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Continuity of the quenching time in a nonlinear parabolic equation with a potential

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Abstract

In this paper, we consider the following initial-boundary value problem

$$\begin{cases} (u^m)_t = \Delta u - a(x)u^{-q} \text{ in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain in IR^N with smooth boundary $\partial\Omega$ q>0, m>1, Δ is the Laplacian, ν is the exterior normal unit vector on $\partial\Omega$, $a \in C^0(\overline{\Omega})$, a(x)>0, $x \in \overline{\Omega}$, $u_0 \in C^1(\overline{\Omega})$, $u_0(x) > 0$, $x \in \overline{\Omega}$. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of the initial datum u_0(x) and the potential a. Finally, we give some numerical results to illustrate our analysis.

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1. Introduction

Let Ω be a bounded domain in IR^N with smooth boundary $\partial\Omega$ Consider the following initial boundary value problem

$$(u^m)_t = \Delta u - a(x)u^{-q} \ in \ \Omega \times (0,T), \quad (1.1)$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T), \quad (1.2)$$
$$u(x,0) = u_0(x) \ in \ \overline{\Omega}, \quad (1.3)$$

which model flow and heat transfer in porous media. In particular the above problem has a lot of applications in the theory of heat transfer in biological tissues. The initial datum $u_0(x)$ is continious, q>0, m>1, Δ is the Laplacian, ν is the exterior normal unit vector on $\partial\Omega$. The potential $a \in C^0(\overline{\Omega})$, a(x)>0, $x \in \overline{\Omega}$, and the initial datum $u_0 \in C^1(\overline{\Omega})$, $u_0(x) > 0$, $x \in \overline{\Omega}$ and u_0 satisfies the compatibility conditions $\frac{\partial u_0}{\partial \nu} = 0$, $x \in \overline{\Omega}$. Here (0,T) is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t \to T} u_{min}(t) = 0,$$

where $u_{min}(t) = \min_{x \in \overline{\Omega}} u(x,t)$. In this last case, we say that the solution u quenches in a finite time, and the time *T* is called the quenching time of the solution *u*. Thus, we have *u(x,t)>0* in $\overline{\Omega} \times [0,T)$.

The equation (1.1) may be rewritten as
$u_t = \frac{1}{m} u^{1-m} \Delta u - \frac{a(x)}{m} u^{-q+1-m}$ in$\Omega \times (0,T)$. Setting $c = \frac{1}{m}$, $\alpha = -1 + m$ and *p=q-1+m*, the problem (1.1)—(1.3) becomes

$$u_t = cu^{-\alpha} \Delta u - ca(x)u^{-p} \ in \ \Omega \times (0,T), \quad (1.4)$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T), \quad (1.5)$$
$$u(x,0) = u_0(x) > 0 \ in \ \overline{\Omega}. \quad (1.6)$$

Solutions of nonlinear parabolic equations which quench in a finite time have been the subject of investigation of many authors (see [2]— [6], [8]— [10], [15], [18], [20], and the references cited therein). By standard methods, it is easy to prove the local in time existence and uniqueness of a classical solution (see, for instance [3], [19]). In this paper, we are interested in the continuity of the quenching time as a function of the initial datum u_0 and the potential *a*. More precisely, we consider the following initial-boundary value problem:

$$v_t = cv^{-\alpha}\Delta v - ca(x)v^{-p} \text{ in } \Omega \times \left(0, T_h^k\right), \tag{1.7}$$

$$\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times \left(0, T_h^k\right), \tag{1.8}$$

$$u(x,0) = u_0^h(x) > 0 \text{ in } \overline{\Omega}, \tag{1.9}$$

where $u_0^h(x) \in C^1(\overline{\Omega})$, $u_0^h(x) \geq u_0(x)$, $x \in \overline{\Omega}$, $\frac{\partial u_0^h}{\partial \nu} = 0$, $x \in \partial\Omega$, $\lim_{h\to 0} u_0^h = u_0$, $a_k \in C^0(\overline{\Omega})$, $0 < a_k(x) \leq a(x)$, $x \in \overline{\Omega}$, $\lim_{k\to 0} a_k = a$.

