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Oscillation of Neutral Delay Partial Difference Equation

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Abstract

In this paper, some sufficient conditions for oscillation of the neutral delay partial equation :

$$\Delta_{1,2}(A_{m,n} - cA_{m-\tau,\, n-\sigma}) + \sum_{i=1}^{\mu} P_i(m,n)A_{m-k_i,\, n-l_i} = 0$$

are established. Our results as a special case when $c = 0$, $\mu = 1$, involve and improve some well-known oscillation results.

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1 Introduction

It is well known that the partial difference equations appear in considerations of random walk problems, molecular structure and chemical reactions problems

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[1-3].Oscillation and nonoscillation of solutions of delay partial difference equations is receiving much attention [4-7].

In this paper, we consider the neutral delay partial difference equation

$$\Delta_{1,2}(A_{m,n} - cA_{m-\tau,n-\sigma}) + \sum_{i=1}^{\mu} P_i(m,n)A_{m-k_i,n-l_i} = 0, \tag{1.1}$$

where $m,n \in N_0 = \{0,1,2,\ldots\}$ and $\tau,\sigma,k_i,l_i (i = 1,2,\cdots,\mu)$ are nonnegative integers, the coefficients $\{P_i(m,n)\} \in N_0^2 = \{0,1,2,\ldots\}^2$ is a sequences of nonnegative real numbers, and $0 \le c \le 1$. We defined

$$\Delta_{1,2}(Z_{m,n}) = Z_{m+1,n} + Z_{m,n+1} - Z_{m,n}, \quad Z_{m,n} = A_{m,n} - cA_{m-\tau,n-\sigma}.$$

A solution $\{A_{m,n}\}$ of (1.1) is said to be eventually positive if $A_{m,n} > 0$ for all large m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

As a special case of Eq. (1.1), B.G.Zhang et al.[5] considered partial difference equation

$$A_{m+1,n} + A_{m,n+1} - A_{m,n} + P_{m,n}A_{m-k,n-l} = 0, \tag{1.2}$$

And proved that: for all large m, n, and there exists ξ such that

$$P_{m,n} \ge \xi > \frac{(k+l)^{k+l}}{(k+l+1)^{k+l+1}}, \tag{1.3}$$

Then every solution of equation (1.2) oscillates.

2 Main Results

In this section, we give some oscillation for Eq.(1.1). In order to prove our main results, we need the following auxiliary results.

Lemma 1. *Suppose that* $\{A_{m,n}\}$ *is an eventually positive solution of equation* (1.1), *then* :

(i) $\Delta_{1,2}(Z_{m,n}) \leq 0$, and $Z_{m,n}$ is monotone decreasing in m , n , that is $Z_{m+1,n} \leq Z_{m,n}, Z_{m,n+1} \leq Z_{m,n}$;

(ii) $Z_{m,n} > 0$.

Proof. Since $\{A_{m,n}\}$ is an eventually positive solution of (1.1), then there exists enough M, N, when $m \geq M$, $n \geq N$, such that

$$A_{m,n} > 0,\ A_{m-\tau,n-\sigma} > 0,\ A_{m-k_i,n-l_i} > 0,\ i = 1, 2, \ldots, \mu.$$

From (1.1), we obtain

$$\Delta_{1,2}(Z_{m,n}) = -\sum_{i=1}^{\mu} P_i(m,n) A_{m-k_i,n-l_i} \leq 0.$$

That is

$$Z_{m+1,n} + Z_{m,n+1} - Z_{m,n} \leq 0,\ Z_{m+1,n} \leq Z_{m,n}, Z_{m,n+1} \leq Z_{m,n}.$$

Next, we show that $Z_{m,n}$ is eventually positive in m, n. If $Z_{m,n} \leq 0$, then there exists $d > 0$, for all large M_1, N_1, when $m \geq M_1$, $n \geq N_1$, such that $Z_{m,n} \leq -d$.

$$A_{m,n} - A_{m-\tau,n-\sigma} \leq A_{m,n} - cA_{m-\tau,n-\sigma} = Z_{m,n} \leq -d,\ A_{m,n} \leq -d + A_{m-\tau,n-\sigma}.$$

Therefore,

$$A_{m+h\tau,n+h\sigma} \leq -d + A_{m+(h-1)\tau,n+(h-1)\sigma} \leq -2d + A_{m+(h-2)\tau,n+(h-2)\sigma} \leq \cdots \leq -(h+1)d + A_{m-\tau,n-\sigma}$$

as $h \to \infty, A_{m+h\tau,n+h\sigma} \to -\infty$ .Which contradiction to $\{A_{m,n}\}$ is an eventually positive solution. This completes the proof. □

