Theoretical Mathematics & Applications, vol.4, no.3, 2014, 53-78
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2014

The sub-equation method and new travelling wave analytical solutions of nonlinear Schrödinger's equation with Kerr law nonlinearity

H. Yépez-Martínez[1], I.O. Sosa[2] and J.M. Reyes[3]

Abstract

In this paper we construct new travelling wave solutions to the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity by introducing the sub-equation method and the generalized Riccati equation. The travelling wave solutions are expressed by hyperbolic functions, trigonometric functions and rational functions. We also discuss the reliability of the proposed method and compared the obtained solutions with those previously reported in the literature. The presented method provides a comprehensive approach to construct many new solutions for nonlinear partial differential equations.

---

[1] Universidad Autónoma de la Ciudad de México, Prolongación San Isidro 151, Col. San Lorenzo Tezonco, Del. Iztapalapa, P.O. Box 09790 México D.F., Mexico.
E-mail: huitzilin.yepez.martinez@uacm.edu.mx

[2] Universidad Autónoma de la Ciudad de México, Prolongación San Isidro 151, Col. San Lorenzo Tezonco, Del. Iztapalapa, P.O. Box 09790 México D.F., Mexico.
E-mail: ivan.sosa@uacm.edu.mx

[3] Universidad Autónoma de la Ciudad de México, Prolongación San Isidro 151, Col. San Lorenzo Tezonco, Del. Iztapalapa, P.O. Box 09790 México D.F., Mexico.
E-mail: manuel.reyes@uacm.edu.mx

Article Info: *Received:* June 18, 2014. *Revised:* July 26, 2014.
Published online : August 31, 2014.

Mathematics Subject Classification: 35G20; 35A35; 35Q55

Keywords: Nonlinear Schrödinger’s equation; Kerr law nonlinearity; sub-equation method; travelling wave solutions

1 Introduction

The nonlinear equations have many wide array of application in many fields, analytical solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more inside into the physical aspects of the problem and thus lead to further applications.

Phenomena in physics and other fields are often described by nonlinear partial differential equations (NPDEs). When we want to understand the physical mechanism of phenomena in nature, described by NPDEs, exact solutions for the NPDEs have to be explored. For example, the wave phenomena observed in fluid dynamics [1], elastic media [2], optical fibers [3], etc. Thus the methods for deriving exact solutions for the NPDEs need to be developed. Recently, many powerful methods have been established and improved. Among these methods we can mention the homogeneous balance method [4], the tanh-function method [5], the extended tanh-function method [6,7], the Jacobi elliptic function expansion method [8,9], the auxiliary equation method [10], the trigonometric function series method [11], the modified mapping method and the extended mapping method [12], hyperbolic function method [13], rational expansion method [14], sine-cosine method [15], F-expansion method [16], the transformed rational function method [17], the multiple exp-function method [18], and so on.

Recently, Wang et al. proposed a new method called the $\left(\frac{G'}{G}\right)$-expansion method to construct traveling wave solutions for NPDEs, which are expressed by the hyperbolic, trigonometric and rational functions [19-21]. Recently, the

$\left(\frac{G'}{G}\right)$-expansion method has been successfully applied to obtain exact solutions for a variety of NPDEs [22-30].

Among these methods, the tanh-function method, the extended tanh-function method and the auxiliary equation method belong to a class called sub-equation method. These sub-equation methods consist of looking for the solutions of the nonlinear evolution equations in consideration as a polynomial in a variable that satisfies an ordinary differential equation, named the sub-equation.

In the present article, we will consider the sub-equation method [31], by introducing the generalized Riccati equation and its twenty seven solutions [32], in order to obtain new travelling wave solutions to the nonlinear Schrödinger's equation (NLSE) with Kerr law nonlinearity. Furthermore, we will show that new travelling wave solutions can be expressed in terms of some elemental functions.

The outline of this work is as follows: in section 2 a brief review of the nonlinear Schrödinger's equation with Kerr law nonlinearity is presented, in section 3, the sub-equation method is considered. Section 4 contains the application of the method to solve the NLSE with Kerr law nonlinearity and we discuss the reliability of the proposed method for the new exact solutions which can be compared with those solutions previously reported in the literature, see for example [33,34]. Finally in section 5 some conclusions are presented.

2 The Nonlinear Schrödinger's Equation with Kerr law Nonlinearity

In this paper, we will consider the perturbed NLSE with Kerr law nonlinearity [16]

$$iu_t + u_{xx} + \alpha |u|^2 u + i\left[\gamma_1 u_{xxx} + \gamma_2 |u|^2 u_x + \gamma_3 \left(|u|^2\right)_x u\right] = 0, \tag{1}$$

the above equation describes the propagation of optical solitons in nonlinear optical fibers that exhibits Kerr law nonlinearity. Where γ_1 is the third order dispersion, γ_2 is the nonlinear dispersion coefficient, while γ_3 is also a version of nonlinear dispersion [12]. The Kerr law nonlinearity originates from the fact that a light wave in an optical fiber faces nonlinear responses from nonharmonic motion of electrons bound in molecules, caused by an external electric field. Even though the nonlinear responses are extremely weak, their effects appear in various ways over long distances of propagation that are measured in terms of light wavelength. More details are presented in Refs. [33-41]. Recently, there are lots of contributions about Eq. (1), see for instance [36-56]. These papers have been concerned with finding various types of solutions, including fronts (kinks), bright solitary waves, and dark solitary waves in various media, such as power law (or dual-power law), parabolic law and Kerr law. The NLSE with Kerr law nonlinearity has important applications in various fields, such as semiconductor materials, optical fiber communications, plasma physics, fluid and solid mechanics. It is well known that in the absence of $\gamma_1, \gamma_2, \gamma_3$ (i.e. $\gamma_1 = \gamma_2 = \gamma_3 = 0$), Eq. (1) reduces to

$$iu_t + u_{xx} + \alpha |u|^2 u = 0 \quad , \tag{2}$$

Eq. (2) admits the bright soliton solution [57]:

$$u(x,t) = k\sqrt{\frac{2}{\alpha}}\sec h\big(k(x-2\mu t)\big)e^{i\left[\mu x-\left(\mu^2-k^2\right)t\right]} \quad , \tag{3}$$

where α, μ and k are arbitrary real constants, for the self-focusing case $\alpha > 0$, and the dark soliton solution [58]:

$$u(x,t)=k\sqrt{-\frac{2}{\alpha}}\tanh\big(k(x-2\mu t)\big)e^{i\left[\mu x-\left(\mu^2+2k^2\right)t\right]} \quad , \tag{4}$$

for the de-focusing case $\alpha < 0$.

3 Description of the Sub-equation Method and its Applications to the Space-Time Partial Differential Equations

Suppose the general nonlinear partial differential equation,

$$P\left(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots\right) = 0 \quad , \tag{5}$$

where $u=u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved. The main steps of the sub-equation method combined with the generalized Riccati equation are described as follows [31]:

Step1: The travelling wave variable ansätz

$$u(x,t) = u(\xi), \quad \xi = k\left(x - ct\right) \quad , \tag{6}$$

where c is the speed of the traveling wave, permits us to transform the equation (5) into an ordinary differential equation (ODE):

$$Q\left(u, u', u'', \ \ldots \ \right) = 0 \quad , \tag{7}$$

where the superscripts stands for the ordinary derivatives with respect to ξ.