Here $(0, T_h^k)$ is the maximal time interval on which the solution v of (1.7)—(1.9) exists.
When T_h^k is finite, then we say that the solution v of (1.7)—(1.9) quenches in a finite time, and the time T_h^k is called the quenching time of the solution v. Due to the fact that $0 < a_k(x) \leq a(x)$ and $u_0^h(x) \geq u_0(x)$, $x \in \overline{\Omega}$, the problem (1.7)—(1.9) becomes

$$\frac{dv}{dt} \geq cv^{-\alpha}\Delta v - ca(x)v^{-p} \text{ in } \Omega \times \left(0, T_h^k\right),$$

$$\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times \left(0, T_h^k\right),$$

$$u(x,0) \geq u(x,0) > 0 \text{ in } \overline{\Omega}.$$

It follows from the maximum principle that $v \geq u$ as long as all of them are defined. We deduce that $T_h^k \geq T$. In the present paper, we prove that if *h* and *k* are small enough, then the solution *v* of (1.7)—(1.9) quenches in a finite time, and its quenching time T_h^k goes to *T* as *h* and *k* go to zero, where *T* is the quenching time of the solution *u* of (1.1)—(1.3). Recently, in [5], Boni and N'gohisse have done an analogous study considering the problem (1.1)—(1.3) in the case where *m=1, a(x)=1, q>0*. Let us notice that in [5], the authors have only studied the continuity of the quenching time as a function of the initial datum. Similar results have been obtained in [1], [7], [12], [13] where the authors have considered the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). The rest of the paper is organized as follows. In the next section, under some assumptions, we show that the solution *v* of (1.7)—(1.9) quenches in a finite time and estimate its quenching time. In the third section, we prove the continuity of the quenching time, and finally, in the last section, we give some numerical results to illustrate our analysis.

2. Quenching Time

In this section, under some assumptions, we show that the solution *v* of (1.7)—(1.9) quenches in a finite time and estimate its quenching time.
We borrow an idea of Friedman and McLeod in [11] and prove the following result.

Theorem 2.1. Let $p > \alpha$. Suppose that there exist constants $A \in (0,1], \beta \in (0, p-\alpha]$ such that the initial datum at (1.9) satisfies

$$c\left(u_0^h(x)\right)^{-\alpha} \Delta u_0^h(x) - c a_k(x)\left(u_0^h(x)\right)^{-p} \leq -A\left(u_0^h(x)\right)^{-\beta} \ \ in \ \ \Omega. \tag{2.1}$$

Then, the solution v of (1.7)— (1.9) quenches in a finite time T_h^k which obeys the following estimate

$$T_h^k \leq \frac{\left(u_{0min}^h\right)^{\beta+1}}{A(\beta+1)},$$

where $u_{0min}^h = \min_{x\in\overline{\Omega}} u_0^h(x)$.

PROOF. Since $\left(0, T_h^k\right)$ is the maximal time interval of existence of the solution *v*, our aim is to show that T_h^k is finite and satisfies the above inequality. Introduce the function *J(x,t)* defined as follows

$$J(x,t) = v_t(x,t) + Av^{-\beta}(x,t) \quad \text{ in } \overline{\Omega} \times \left[0, T_h^k\right).$$

A direct calculation reveals that

$$\begin{aligned} J_t - cv^{-\alpha}\Delta J = (v_t - cv^{-\alpha}\Delta v)_t - A\beta v^{-\beta-1}v_t - Acv^{-\alpha}\Delta v^{-\beta} \\ - c\alpha v^{-\alpha-1}v_t\Delta v \ \text{ in } \Omega \times \left(0, T_h^k\right). \end{aligned} \tag{2.2}$$