Theorem 1. *Assume that*

$$\liminf_{m,n\to\infty}(1+c)\sum_{i=0}^{\mu}P_i(m,n) > \frac{\hat{k}}{(\hat{k}+1)^{\hat{k}+1}}\frac{\hat{l}}{(\hat{l}+1)^{\hat{l}+1}}, \tag{2.1}$$

where $\hat{k}=\min(k_1,k_2,\ldots,k_\mu)$, $\hat{l}=\min(l_1,l_2,\ldots,l_\mu)$, *then every solution of equation* (1.1) *oscillates.*

Proof. Suppose to the contrary that the equation (1.1) has a nonoscillatory solution $\{A_{m,n}\}$.Without loss of generality, we may assume that $\{A_{m,n}\}$ is an eventually positive solution of equation (1.1), then from (1.1), we have

$$\begin{aligned}0&=\Delta_{1,2}(Z_{m,n})+\sum_{i=1}^{\mu}P_i(m,n)A_{m-k_i,n-l_i}\\&=\Delta_{1,2}(Z_{m,n})+\sum_{i=1}^{\mu}P_i(m,n)(Z_{m-k_i,n-l_i}+cA_{m-k_i-\tau,n-l_i-\sigma})\\&>\Delta_{1,2}(Z_{m,n})+\sum_{i=1}^{\mu}P_i(m,n)(Z_{m-k_i,n-l_i}+cZ_{m-k_i-\tau,n-l_i-\sigma})\end{aligned}$$

By Lemma 1, we have

$$\Delta_{1,2}(Z_{m,n})+\sum_{i=1}^{\mu}(1+c)P_i(m,n)Z_{m-\hat{k},n-\hat{l}}<0,$$

That is

$$Z_{m+1,n}+Z_{m,n+1}-Z_{m,n}<-\sum_{i=1}^{\mu}(1+c)P_i(m,n)Z_{m-\hat{k},n-\hat{l}}, \tag{2.2}$$

dividing the both sides of (2.2) by $Z_{m,n}$, we have

$$\frac{Z_{m+1,n}}{Z_{m,n}}+\frac{Z_{m,n+1}}{Z_{m,n}}<1-\sum_{i=1}^{\mu}(1+c)P_i(m,n)\frac{Z_{m-\hat{k},n-\hat{l}}}{Z_{m,n}}, \tag{2.3}$$

Set $r_{m,n}=\dfrac{Z_{m,n}}{Z_{m+1,n}}, t_{m,n}=\dfrac{Z_{m,n}}{Z_{m,n+1}}$. It is easy to see that $r_{m,n}\ge 1$, $t_{m,n}\ge 1$, for all large m and n. $\{r_{m,n}\},\{t_{m,n}\}$ are bounded. Let $\liminf_{m,n\to\infty} t_{m,n}=b\ge 1$, then from (2.3), we have

$$\frac{1}{r_{m,n}}+\frac{1}{t_{m,n}}<1-\sum_{i=1}^{\mu}(1+c)P_i(m,n)\frac{Z_{m-\hat{k},n-\hat{l}}}{Z_{m,n}}$$

$$\le 1-\sum_{i=1}^{\mu}(1+c)P_i(m,n)r_{m-\hat{k},n}r_{m-\hat{k}+1,n}\cdots r_{m-1,n}t_{m-\hat{l},n}t_{m-\hat{l}+1,n}\cdots t_{m-1,n} \quad (2.4)$$

from (2.4), we get

$$\limsup_{m,n\to\infty}\frac{1}{r_{m,n}}+\limsup_{m,n\to\infty}\frac{1}{t_{m,n}}=\frac{1}{a}+\frac{1}{b}\le 1-\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)a^{\hat{k}}b^{\hat{l}} ,$$

or

$$\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)\le(1-\frac{1}{a}-\frac{1}{b})\frac{1}{a^{\hat{k}}b^{\hat{l}}}\le\frac{(a-1)(b-1)}{a^{\hat{k}+1}b^{\hat{l}+1}}=f(a,b).$$

Now it is easy to see that

$$\max_{a\ge1,b\ge1}f(a,b)=\frac{\hat{k}}{(\hat{k}+1)^{\hat{k}+1}}\frac{\hat{l}}{(\hat{l}+1)^{\hat{l}+1}},$$

hence we obtain

$$\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)\le\frac{\hat{k}}{(\hat{k}+1)^{\hat{k}+1}}\frac{\hat{l}}{(\hat{l}+1)^{\hat{l}+1}},$$

which is a contradiction to (2.1).this completes the proof. □

According to theorem 1, if $c=0,\mu=1$, we obtain the following result:

Corollary 1. *For all large* m, n, *there exists* ξ *such that*

$$P_{m,n}\ge\xi>\frac{k^k}{(k+1)^{k+1}}\frac{l^l}{(l+1)^{l+1}}. \quad (2.5)$$

Then every solution of equation oscillates.