Step 2: Suppose the traveling wave solution of equation (5) can be expressed by a polynomial in $G(\xi)$ as follows:

$$u(\xi) = \sum_{n=0}^{m_{\max}} a_n \left(G\right)^m \quad , \quad a_n \neq 0 \quad , \tag{8}$$

where G satisfies the generalized Riccati equation,

$$G' = r + pG + qG^2 = 0 \quad , \tag{9}$$

a_n $(n=1,2,3,\ldots,m_{\max})$, r, p and q are arbitrary constants to be determined later. The generalized Riccati equation (9) has twenty seven solutions, (see for example [32]), which can be expressed as follow:

Family 1: When $p^2 - 4qr < 0$ and $pq \neq 0$ (or $rq \neq 0$), the solutions of equation (9) are:

$$G_1\left(\xi\right) = \frac{1}{2q}\left(-p + h\tan\left(\frac{1}{2}h\xi\right)\right)$$

$$G_2(\xi) = -\frac{1}{2q}\left(p + h\cot\left(\frac{1}{2}h\xi\right)\right)$$

$$G_3(\xi) = \frac{1}{2q}\left(-p + h\left(\tan(h\xi) \pm \sec(h\xi)\right)\right)$$

$$G_4(\xi) = -\frac{1}{2q}\left(p + h\left(\cot(h\xi) \pm \csc(h\xi)\right)\right)$$

$$G_5(\xi) = \frac{1}{4q}\left(-2p + h\left(\tan\left(\frac{1}{4}h\xi\right) - \cot\left(\frac{1}{4}h\xi\right)\right)\right)$$

$$G_6(\xi) = \frac{1}{2q}\left(-p + \frac{\sqrt{(X^2 - Y^2)\left(h^2\right)} - X\, h\cos(h\xi)}{X\sin(h\xi) + Y}\right)$$

$$G_7(\xi) = \frac{1}{2q}\left(-p + \frac{\sqrt{(X^2 - Y^2)\left(h^2\right)} + X\, h\cos(h\xi)}{X\sin(h\xi) + Y}\right) \quad , \qquad (10)$$

with $h = \sqrt{4qr - p^2}$, where X and Y are two non-zero real constants and satisfies the condition $X^2 - Y^2 > 0$.

$$G_8(\xi) = \frac{-2r\cos\left(\frac{1}{2}h\xi\right)}{h\sin\left(\frac{1}{2}h\xi\right) + p\cos\left(\frac{1}{2}h\xi\right)}$$

$$G_9(\xi) = \frac{2r\sin\left(\frac{1}{2}h\xi\right)}{-p\sin\left(\frac{1}{2}h\xi\right) + h\cos\left(\frac{1}{2}h\xi\right)}$$

$$G_{10}(\xi) = \frac{-2r\cos(h\xi)}{h\sin(h\xi) + p\cos(h\xi) \pm h}$$

$$G_{11}(\xi) = -\frac{2r\sin(h\xi)}{-p\sin(h\xi) + h\cos(h\xi) \pm h}$$

$$G_{12}(\xi) = \frac{4r\sin\left(\frac{1}{4}h\xi\right)\cos\left(\frac{1}{4}h\xi\right)}{-2p\sin\left(\frac{1}{4}h\xi\right)\cos\left(\frac{1}{4}h\xi\right) + 2h\cos^2\left(\frac{1}{4}h\xi\right) - h} \quad . \qquad (11)$$

Family 2: When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $rq \neq 0$), the solutions of equation (9) are:

$$G_{13}(\xi)=\frac{1}{2q}\left(p+\sqrt{-h^2}\tanh\left(\frac{1}{2}\sqrt{-h^2}\xi\right)\right)$$

$$G_{14}(\xi)=-\frac{1}{2q}\left(p+\sqrt{-h^2}\coth\left(\frac{1}{2}\sqrt{-h^2}\xi\right)\right)$$

$$G_{15}(\xi)=-\frac{1}{2q}\left(p+\sqrt{-h^2}\left(\tanh\left(\sqrt{-h^2}\xi\right)\pm i\sec h\left(\sqrt{-h^2}\xi\right)\right)\right)$$

$$G_{16}(\xi)=-\frac{1}{2q}\left(p+\sqrt{-h^2}\left(\coth\left(\sqrt{-h^2}\xi\right)\pm \csc h(\sqrt{-h^2}\xi)\right)\right)$$

$$G_{17}(\xi)=-\frac{1}{4q}\left(2p+\sqrt{-h^2}\left(\tanh\left(\frac{1}{4}\sqrt{-h^2}\xi\right)+\coth\left(\frac{1}{4}\sqrt{-h^2}\xi\right)\right)\right)$$

$$G_{18}(\xi)=\frac{1}{2q}\left(-p+\frac{\sqrt{(X^2+Y^2)\left(-h^2\right)}-X\sqrt{-h^2}\cosh\left(\sqrt{-h^2}\xi\right)}{X\sinh\left(\sqrt{-h^2}\xi\right)+Y}\right)$$

$$G_{19}(\xi)=\frac{1}{2q}\left(-p-\frac{\sqrt{(Y^2-X^2)\left(-h^2\right)}+X\sqrt{-h^2}\sinh\left(\sqrt{-h^2}\xi\right)}{X\cosh\left(\sqrt{-h^2}\xi\right)+Y}\right), \qquad (11)$$

where X and Y are two non-zero real constants and satisfies the condition $Y^2 - X^2 > 0$.

$$G_{20}(\xi)=\frac{2r\cosh\left(\frac{1}{2}\sqrt{-h^2}\xi\right)}{\sqrt{-h^2}\sinh\left(\frac{1}{2}\sqrt{-h^2}\xi\right)-p\cosh\left(\frac{1}{2}\sqrt{-h^2}\xi\right)}$$

$$G_{21}(\xi)=\frac{2r\sinh\left(\frac{1}{2}\sqrt{-h^2}\xi\right)}{\sqrt{-h^2}\cosh\left(\frac{1}{2}\sqrt{-h^2}\xi\right)-p\sinh\left(\frac{1}{2}\sqrt{-h^2}\xi\right)}$$

$$G_{22}(\xi)=\frac{2r\cosh\left(\sqrt{-h^2}\xi\right)}{\sqrt{-h^2}\sinh\left(\sqrt{-h^2}\xi\right)-p\cosh\left(\sqrt{-h^2}\xi\right)\pm i\sqrt{-h^2}}$$

$$G_{23}(\xi)=-\frac{2r\sinh\left(\sqrt{-h^2}\xi\right)}{-p\sinh\left(\sqrt{-h^2}\xi\right)+\sqrt{-h^2}\cosh\left(\sqrt{-h^2}\xi\right)\pm\sqrt{-h^2}}$$

$$G_{24}(\xi)=\frac{2r\sinh\left(\frac{1}{4}\sqrt{-h^2}\xi\right)\cosh\left(\frac{1}{4}\sqrt{-h^2}\xi\right)}{-2p\sinh\left(\frac{1}{4}\sqrt{-h^2}\xi\right)\cosh\left(\frac{1}{4}\sqrt{-h^2}\xi\right)+2\sqrt{-h^2}\left(\cosh^2\left(\frac{1}{4}\sqrt{-h^2}\xi\right)-\frac{1}{2}\right)}\quad . \tag{12}$$

Family 3: When $r=0$ and $pq\neq 0$, the solutions of equation (9) are:

$$\begin{aligned}G_{25}(\xi)&=\frac{-pd}{q\left[d+\cosh(p\xi)-\sinh(p\xi)\right]}\\G_{26}(\xi)&=-\frac{p\left[\cosh(p\xi)+\sinh(p\xi)\right]}{q\left[d+\cosh(p\xi)+\sinh(p\xi)\right]}\quad ,\end{aligned} \tag{13}$$

where d is an arbitrary constant.