A straightforward computation shows that

$$\Delta v^{-\beta} = \beta(\beta+1)v^{-\beta-2}|\nabla v|^2 - \beta v^{-\beta-1}\Delta v \ \ \text{ in } \Omega \times \left(0, T_h^k\right).$$

which implies that

$$\Delta v^{-\beta} \geq -\beta v^{-\beta-1}\Delta v \ in \ \ \Omega \times \left(0, T_h^k\right). \tag{2.3}$$

On the other hand, multiply both sides of (1.7) by v_t to obtain

$$v_t^2 = cv^{-\alpha}v_t\Delta v - ca_k(x)v^{-p}v_t \ \text{ in } \ \Omega \times \left[0, T_h^k\right).$$

Since v_t^2 is nonnegative in $\Omega \times \left(0, T_h^k\right)$, we see that

$$cv^{-\alpha-1}v_t\Delta v \geq ca_k(x)v^{-p-1}v_t \ \text{ in } \ \Omega \times \left(0, T_h^k\right).$$

It follows from (2.2)—(2.3) that

$$J_t - cv^{-\alpha}\Delta J \le (v_t - cv^{-\alpha}\Delta v)_t - A\beta v^{-\beta-1}(v_t - cv^{-\alpha}\Delta v - ca_k(x)\alpha v^{-p-1}v_t \ \text{ in } \Omega \times \left(0, T_h^k\right).$$

Taking into account (1.7), we arrive at

$$J_t - cv^{-\alpha}\Delta J \le cpa_k(x)v^{-p-1}v_t + ca_k(x)A\beta v^{-\beta-1}v^{-p} - ca_k(x)\alpha v^{-p-1}v_t \ \text{ in } \Omega \times \left(0, T_h^k\right).$$

Since $\beta \le p - \alpha$, we deduce that

$$J_t - cv^{-\alpha}\Delta J \le ca_k(x)(p-\alpha)v^{-p-1}(v_t + Av^{-\beta}) \ \text{ in } \Omega \times \left(0, T_h^k\right).$$

Taking into account the expression of *J*, we find that

$$J_t - cv^{-\alpha}\Delta J \le ca_k(x)(p-\alpha)v^{-p-1}J \ \text{ in } \Omega \times \left(0, T_h^k\right).$$

According to (1.8), we also see

$$\frac{\partial J}{\partial \nu} = \left(\frac{\partial v}{\partial \nu}\right)_t - A\beta v^{-\beta-1}\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times \left(0, T_h^k\right),$$

and due to (2.1), we discover that

$$J(x,0) = c\left(u_0^h\right)^{-\alpha}\Delta u_0^h - ca_k(x)\left(u_0^h\right)^{-p} + A\left(u_0^h\right)^{-\beta} \le 0 \ \text{ in } \ \overline{\Omega}.$$

Apply the maximum principle to get

$$J(x,t) \le 0 \ \text{ in } \ \overline{\Omega} \times \left(0, T_h^k\right),$$

or equivalently

$$v_t(x,t) + Av^{-\beta}(x,t) \le 0 \ \text{ in } \ \overline{\Omega} \times \left(0, T_h^k\right).$$

This estimate may be rewritten as follows

$$v^{\beta}dv \le -Adt \ \text{ in } \ \overline{\Omega} \times \left(0, T_h^k\right). \tag{2.4}$$

Integrating the above inequality over $(0, T_h^k)$, we find that

$$T_h^k \leq \frac{\left(v(x,0)\right)^{\beta+1}}{A(\beta+1)} \quad \text{for } x \in \bar{\Omega},$$

which implies that

$$T_h^k \leq \frac{\left(u_{0min}^h\right)^{\beta+1}}{A(\beta+1)}.$$

Consequently, we deduce that v quenches at the time T_h^k because of the quantity on the right-hand side of the above inequality is finite. This ends the proof.