Remark 1. From Corollary 1, compare (2.5) with (1.3), obviously

$$\frac{k^k}{(k+1)^{k+1}}\frac{l^l}{(l+1)^{l+1}} < \frac{(k+l)^{k+l}}{(k+l+1)^{k+l+1}}.$$

Theorem 2. *Assume that*

$$\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)2^{\xi_i}\frac{(\xi_i+1)^{\xi_i+1}}{(\xi_i)^{\xi_i}} > 1, \qquad (2.6)$$

where $\xi_i = \min(k_i, l_i), i = 1,2,\cdots\mu$, *then every solution of equation* (1.1) *oscillates.*

Proof. Suppose to the contrary that $\{A_{m,n}\}$ is an eventually positive solution of equation (1.1). Set $s_{m,n} = \frac{z_{m,n}}{z_{m+1,n+1}}$. It is easy to see that $s_{m,n} \geq 1$, for all large m and n, $s_{m,n}$ is bounded. Let $\liminf_{m,n\to\infty} s_{m,n} = \beta \geq 1$, then from (1.1) and by Lemma 1, we have

$$\Delta_{1,2}(Z_{m,n}) + \sum_{i=1}^{\mu}(1+c)P_i(m,n)Z_{m-\xi_i,n-\xi_i} \leq 0.$$

That is

$$\frac{2}{s_{m,n}} \leq \frac{Z_{m+1,n}}{Z_{m,n}} + \frac{Z_{m,n+1}}{Z_{m,n}} \leq 1 - \sum_{i=1}^{\mu}(1+c)P_i(m,n)\frac{Z_{m-\xi_i,n-\xi_i}}{Z_{m,n}},$$

$$\frac{2}{s_{m,n}} \leq 1 - \sum_{i=1}^{\mu}(1+c)P_i(m,n)s_{m-\xi_i,n-\xi_i}s_{m-\xi_i+1,n-\xi_i+1}\cdots s_{m-1,n-1}. \qquad (2.7)$$

Taking supremun limit on both sides of (2.7),we have

$$\frac{2}{\beta} \leq 1 - \liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)\beta^{\xi_i},$$

which implies $\beta > 2$ and

$$\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)\frac{\beta^{\xi_i+1}}{\beta-2} \leq 1. \qquad (2.8)$$

Noticed that

$$\min_{\beta>2}\left(\frac{\beta^{\xi_i+1}}{\beta-2}\right)=2^{\xi_i}\frac{(\xi_i+1)^{\xi_i+1}}{\xi_i^{\xi_i}}.$$

Hence we have

$$\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)2^{\xi_i}\frac{(\xi_i+1)^{\xi_i+1}}{\xi_i^{\xi_i}}\le 1,$$

which contradicts (2.6). This completes the proof. □

Corollary 2. *Assume that*

$$\mu\Big(\prod_{i=1}^{\mu}\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)\Big)^{\frac{1}{\mu}}>\frac{\bar{\xi}^{\bar{\xi}}}{2^{\bar{\xi}}(\bar{\xi}+1)^{\bar{\xi}+1}},\tag{2.9}$$

where $\bar{\xi}=\frac{1}{\mu}(\sum_{i=1}^{\mu}\xi_i)$, $\xi_i=\min(k_i,l_i)$, $i=1,2,\ldots,\mu$. Then every solution of (1.1) oscillates.

Proof . In fact, from (2.8) we have

$$1\ge\liminf_{m,n\to\infty}\sum_{i=1}^{\mu}(1+c)P_i(m,n)\frac{\beta^{\xi_i+1}}{\beta-2}\ \ \ge\mu\Big(\prod_{i=1}^{\mu}\liminf_{m,n\to\infty}(1+c)P_i(m,n)\Big)^{\frac{1}{\mu}}.$$

Hence

$$1\ge\mu\Big(\prod_{i=1}^{\mu}\liminf_{m,n\to\infty}(1+c)P_i(m,n)\Big)^{\frac{1}{\mu}}2^{\bar{\xi}}\frac{(\bar{\xi}+1)^{\bar{\xi}+1}}{\bar{\xi}^{\bar{\xi}}}.$$

That is

$$\mu\Big(\prod_{i=1}^{\mu}\liminf_{m,n\to\infty}(1+c)P_i(m,n)\Big)^{\frac{1}{\mu}}\le\frac{\bar{\xi}^{\bar{\xi}}}{2^{\bar{\xi}}(\bar{\xi}+1)^{\bar{\xi}+1}}.$$

which contradicts (2.9).The proof is completed. □

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