Family 4: When $q\neq 0$ and $r=q=0$, the solution of equation (9) is:

$$G_{27}(\xi)=-\frac{1}{q\xi+c_1}\quad , \tag{14}$$

where c_1 is an arbitrary constant.

Step 3: To determine the positive integer $m_{\max}$, substitute equation (8) along with equation (9) into equation (7) and then consider homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (7).

Step 4: Substituting equation (8) along with equation (9) into equation (7) together with the value of $m_{\max}$ obtained in step 3, we obtain polynomials in G^i (i=0,1,2,3 …) and vanishing each coefficient of the polynomial, yields a set of algebraic equations for a_n, p, q and r.

Step 5: Suppose the value of the constants a_n, p, q and r can be determined by solving the set of algebraic equations obtained in step 4. Since the general solutions of equation (8) are known, substituting a_n, p, q and r into equation (7), we obtain analytical exact travelling wave solutions of the nonlinear evolution equation (5).

4 The Sub-equation Method Applied to the NLSE

In this section we apply the sub-equation method to construct the exact analytical solutions for the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity:

$$iu_t + u_{xx} + \alpha |u|^2 u + i\left[\gamma_1 u_{xxx} + \gamma_2 |u|^2 u_x + \gamma_3 \left(|u|^2\right)_x u\right] = 0 \quad . \tag{15}$$

Assume that Eq. (16), has traveling wave solutions in the following form:

$$u(x,t) = F(\xi)\exp\left(i(Kx - \Omega t)\right), \quad \xi = k(x - ct) \quad . \tag{16}$$

By virtue of (16) and (17), we get:

$$\begin{aligned} &i\left(\gamma_1 k^3 F''' - 3\gamma_1 K^2 kF' + \gamma_2 kF^2 F' + 2\gamma_3 kF^2 F' - ckF' + 2KkF'\right) \\ &+\left(\Omega F + k^2 F'' - K^2 F + \alpha F^3 + 3\gamma_1 Kk^2 F'' + \gamma_1 K^3 F - \gamma_2 KF^3\right) = 0 \quad . \end{aligned} \tag{17}$$

Then we have two equations as follows [33,34]:

$$\gamma_1 k^2 F''' + \left(2K - c - 3\gamma_1 K^2\right)F' + \gamma_2 F^2 F' + 2\gamma_3 F^2 F' = 0 \quad , \tag{18}$$

$$k^2\left(1 - 3\gamma_1 K\right)F'' + \left(\Omega - K^2 + \gamma_1 K^3\right)F + \left(\alpha - \gamma_2 K\right)F^3 = 0 \quad . \tag{19}$$

Integrating (19) and taking zero be the integration constant, we have

$$\gamma_1 k^2 F'' + \left(2K - c - 3\gamma_1 K^2\right)F + \left(\frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3\right)F^3 = 0 \quad . \tag{20}$$

By (20) and (21), they have the same solutions. So, we have the following equation:

$$\frac{\gamma_1 k^2}{k^2\left(1 - 3\gamma_1 K\right)} = \frac{2K - c - 3\gamma_1 K^2}{\Omega - K^2 + \gamma_1 K^3} = \frac{\frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3}{\alpha - \gamma_2 K} \quad . \tag{21}$$

From (22), we can obtain:

$$K = \frac{C - \alpha\gamma_1}{3C\gamma_1 - \gamma_1\gamma_2}, \quad \Omega = \frac{\left(1 - 3\gamma_1 K\right)\left(2K - c - 3\gamma_1 K^2\right)}{C} + K^2 - \gamma_1 k^3 \quad , \tag{22}$$

where we assume that:

$$A = \gamma_1 k^2, \quad B = 2K - c - 3\gamma_1 K^2, \quad C = \frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3 \quad . \tag{23}$$

Then the Eq. (20) and Eq. (21) are transformed into the following form:

$$AF'' + BF + CF^3 = 0 \quad . \tag{24}$$

Based on the sub-equation method Refs. [10,11,34], Eq. (25) has the following formal solution:

$$F(\xi) = \sum_{i=0}^{m_{\max}} a_i G^i \quad . \tag{25}$$

where $G(\xi)$ satisfies the generalized Riccati equation (9). To determine the positive integer $m_{\max}$, we substitute the equation (26) along with (9) into (25) and then consider homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (25), obtaining the following ansätz:

$$F(\xi) = a_0 + a_1 G(\xi) \quad , \tag{26}$$

with $m_{\max}$ =1, substituting Eq. (27) into Eq. (25) the left hand sides of this equation is converted into the following polynomial in G^i :

$$\begin{aligned} &\frac{Ba_0}{A} + \frac{Ca_0^3}{A} + pra_1 + \left(\frac{Ba_1}{A} + p^2 a_1 + 2qra_1 + \frac{3Ca_0^2 a_1}{A} \right) G(\xi) \\ &+ \left(3pqa_1 + \frac{3Ca_0 a_1^2}{A} \right) G(\xi)^2 + \left(2q^2 a_1 + \frac{Ca_1^3}{A} \right) G(\xi)^3 = 0 \quad . \end{aligned} \tag{27}$$

Setting each coefficient of this polynomial to zero, we obtain a set of simultaneous algebraic equations for a_0, a_1, p, q and r as follows:

$$\begin{aligned} 0 &= \frac{Ba_0}{A} + \frac{Ca_0^3}{A} + pra_1 \\ 0 &= \frac{Ba_1}{A} + p^2 a_1 + 2qra_1 + \frac{3Ca_0^2 a_1}{A} \\ 0 &= 3pqa_1 + \frac{3Ca_0 a_1^2}{A} \\ 0 &= 2q^2 a_1 + \frac{Ca_1^3}{A} \quad . \end{aligned} \tag{28}$$

Solving the set of algebraic equations by using symbolic computation software, such as *Mathematica*, we obtain:

$$
\begin{aligned}
a_0 &= \frac{\sqrt{-B - Ap^2 - 2Aqr}}{\sqrt{3}\sqrt{C}} \\
a_1 &= -\frac{\sqrt{-B - Ap^2 - 2Aqr}\left(2B - Ap^2 - 2Aqr\right)}{3\sqrt{3CA}\,pr} \\
p &= -\frac{\sqrt{-B + Aqr}}{\sqrt{A}} \\
q &= -\frac{B}{Ar} \\
\sqrt{4qr - p^2} &= \sqrt{\frac{-2B}{A}} \quad ,
\end{aligned}
\tag{29}
$$

where r is an arbitrary constant. Therefore, we find the following solution of (27):

$$
F(\xi) = \frac{\sqrt{B}}{\sqrt{\mathrm{C}}} + \frac{\sqrt{2}B^{3/2}G(\xi)}{\sqrt{A}\sqrt{-B}\sqrt{\mathrm{C}}r} \quad , \tag{30}
$$

where $G(\xi)$ satisfies the generalized Riccati equation Eq. (9).