Remark 2.1. *Let* $t_0 \in (0, T_h^k)$. Integrating the inequality in (2.4) from t_0 *to* T_h^k *we get*

$$T_h^k - t_0 \leq \frac{\left(v(x,t_0)\right)^{\beta+1}}{A(\beta+1)} \quad for\ x \in \bar{\Omega}.$$

We deduce that

$$T_h^k - t_0 \leq \frac{\left(v_{min}(t_0)\right)^{\beta+1}}{A(\beta+1)}.$$

3. Continuity of the Quenching Time

In this section, under some assumptions, we show that the solution v of (1.7)—(1.9) quenches in a finite time and its quenching time goes to that of the solution u of (1.1)—(1.3) when h and k go to zero.
Firstly, we show that the solution v approaches the solution u in $\bar{\Omega} \times [0, T-\tau]$ with $\tau \in (0, T)$ when h and k tend to zero. This result is stated in the following theorem.

Theorem 3.1. Let u be the solution of (1.1)—(1.3). Suppose that $u \in C^{2,1}(\bar{\Omega} \times [0, T-\tau])$ and $\min_{t\in[0,T-\tau]} u_{min}(t) = \rho > 0$ with $\tau \in (0, T)$.

Assume that the potential at (1.7) and the initial datum at (1.9) satisfy

$$\|a_k - a\|_\infty = o(1) \ \ as\ k \to 0, \tag{3.1}$$

$$\left\|u_0^h - u_0\right\|_\infty = o(1) \ \ as\ h \to 0, \tag{3.2}$$

respectively.

Then, the problem (1.7)—(1.9) admits a unique solution $v \in C^{2,1}\left(\bar{\Omega} \times \left[0, T_h^k\right)\right)$, and the following relation holds:

$$\sup_{t\in[0,T-\tau]} \|v(.,t)-u(.,t)\|_\infty = O(\|a_k-a\|_\infty + \|u_0^h-u_0\|_\infty) \ as\ (h,k)\to(0,0).$$

PROOF. The problem (1.7)—(1.9) has for each *(h, k)* a unique solution $v \in C^{2,1}\left(\bar{\Omega}\times\left[0,T_h^k\right)\right)$.
In the introduction of the paper, we have seen that $T_h^k \geq T$. Let $t(h,k)\leq T-\tau$ the greatest value of *t>0* such that

$$\|v(.,t)-u(.,t)\|_\infty \leq \frac{\rho}{2} \ for \quad t\in\left(0,t(h,k)\right). \tag{3.3}$$

By a direct calculation, we see that

$\|v(.,0)-u(.,0)\|_\infty = \|u_0^h-u_0\|_\infty$, which implies that $\|v(.,0)-u(.,0)\|_\infty$ tends to zero as *h* goes to zero because of (3.2). Due to this fact, we deduce that *t(h,k)>0* for *h* sufficiently small. Invoking the triangle inequality, we obtain

$$v_{min}(t)\geq u_{min}(t)-\|v(.,t)-u(.,t)\|_\infty \text{ for } \quad t\in\left(0,t(h,k)\right),$$

which leads us to

$$v_{min}(t)\geq \rho-\frac{\rho}{2}=\frac{\rho}{2}\ for \quad t\in\left(0,t(h,k)\right).$$

Introduce the function *e(x,t)* defined as follows

$$e(x,t)=v(x,t)-u(x,t) \ \text{ in } \bar{\Omega}\times\left[0,t(h,k)\right).$$

A routine computation reveals that

$$e_t - cv^{-\alpha}\Delta e = (c\alpha\eta^{-\alpha-1}\Delta u - ca_k p\theta^{-p-1})e$$

$$+c(a-a_k)u^{-p} \quad \text{in} \quad \Omega\times\left(0,t(h,k)\right),$$

$$\frac{\partial e}{\partial \nu}=0 \text{ on } \partial\Omega\times\left(0,t(h,k)\right),$$

$$e(x,0)=u_0^h(x)-u_0(x)>0 \text{ in } \bar{\Omega},$$

where θ and η are intermediate values between *u* and *v*.

