	$2.2852E-004$	$2.6432E-007$	$3.8940E-012$
0.8	$4.3482E-004$	$4.7009E-007$	$5.7709E-012$
1.0	$9.4787E-004$	$2.4950E-006$	$3.3360E-012$

Example 5. Consider the first-order nonlinear Fredholm integro-differential equation [5]

$$y'(x) = e^x - \frac{1}{5}e^{-x^2}(e^5 - 1) + \int_0^1 e^{2t-x^2} y^3(t)\,dt, \qquad 0 \le x \le 1, \tag{24}$$

with the initial condition $y(0) = 1$ and the exact solution $y(x) = e^x$. The absolute difference error for $m = 3, 5, 9$ in Table 3 is being observed. As we observed in this Table with increasing the value of m the resulted accuracy increased as well. It is noted that with $N = 5$, the maximum absolute error on the grid points Sinc in [5], is 3.72499×10^{-3}; but in the present method with $m = 10$, for equality basis function(11 basis function) the maximum absolute error on the grid points Sinc is 3.4916×10^{-11}. Figure 1(e, f).

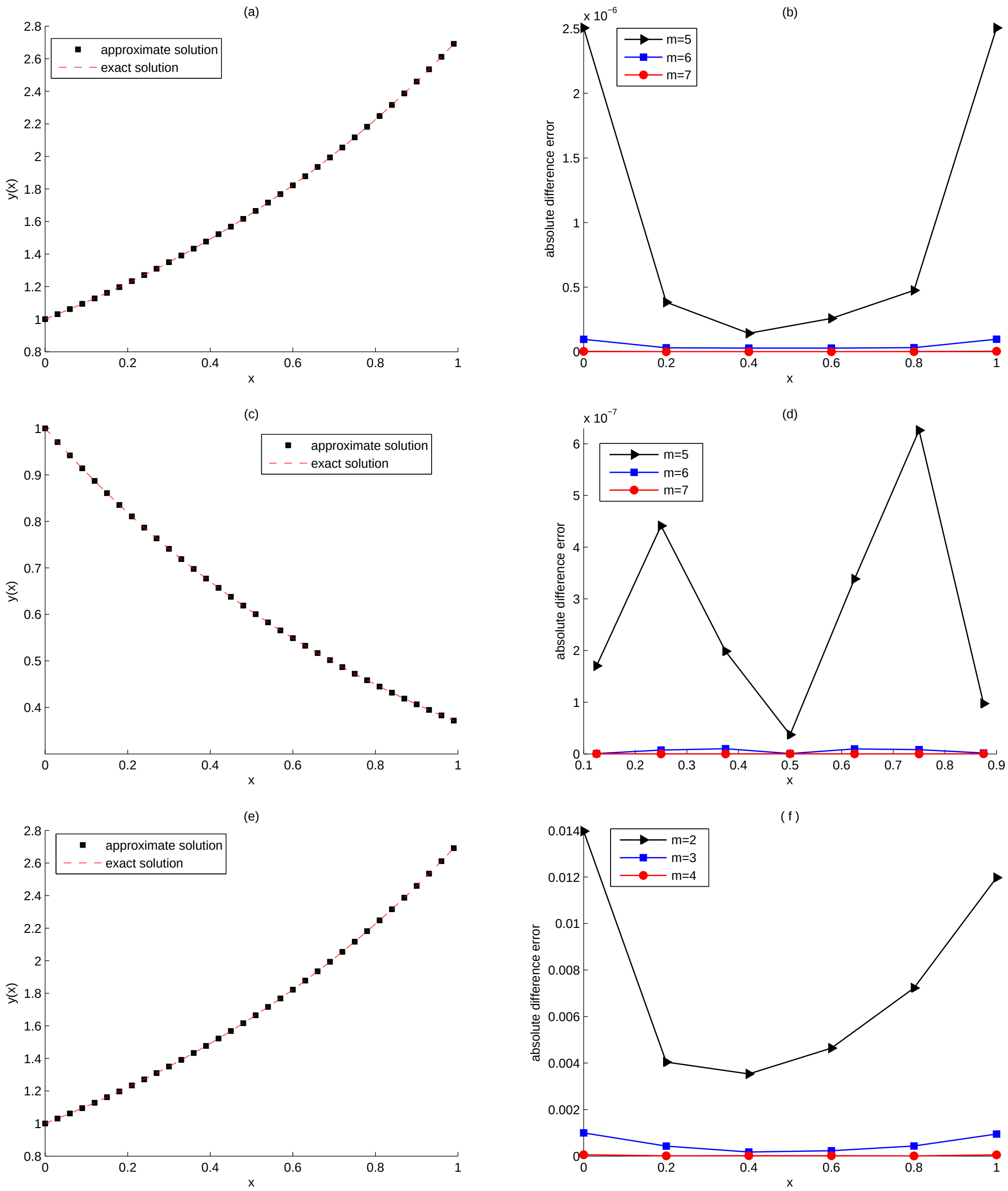

Figure 1: (a, c) Exact and approximate solution of Examples 3 and 4 for $m = 7$, respectively; (b, d) Absolute difference error of Examples 3 and 4 for $m = 5, 6, 7$, respectively; (e) Exact and approximate solution of Example 5 for $m = 9$ and (f) Absolute difference error of Example 5 for $m = 2, 3, 4$.

8 Conclusion

In this article, a numerical method for solving nonlinear Fredholm integro-differential equations was purposed. As observed in this method, at first the result of equation was considered in the form of expansion of Bernstein basis functions of m-th degree. Then by using operational matrices of these polynomials the mentioned equation was converted to a system of nonlinear algebraic equations. One of the most important properties of this method is that when the result of equation is in the form of a polynomial of degree $\leq m$, the exact solution is obtained. In addition this method has high relative accuracy for small values of m, specially the time of calculations is short. Also between value of m and accuracy of result is a direct relation, namely by increasing the value of m the accuracy of result is increased as well.

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