Now on the basis of the solutions of the Riccati equation (9), we obtain some new solutions of the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity (16).

When p^2 - 4$qr > 0 \Rightarrow \dfrac{B}{A} > 0$ and $pq \neq 0$ (or $rq \neq 0$), the solutions (12), (13) and the equation (17) lead us to the hyperbolic travelling form solutions to the NLSE (16) given by the equation (31).

Hence we get the exact solution of Eq. (16), for the case of the Riccati solutions $G_{13}(\xi)$ and $G_{16}(\xi)$, is given by:

$$
\left|u_1(x,t)\right| = \left|\sqrt{-\frac{B}{C}}\tanh\left(\sqrt{\frac{B}{2A}}k(x+ct)\right)\right| \quad , \tag{31}
$$

see Figure 1.

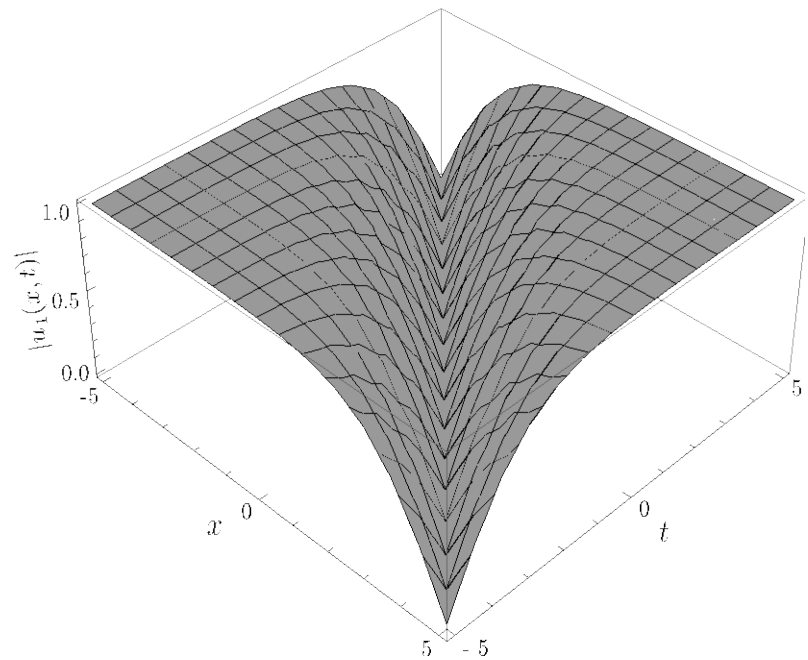

Figure 1: The graphic of solution (32), taking $k=1$, $c=1$, $A=2$, $B=1$ and $C=-1$.

with $\frac{A}{C}<0$. Here $|u|$ is the norm of u. The exact solution of Eq. (16), for the case of the Riccati solutions $G_{14}(\xi)$ and $G_{17}(\xi)$, is given by:

$$|u_2(x,t)|=\left|\sqrt{-\frac{B}{C}}\coth\left(\sqrt{\frac{B}{2A}}k(x+ct)\right)\right| \quad , \tag{32}$$

see Figure 2.

with $\frac{A}{C}<0$, also the exact solution of Eq. (16) for the case of the Riccati solution $G_{15}(\xi)$ is given:

$$|u_3(x,t)|=\left|\sqrt{-\frac{B}{C}}\left(\tanh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)\pm i\operatorname{sech}\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)\right)\right| \quad , \tag{33}$$

with $\frac{A}{C}<0$, and the exact solutions of Eq. (16), for the case of the Riccati solutions $G_{18}(\xi)$ - $G_{19}(\xi)$, are given by:

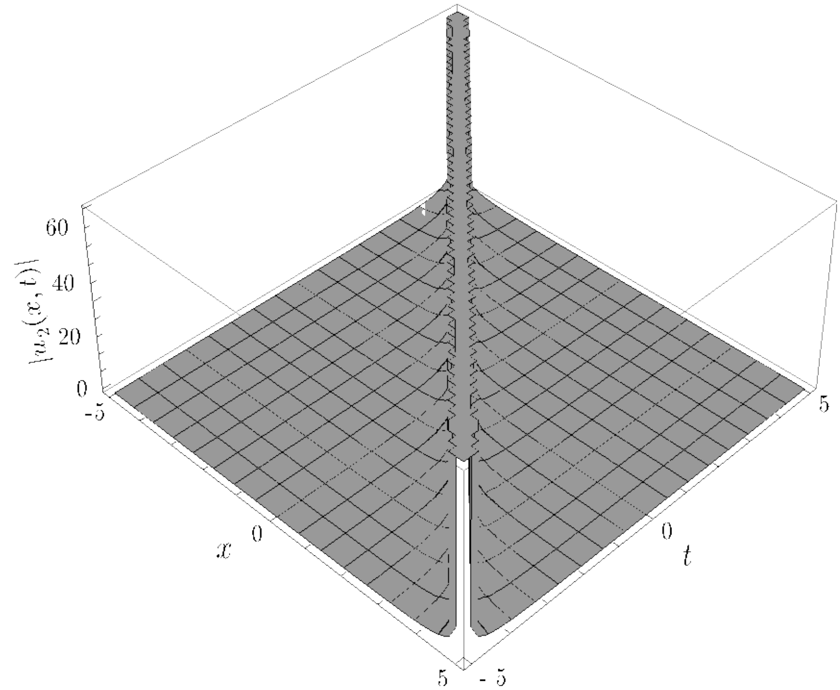

Figure 2: The graphic of solution (33), taking $k=1$, $c=1$, $A=2$, $B=1$ and $C=-1$.

$$\left|u_4(x,t)\right| = \left|\sqrt{-\frac{B}{C}}\left(\frac{\sqrt{X^2+Y^2} \pm X\cosh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}{Y \pm X\sinh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}\right)\right| , \tag{34}$$

$$\left|u_5(x,t)\right| = \left|\sqrt{-\frac{B}{C}}\left(\frac{Y \pm X\sinh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}{\sqrt{X^2+Y^2} \pm X\cosh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}\right)\right| , \tag{35}$$

$$\left|u_6(x,t)\right| = \left|\sqrt{-\frac{B}{C}}\left(\frac{\sqrt{Y^2-X^2} \pm X\sinh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}{Y \pm X\cosh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}\right)\right| , \tag{36}$$

and

$$\left|u_7(x,t)\right| = \left|\sqrt{-\frac{B}{C}}\left(\frac{Y \pm X\cosh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}{\sqrt{Y^2-X^2} \pm X\sinh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)}\right)\right| , \tag{37}$$

with $\frac{A}{C} < 0$, where X and Y are two non-zero real constants which satisfy the condition $Y^2 - X^2 > 0$, see Figures 3, 4, 5 and Figure 6.