Let M be such that $c\left(\frac{\rho}{2}\right)^{-p} \leq M$ and $|\Delta u| \leq M$ for $(x,t) \in \Omega \times \big(0, t(h,k)\big)$.
We deduce that
$$e_t - cv^{-\alpha}\Delta e \leq (c\alpha\eta^{-\alpha-1}M - ca_k p\theta^{-p-1})e + (a-a_k)M \text{ in } \Omega \times \big(0,t(h,k)\big),$$

$$\frac{\partial e}{\partial \nu} = 0 \text{ on } \partial\Omega \times \big(0,t(h,k)\big),$$

$$e(x,0) \geq u_0^h(x) - u_0(x) > 0 \text{ in } \overline{\Omega}.$$

Let L be such that $L = c\|a_k\|p\left(\frac{\rho}{2}\right)^{-p-1} + c\alpha M\left(\frac{\rho}{2}\right)^{-p-1}$.
It is not hard to see that
$L \geq c\alpha\eta^{-\alpha-1}M - ca_k p\theta^{-p-1}$ for (x,t) $\Omega \times \big(0,t(h,k)\big)$.
Set
$$z(x,t) = e^{(L+M)t}\left(\|a_k - a\|_\infty + \left\|u_0^h - u_0\right\|_\infty\right) \text{ in } \overline{\Omega} \times [0,T].$$

Thanks to this observation, a straightforward calculation yield
$$z_t - cv^{-\alpha}\Delta z \geq (ca_k p\theta^{-p-1} - c\alpha\eta^{-\alpha-1}\Delta u)z$$

$$+\|a_k - a\|_\infty M \text{ in } \Omega \times \big(0,t(h,k)\big),$$

$$\frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega \times \big(0,t(h,k)\big),$$

$$z(x,0) \geq e(x,0) \qquad \text{in } \overline{\Omega}.$$

It follows from the maximum principle that
$$z(x,t) \geq e(x,t) \quad \text{in } \Omega \times \big(0,t(h,k)\big).$$
In the same way, we also prove that
$$z(x,t) \geq -e(x,t) \quad \text{in } \Omega \times \big(0,t(h,k)\big),$$
which implies that
$$\|e(.,t)\|_\infty \leq e^{(L+M)t}\left(\|a_k - a\|_\infty + \left\|u_0^h - u_0\right\|_\infty\right) \text{ for } t \in \big(0,t(h,k)\big).$$
Let us show that t(h,k)=T-τ. Suppose that t(h,k)<T-τ. From (3.3),

we obtain
$$\frac{\rho}{2} = \left\|v\big(.,t(h,k)\big) - u\big(.,t(h,k)\big)\right\|_\infty \leq e^{(L+M)T}\left(\|a_k - a\|_\infty + \left\|u_0^h - u_0\right\|_\infty\right).$$

Since the term on the right-hand side of the above inequality goes to zero as h and k go to zero, we deduce that $\frac{\rho}{2} \leq 0$, which is impossible. Consequently, t(h,k)=T-τ and the proof is complete.

Now, we are in a position to prove the main result of the paper.

Theorem 3.2. Suppose that the problem (1.1)—(1.3) has a solution u which quenches at the time T and $u \in C^{2,1}(\bar{\Omega} \times [0,T))$. Assume that the potential at (1.7) and the initial datum at (1.9) obey the conditions (3.1) and (3.2), respectively. Under the assumptions of Theorem 2.1, the problem (1.7)—(1.9) has a unique solution v which quenches in a finite time T_h^k, and the following relation holds

$$\lim_{(h,k)\to(0,0)} T_h^k = T.$$

PROOF. Let $\varepsilon \in \left(0, \frac{T}{2}\right)$. There exists $\rho > 0$ such that

$$\frac{y^{\beta+1}}{A(\beta+1)} \leq \frac{\varepsilon}{2}, \qquad 0 \leq y \leq \rho. \tag{3.4}$$

Since u quenches in a finite time T, there exists $T_0 \in (T - \frac{\varepsilon}{2}, T)$ such that

$$0 < u_{min}(t) < \frac{\rho}{2} \quad \text{for} \quad t \in [T_0, T).$$

Set $T_1 = \frac{T_0+T}{2}$. It is not hard to see that

$$0 < u_{min}(t) < \frac{\rho}{2} \quad \text{for} \quad t \in [T_0, T_1].$$

Making use of Theorem 3.1, we see that the problem (1.7)—(1.9) has a unique solution v, and the following estimate holds

$$\|v(.,t) - u(.,t)\|_\infty \leq \frac{\rho}{2} \quad \text{for} \quad t \in [0, T_1],$$

which implies that $\|v(.,T_1) - u(.,T_1)\|_\infty \leq \frac{\rho}{2}$.

An application of the triangle inequality leads us to

$$v_{min}(T_1) \leq \|v(.,T_1) - u(.,T_1)\|_\infty + u_{min}(T_1) \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

Invoking Theorem 2.1, we discover that v quenches at the time T_h^k. On the other hand, in the introduction of the paper, we have shown that $T_h^k \geq T$.

We infer from Remark 2.1 and (3.4) that

$$0 \leq T_h^k - T = \left|T_h^k - T_1\right| + |T_1 - T| \leq \frac{\left(v_{min}(T_1)\right)^{\beta+1}}{A(\beta+1)} + \frac{\varepsilon}{2} \leq \varepsilon,$$

and the proof is complete.

4. Numerical Results

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = cu^{-\alpha}\Delta u - ca(x)u^{-p} \text{ in } \mathrm{B}\times(0,T),$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \mathrm{S}\times(0,T),$$
$$u(x,0) = u_0(x) > 0 \text{ in } \overline{\mathrm{B}},$$

where $B = \{x \in IR^N, \|x\| < 1\}, S = \{x \in IR^N, \|x\| = 1\},\ \alpha = 1, p = 2, c = 1/2$. If we look at the original problem, the choice of $\alpha, p\ and\ c$ corresponds to the case where m=2 and q=1. The above problem may be rewritten in the following form

$$u_t = cu^{-\alpha}\left(u_{rr} + \frac{N-1}{r}u_r\right) - ca(r)u^{-p}, \qquad r \in (0,1), \qquad \mathrm{t} \in (0,T), \tag{4.1}$$
$$u_r(0,t) = 0,\ \ u_r(1,t) = 0, \qquad \mathrm{t} \in (0,T), \tag{4.2}$$
$$u(r,0) = \varphi(r),\ \ \mathrm{r} \in [0,1]. \tag{4.3}$$

Here, we take $\varphi(r) = \frac{2+\varepsilon\cos(\pi r)}{4}, a(r) = 2 - \varepsilon\sin(\pi r)$,where $\varepsilon \in [0,1]$. We start by the construction of an adaptive scheme as follows. Let I be a positive integer and let h=1/I. Define the grid $x_i = ih, 0 \le i \le I$, and approximate the solution u of (4.1)—(4.3) by the solution $U_h^{(n)} = \left(U_0^{(n)}, \dots, U_I^{(n)}\right)^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = Nc\left(U_0^{(n)}\right)^{-\alpha}\left(\frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2}\right) - ca(x_0)\left(U_0^{(n)}\right)^{-p},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = c\left(U_i^{(n)}\right)^{-\alpha}\left(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}\right)$$
$$- ca(x_i)\left(U_i^{(n)}\right)^{-p}, \qquad 1 \le i \le I-1,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = c\left(U_I^{(n)}\right)^{-\alpha}\left(\frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2}\right) - ca(x_I)\left(U_I^{(n)}\right)^{-p},$$
$$U_i^{(0)} = \varphi_i, \qquad 0 \le i \le I,$$

where $n \ge 0$, $\varphi_i = \frac{2+\varepsilon\cos(i\pi h)}{4}$, $a(x_i) = 2 - \varepsilon\sin(\pi ih)$.