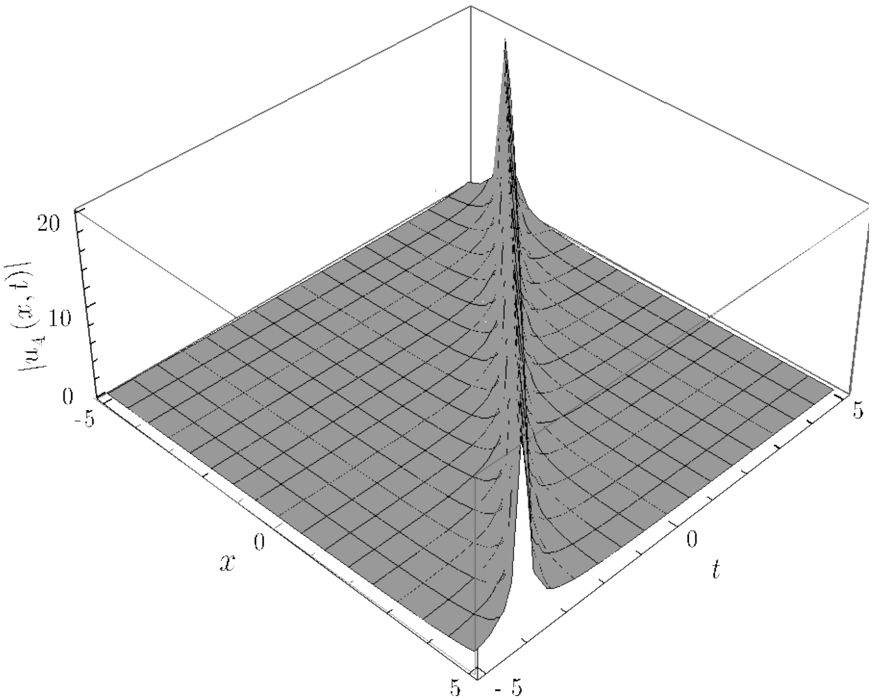

Figure 3: The graphic of solution (35), taking $k=1$, $c=1$, $A=2$, $B=1$, $C=-1$, $X=1$ and $Y=\sqrt{2}$.

For the case of the Riccati solutions $G_{20}(\xi)$ – $G_{24}(\xi)$, the exact solutions of Eq. (16) are given by:

$$|u_8(x,t)| = \left| \sqrt{-\frac{B}{C}} \left(\frac{1 \pm i \tanh\left(\sqrt{\frac{B}{2A}} k(x+ct)\right)}{1 \mp i \tanh\left(\sqrt{\frac{B}{2A}} k(x+ct)\right)} \right) \right| , \tag{38}$$

with $\frac{A}{C} < 0$.

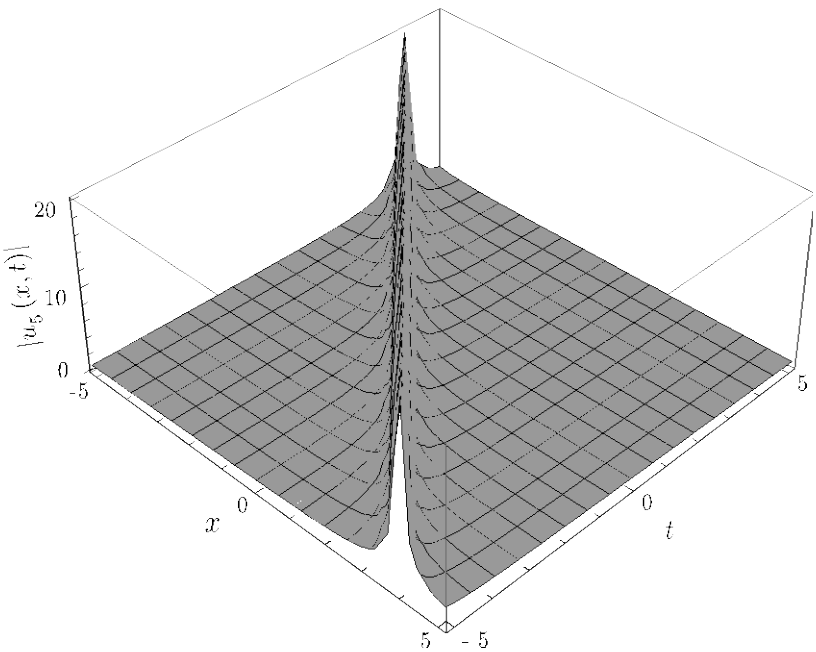

Figure 4: The graphic of solution (36), taking $k=1$, $c=1$, $A=2$, $B=1$, $C=-1$, $X=1$ and $Y=\sqrt{2}$.

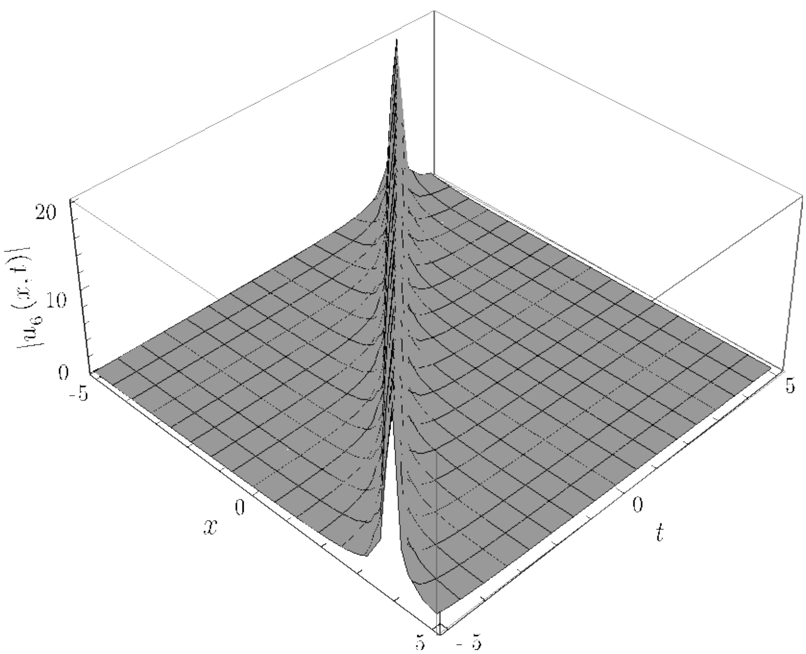

Figure 5: The graphic of solution (37), taking $k=1$, $c=1$, $A=2$, $B=1$, $C=-1$, $X=1$ and $Y=\sqrt{2}$.

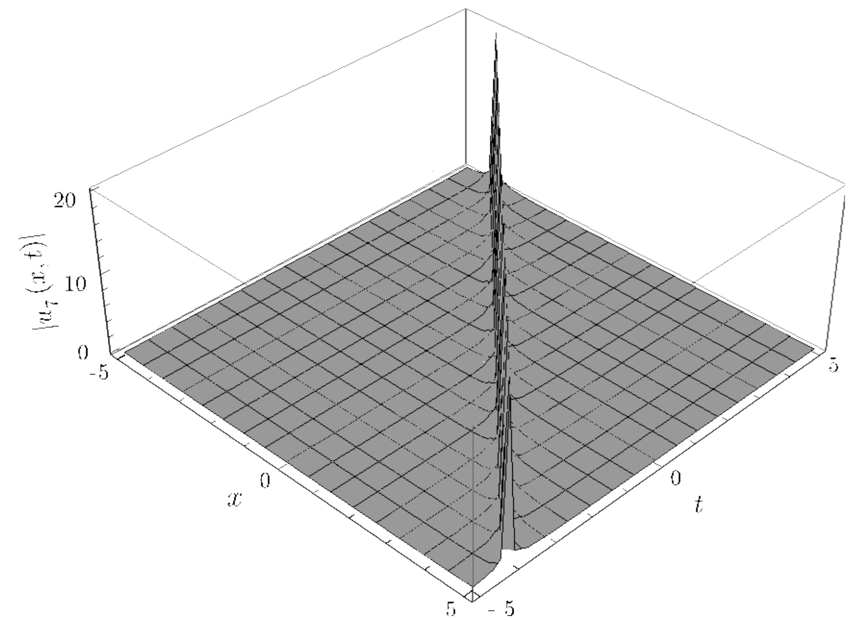

Figure 6: The graphic of solution (38), taking $k=1$, $c=1$, $A=2$, $B=1$, $C=-1$, $X=1$ and $Y=\sqrt{2}$.

When $r=0 \Rightarrow \frac{A}{C}<0$ and $pq\neq 0$, the solutions (14) and the equation (17) lead us to the hyperbolic travelling form solution to the NLSE (16) given by:

$$|u_9(x,t)|=\left|\sqrt{-\frac{B}{C}}\left(\frac{1\pm d\left(\cosh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)+\sinh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)\right)}{1\mp d\left(\cosh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)+\sinh\left(\sqrt{\frac{2B}{A}}k(x+ct)\right)\right)}\right)\right| , \quad (39)$$

with $\frac{A}{C}<0$, where d is an arbitrary constant, see Figure 7.

When $q\neq 0$ and $r=p=0 \Rightarrow \frac{B}{A}=0$, the solution (15) and the equation (17) lead us to the travelling form solutions to the NLSE (16) given by:

$$|u_{10}(x,t)|=\left|\sqrt{\frac{-2A}{C}}\frac{1}{k(x+ct)+c_1}\right| , \quad (40)$$

with $\frac{A}{C}<0$, where c_1 is an arbitrary constant, see Figure 8.

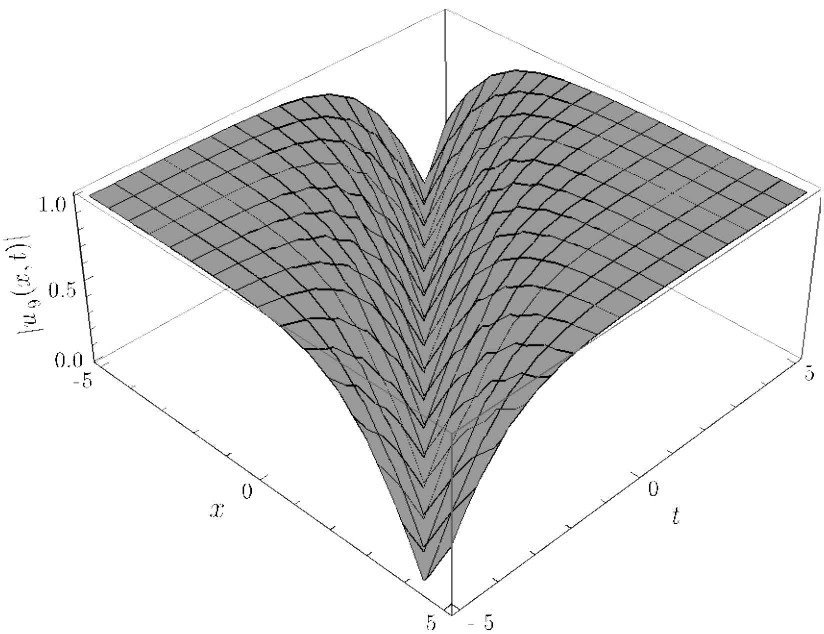

Figure 7: The graphic of solution (40), taking $k=1$, $c=1$, $A=2$, $B=1$, $C=-1$ and $d=2$.

If we compared our solutions Eqs. (32)-(41) with those previously obtained in the literature (see, for example Zhang, et. al. [34]), some important differences can be observed for the new analytical exact solutions and we need to be sure that the solutions (32)-(41) are correct. For the case $\frac{A}{C}<0$ and $\frac{B}{A}>0$, the solutions of the present work can be compared with the results obtained with the aid of symbolic mathematical software, like *Mathematica*. The results obtained with *Mathematica* show that the solutions (32)-(41) are correct. The *Mathematica* code is given by:

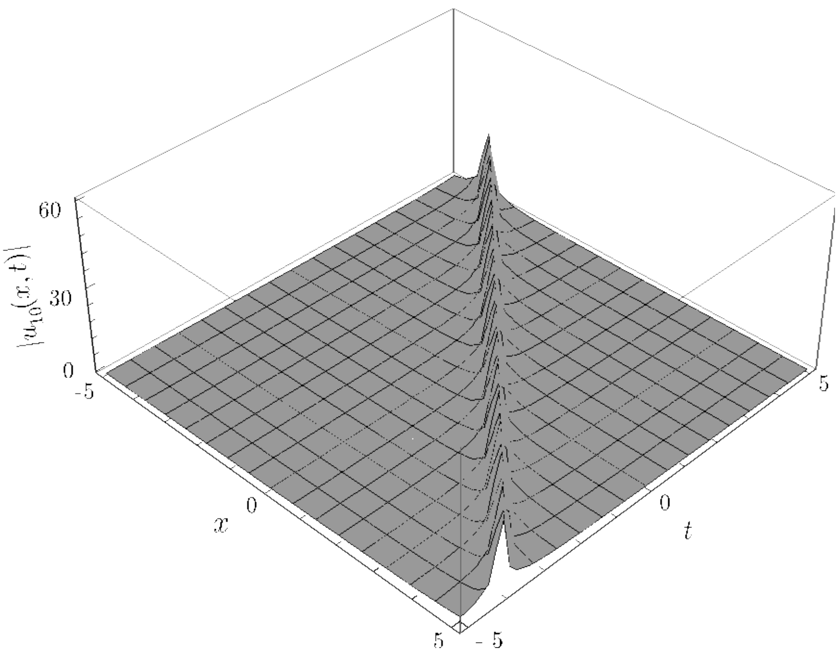

Figure 8: The graphic of solution (41), taking $k=1$, $c=1$, $A=2$, $C=-1$ and $c_1=-1$.

```
In [1]:=Subscript[F,1][y_]:=Sqrt[-B/C]Tanh[Sqrt[B/(2A)]y];

In[2]:=FullSimplify[D[Subscript[F,1][y],{y, 2}]
        + (B/A)Subscript[F,1][y]
        +(C/A)Subscript[F,1][y]^3]

Out[2]=0

In[3]:= Subscript[F,4][y]:=
        (Sqrt[-(B/CC)](Sqrt[X^2+Y^2]+XCosh[Sqrt[2B/A]y]))
        /(Y +   X Sinh[Sqrt[2]Sqrt[B/A] y])
```

```
In[4]:= FullSimplify[D[Subscript[F,4][y], {y, 2}]
            + (B/A)Subscript[F,4][y]
            +(C/A)Subscript[F,4][y]^3]

Out[4]=0

In[5]:= Subscript[F,5][y]:=
      (Sqrt[-(B/CC)] (Y+X Sinh[Sqrt[2] Sqrt[B/A] y]))
             /(Sqrt[X^2 + Y^2] + X Cosh[Sqrt[2] Sqrt[B/A] y]);

In[6]:= FullSimplify[D[Subscript[F,5][y], {y, 2}]
            + (B/A)Subscript[F,5][y]
            +(C/A)Subscript[F,5][y]^3]
Out[6]=0

In[7]:=Subscript[F, 7][y_] :=
      (Sqrt[-(B/CC)] (Y + X Cosh[Sqrt[2] Sqrt[B/A] y]))
       /(Sqrt[-X^2 + Y^2] - X Sinh[Sqrt[2] Sqrt[B/A] y]);

In[8]:= FullSimplify[D[Subscript[F, 7][y], {y, 2}]
            + (B/A)Subscript[F, 7][y]
            +(C/A)Subscript[F, 7][y]^3]

Out[8]=0.

In[9]:= Subscript[F, 9][y_] := -(Sqrt[-B]/Sqrt[CC])
((1 - d (1Cosh[(Sqrt[2] Sqrt[B] y)/Sqrt[A]] +
          Sinh[(Sqrt[2] Sqrt[B] y)/Sqrt[A]]))/
(1 + d(Cosh[(Sqrt[2] Sqrt[B] y)/Sqrt[A]] +
```

```
            Sinh[(Sqrt[2] Sqrt[B] y)/Sqrt[A]])));
In[10]:= FullSimplify[D[Subscript[F, 9][y], {y, 2}]
             + (B/A)Subscript[F, 9][y]
             +(C/A)Subscript[F, 9][y]^3]

Out[10]=0.
```

Where “Subscript[F, i][y_]” stands for:

$$F_i(\xi) = \frac{u_i(\xi)}{\exp\left(i\left(Kx - \Omega t\right)\right)} \quad , \tag{41}$$

and therefore the functions $F_i(\xi)$ (see equation (31)), associated with the solutions (32)-(41), satisfy the required equation (25).

5 Conclusions

We have presented a direct method to obtain new explicit analytical exact solutions for NLSE with Kerr law nonlinearity. These solutions are obtained using the sub-equation method and the generalized solutions to the Riccati equation. These solutions have not been reported in the literature by applying directly the sub-equation method. This method is one of the most effective approaches to obtain explicit and exact solutions of nonlinear equations. Various methods have some merits and deficiencies with respect to the problem considered and there is no single best method to investigate the exact analytical traveling wave solutions of nonlinear equations. The presented method provides a comprehensive approach to construct many new solutions of NPDEs. We have confirmed the accuracy of the NLSE solutions here presented with the aid of a symbolic code in *Mathematica*, for the special case when $\frac{A}{C} < 0$ and $\frac{B}{A} > 0$, in the Eq. (25).

Acknowledgements. We gratefully acknowledge to the Universidad Autónoma de la Ciudad de México for supporting this research work.

References

[1] A.K. Ray, J. K. Bhattacharjee, Standing and traveling waves in the shallow water circular hydraulic jump, *Phys. Lett. A*, **371**(3), (2007), 241-248.

[2] P.M. Jordan and A. Puri, A note on traveling wave solutions for a class of nonlinear viscoelastic media, *Phys. Lett. A*, **335**(2-3), (2005), 150-156.

[3] Z.Y. Yan, Generalized method and its application in the higher-order nonlinear Schrödinger equation in nonlinear optical fibres, *Chaos Solitons & Fractals*, **16**(5), (2003), 759-766.

[4] M.L. Wang, Y. B. Zhou and Z. B. Li, Applications of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, *Phys. Lett. A*, **216**(1-5), (1996), 67-75.

[5] E.J. Parkes and B. R. Duffy, An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations, *Comput. Phys. Commun.*, **98**(3), (1996), 288-300.

[6] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, *Phys. Lett. A*, **277**(4-5), (2000), 212-218.

[7] E.G. Fan and H. Q. Zhang, A note on the homogeneous balance method, *Phys. Lett. A*, **246(**5), (1998), 403-406.

[8] S. Liu, Z. Fu, S. Liu and Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, *Phys. Lett. A*, **289**(1-2), (2001), 69-74.

[9] Z. Fu, S. Liu, S. Liu and Q. Zhao, The JEFE method and periodic solutions of two kinds of nonlinear wave equations, *Commun. Nonlinear Sci. Numer.Simul*, **8(**3), (2003), 67-75.

[10] Sirendaoreji, New exact travelling wave solutions for the Kawahara and modified Kawahara equations, *Chaos Solitons & Fractals*, **19**(1), (2004), 147-150.

[11] Z. Zhang, New exact traveling wave solutions for the nonlinear Klein-Gordon equation, *Turk. J. Phys.*, **32**(5), 2008, 235-340.

[12] Z. Y. Zhang, Z. H. Liu, X. J. Miao and Y. Z. Chen, New exact solutions to the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity, *Appl. Math. Comput.*, **216**(10), (2010), 3064-3072.

[13] Y. Shang, Y. Huang and W. Yuan, The extended hyperbolic functions method and new exact solutions to the Zakharov equations, *Appl. Math. Comput.*, **200**(1), (2008), 110-122.

[14] Q. Wang, Y. Chen and H. Zhang, A new Riccati equation rational expansión method and its application to (2+1)-dimensional Burgers equation, *Chaos Solitons & Fract.*, **25**(5), (2005), 1019-1028.

[15] C. Yan, A simple transformation for nonlinear waves, *Phys. Lett. A*, **224**(1-2), (2005), 77-84.

[16] M. L. Wang and Y. B. Zhou, The periodic wave solutions for the Klein-Gordon-Schrödinger equations, *Phys. Lett. A*, **318**(1-2), (2003), 84-92.

[17] W. X. Ma and J. H. Lee, A transformed rational function method and exact solutions to the 3 + 1 dimensional Jimbo-Miwa equation, *Chaos Solitons & Fract.*, **42**(3), (2009), 1356-1363.

[18] W. X. Ma, T. Huang and Y. Zhang, A multiple exp-function method for nonlinear differential equations and its application, *Phys. Scr.*, **82**(6), (2010), 065003.

[19] M. Wang, X. Li and J. Zhang, The $\left(\frac{G'}{G}\right)$-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A*, **372**(4), (2008), 417-423.

[20] M. Wang, J. Zhang and X. Li, Application of the $\left(\frac{G'}{G}\right)$-expansion to travelling wave solutions of the Broer-Kaup and the approximate long water wave equations, *Appl. Math. Comput.*, **206**(1), (2008), 321-326.

[21] L. Ling-Xiao and W. Ming-Liang, The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions for a higher-order nonlinear Schrödinger equation, *Appl. Math. Comput.*, **208**(2), (2009), 440-445.

[22] A. Bekir, Application of the $\left(\frac{G'}{G}\right)$-expansion method for nonlinear evolution equations, *Phys. Lett. A*, **372**(19), (2008), 3400-3406.