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = min\left\{\frac{(1-h^2)h^2\left(U_{hmin}^{(n)}\right)^{\alpha}}{2Nc}, h^2 c\left(U_{hmin}^{(n)}\right)^{p+1}\right\}$$

with $U_{hmin}^{(n)} = \min_{0\leq i\leq I} U_i^{(n)}$. Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution (4.1)--(4.3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = Nc\left(U_0^{(n)}\right)^{-\alpha}\left(\frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2}\right) - ca(x_0)\left(U_0^{(n)}\right)^{-p-1} U_0^{(n+1)},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = c\left(U_i^{(n)}\right)^{-\alpha}\left(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2}\right.$$

$$\left. + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h}\right) - ca(x_i)\left(U_i^{(n)}\right)^{-p-1} U_i^{(n+1)},$$

$$1 \leq i \leq I-1,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = c\left(U_I^{(n)}\right)^{-\alpha}\left(\frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2}\right) - ca(x_I)\left(U_I^{(n)}\right)^{-p-1} U_I^{(n+1)},$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,$$

where $\Delta t_n = h^2 c\left(U_{hmin}^{(n)}\right)^{p+1}$.

Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution are also guaranteed using standard methods (see, for instance [2]). It is not hard to see that $u_{rr}(0,t) = \lim_{r\to 0}\frac{u_r(r,t)}{r}$. Hence, if r=0 and r=1, then we see that

$$u_r(0,t) = cNu^{-\alpha}(0,t)u_{rr}(0,\mathrm{t}) - ca(0)u^{-p}(0,t), \quad \mathrm{t} \in (0,T),$$

$$u_r(1,t) = cu^{-\alpha}(1,t)u_{rr}(1,\mathrm{t}) - ca(1)u^{-p}(1,t), \quad \mathrm{t} \in (0,T).$$

These observations have been taken into account in the construction of our schemes at the first and last nodes. We need the following definition.

Definition 4.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to\infty} U_h^{(n)} = 0,$ and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = |T^{n-1} - T^n| \leq 10^{-16}$.

The order(s) of the method is computed from

$$s = \frac{\log\left((T_{4h} - T_{2h})/(T_{2h} - T_h)\right)}{\log(2)}$$

Numerical experiments for α=1, p=2, N=2.

First case: ε=1

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUt	n
16	0.009192	3760	10	-
32	0.009302	15458	80	-
64	0.009326	62004	649	2.20
128	0.009332	248015	5023	2.00

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUt	n
16	0.009201	3759	12	-
32	0.009307	15458	67	-
64	0.009327	62003	738	2.20
128	0.009332	248015	7425	2.00

Second case: $\varepsilon = 1/100$

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUt	n
16	0.041445	2971	8	-
32	0.041368	11050	58	-
64	0.041371	40609	410	1.98
128	0.041367	149197	2911	1.91

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUt	n
16	0.041445	2971	9	-
32	0.041386	11050	47	-
64	0.041371	40609	537	1.98
128	0.041367	149197	3813	1.91

Third case: $\varepsilon = 1/10000$

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUt	n
16	0.041741	2903	7	-
32	0.041684	10675	85	-
64	0.041669	38907	661	2.02
128	0.041665	125386	5087	1.90

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUt	n
16	0.041745	2903	9	-
32	0.041684	10675	46	-
64	0.041669	38907	478	2.02
128	0.041665	125386	5124	1.90

Remark 4.1. If we consider the problem (4.1)— (4.3) in the case where $\varepsilon = 0$, it is well known that, theoretically, the quenching time of the solution u of (4.1)— (4.3) is the same as the one of the solutions β(t) of the following ordinary differential equation $\beta'(t) = -2\beta^{-p}(t)$ t>0, $\beta(0) = \frac{1}{2}$. A simple calculation shows that the quenching time of $\beta(t)$ is 0.041666. We observe from Tables 1 to 6 that if ε diminishes, then the numerical quenching time goes to 0.041666. This result confirms the theory established in the previous section.

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