[23] S. Zhang and J. L. Tong, A generalized $\left(\frac{G'}{G}\right)$-expansion method for the mKdV equation with variable coefficients, *Phys. Lett. A*, **372**(13), (2008), 2254-2257.

[24] S. Zhang, W. Wang and J. L. Tong, A generalized $\left(\frac{G'}{G}\right)$-expansion method and its application to the (2 + 1)-dimensional Broer-Kaup equations, *Appl. Math. Comput.*, **209**(2), (2009), 399-404.

[25] E.M.E. Zayed, The $\left(\frac{G'}{G}\right)$-expansion method and its applications to some nonlinear evolution equations in the mathematical physics, *J. Appl. Math. Comput.*, **30**(1-2), (2009), 89-103.

[26] I. Aslan and T. Özis, Analytic study on two nonlinear evolution equations by using the $\left(\frac{G'}{G}\right)$-expansion method, *Appl. Math. Comput.*, **209**(2), (2009), 425-429.

[27] I. Aslan and T. Özis, On the validity and reliability of the $\left(\frac{G'}{G}\right)$-expansion method by using higher-order nonlinear equations, *Appl. Math. Comput.*, **211**(2), (2009), 531-536.

[28] B.Q. Li and Y.L. Ma, $\left(\frac{G'}{G}\right)$-expansion method and new exact solutions for (2 + 1)-dimensional asymmetrical Nizhnik-Novikov-Veselov system, *Acta Phys Sin*, **58**(7), (2009), 4373-4378.

[29] Y.L. Ma, B.Q. Li and J.Z. Sun, New application of $\left(\frac{G'}{G}\right)$-expansion method to high-dimensional nonlinear physical equations, *Acta Phys Sin*, **58**(11), (2009), 7402-7409.

[30] G. Ebadi and A. Biswas, Application of the $\left(\frac{G'}{G}\right)$-expansion method for nonlinear diffusion equations with nonlinear source, *J. Franklin Inst.*, **347**(7), (2010), 1391-1398.

[31] H. Zhang, A Note on some sub-equation methods and new types of exact travelling wave solutions for two nonlinear partial differential equations, *Acta Applicandae Mathematicae,* **106**(2), (2009), 241-249.

[32] S. Zhu, The generalized Riccati equation mapping method in nonlinear evolution equation: application to (2+1)-dimensional Boiti-Leon-Pempinelle equation, *Chaos Soliton & Fract.*, **37**(5), (2008), 1335-1342.

[33] Z.Y. Zhang, Y.X. Li, Z.H. Liu and X.J. Miao, New exact solutions to the perturbed nonlinear Schrödinger’s equation with Kerr law nonlinearity via modified trigonometric function series method, *Commun. Nonlinear Sci. Numer. Simulat.*, **16**(8), (2011), 3097-3106.

[34] Z.Y. Zhang, X.Y. Gan, D.M. Yu, Y.H. Zhang and X.P. Li, A note on exact traveling wave solutions of the perturbed nonlinear Schrödinger’s equation with Kerr law nonlinearity, *Commun. Theor. Phys.*, **57**(5), (2012), 764-770.

[35] A. Biswas and S. Konar, Introduction to non-Kerr law optical solitons, CRC Press, Florida, 2007.

[36] A. Biswas, Perturbation of solitons with non-Kerr law nonlinearity, *Chaos Solitons & Fract.*, **13**(4), (2002), 815-823.

[37] A. Biswas and D. Milovic, Bright and dark solitons of the generalized nonlinear Schrödinger's equation, *Commun. Nonlinear Sci. Numer. Simul.*, **15**(6), (2010), 1473-1484.

[38] A. Biswas and K. Porsezian, Soliton perturbation theory for the modified nonlinear Schrödinger's equation, *Commun. Nonlinear Sci. Numer. Simul.*, **12**(6), (2007), 886-903.

[39] A. Biswas, H. Ren and S. Konar, Stochastic perturbation of non-Kerr law optical solitons, *Optik-Int. J. Light Electron Opt.*, **118**(10), (2007), 471-480.

[40] R. Kohl, A. Biswas, D. Milovic and E. Zerrad, Optical soliton perturbation in a non-Kerr law media, *Opt. Laser Technol.*, **40**(4), (2008), 647-662.

[41] A. Biswas, Quasi-stationary non-Kerr law optical solitons, *Opt. Fiber Technol.*, **9**(4), (2003), 224-259.

[42] A. Biswas and S. Konar, Quasi-particle theory of optical soliton interaction, *Commun. Nonlinear Sci. Numer. Simul.*, **12**(7), (2007), 1202-1228.

[43] A. Biswas, Perturbation of chiral solitons, *Nucl. Phys. B*, **806**(3), (2009), 457-461.

[44] C. Khalique and A. Biswas, Optical solitons with power law nonlinearity using Lie group analysis, *Phys. Lett. A*, **373**(23-24), (2009), 2047-2049.

[45] A. Biswas, 1-soliton solution of (1 + 2)-dimensional nonlinear Schrödinger's equation in dual-power law media, *Phys. Lett. A*, **372**(38), (2008), 5941-5943.

[46] M. Saha, A. K. Sarma and A. Biswas, Dark optical solitons in power law media with time-dependent coefficients, *Phys. Lett. A*, **373**(48), (2009), 4438-4441.

[47] A. Biswas, Theory of non-Kerr law solitons, *Appl. Math. Comput.*, **153**(2), (2004), 369-385.

[48] A. Biswas, Adiabatic dynamics of non-Kerr law solitons, *Appl. Math. Comput.*, **151**(1), (2004), 41-52.

[49] A. Biswas, Soliton-soliton interaction with dual-power law nonlinearity, *Appl. Math. Comput.*, **198**(2), (2008), 605-612.

[50] A. Biswas, Topological 1-soliton solution of the nonlinear Schrödinger's with Kerr law nonlinearity in (1 + 2) dimensions, *Commun. Nonlinear Sci. Numer. Simul.*, **14**(7), (2009), 2845-2847.

[51] A. Biswas, 1-Soliton solution of (1 + 2) dimensional nonlinear Schrödinger's equation in power law media, *Commun. Nonlinear Sci. Numer. Simul.*, **14**(5), (2009), 1830-1833.

[52] A. Biswas, Perturbation of solitons due to power law nonlinearity, *Chaos Solitons & Fract.*, **12**(3), (2001), 579-588.

[53] A. Biswas, Quasi-stationary optical solitons with parabolic law nonlinearity, *Opt. Commun.*, **216**(4-6), (2003), 427-437.

[54] A. Biswas, Quasi-stationary optical solitons with dual-power law nonlinearity, *Opt. Commun.*, **235**(1-3), (2004), 183-194.

[55] A. Biswas, Solitons in magneto-optic waveguides, *Appl. Math. Comput.*, **153**(2), (2004), 387-393.

[56] T. Özis and A. Yıldırım, Reliable analysis for obtaining exact soliton solutions of nonlinear Schrödinger (NLS) equation, *Chaos Solitons & Fract.*, **38**(1), (2008), 209-212.

[57] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov. Phys. JETP*, **34**, (1972), 62-69.

[58] V. E. Zakharov and A. B. Shabat, Interaction between solitons in a stable medium, *Sov. Phys. JETP*, **37**, (1973), 823-828.