

On Linear Operators with Closed Range

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Abstract

Linear operators between Fréchet spaces such that the closedness of range of an operator implies the closedness of range of another operator are discussed when the operators are topologically dominated by each other.

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1 Introduction

Let X and Y be normed spaces, $\mathcal{B}(X, Y)$ will denote the normed space of all continuous linear operators from X to Y . $R(T)$ and $N(T)$ will denote the range and null spaces of a linear operator T respectively.

The Banach's closed range theorem [5] reads as follows : If X and Y are Banach spaces and if $T \in \mathcal{B}(X, Y)$, then $R(T)$ is closed in Y if and only if $R(T^*)$ is closed in X^* . Given a differential operator T defined on some subspace of $L^p(\Omega)$, one may be interested in determining the family of functions $y \in L^p(\Omega)$ for which $Tf = y$ has a solution. It is well known that if $T \in \mathcal{B}(X, Y)$ has closed range, then the space of such y is the orthogonal complement of the solutions to the homogeneous equation $T^*g = 0$. There are many important

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applications of “closed range unbounded operators” in the spectral study of differential operators and also in the context of perturbation theory (see e.g. [1], [2]). In this paper, we deal with continuous linear operators between Banach and Fréchet spaces.

2 Norm Equivalent Operators

A characterization of closed range bounded linear operator between two Banach spaces is given in [3]. Using this characterization, we derive results which have conclusions of having closed range of an operator when the other operator has closed range. In this section we assume that the spaces are Banach unless otherwise specified.

Theorem 2.1. [3] *Let $T \in \mathcal{B}(X, Y)$. Then $R(T)$ is closed in Y if and only if there is a constant $c > 0$ such that for given $x \in X$, there is a $y \in X$ such that (i) $Tx = Ty$ and (ii) $\|y\| \leq c\|Tx\|$.*

Lemma 2.2. *Let $S, T \in \mathcal{B}(X, Y)$. If $\|Sx\| \leq k\|Tx\|$ for all $x \in X$, for some $k > 0$ and if S has closed range with $N(T) = N(S)$, then T has closed range.*

Proof. Note that $N(T) \subseteq N(S)$. S and T are one-to-one on $N(S)^\perp$. Let U' be an open ball with center 0 in X . Find a neighborhood U of 0 in Y such that for given $Sx \in U$, there is some $x_1 \in U'$ such that $Sx_1 = Sx$. Find a neighborhood V of 0 in Y such that $T^{-1}(V) \subseteq S^{-1}(U)$.

Fix $Tx \in V$ with $x \in N(S)^\perp$. Then $Sx \in U$ and $x \in N(S)^\perp$ and hence $x \in U'$, because S is one-to-one on $N(S)^\perp$. Therefore $T(N(S)^\perp)$ is closed in Y and hence $T(X) = T(N(S)^\perp)$ is closed. \square

Definition 2.3. *Let $S, T \in \mathcal{B}(X, Y)$. Two operators S and T are said to be norm equivalent if there exist two positive real numbers k_1 and k_2 such that $k_1\|Sx\| \leq \|Tx\| \leq k_2\|Sx\|$, for all $x \in X$.*

Remark 2.4. If S and T are norm equivalent, then $N(T) = N(S)$ and hence S has closed range if and only if T has closed range.

Lemma 2.5. *Let S and T be continuous linear operators from a Hilbert space X to Y . If S has closed range in Y with $T(N(S)) \subseteq T(N(S)^\perp)$ and $\|Sx\| \leq k\|Tx\|$ for all $x \in X$, for some $k > 0$, then T has closed range in Y .*

Proof. Let $c > 0$ be such that $\|Sx\| \geq c\|x\|$, for all $x \in N(S)^\perp$. Then $\|Tx\| \geq \frac{c}{k}\|x\|$, for all $x \in N(S)^\perp$. Thus $T(N(S)^\perp)$ is closed in Y . Since $T(N(S)) \subseteq T(N(S)^\perp)$, we have $T(X) = T(N(S)^\perp)$ and hence $T(X)$ is closed in Y . \square

Lemma 2.6. *Let T be a one-to-one continuous linear operator from X to Y and A be a compact linear operator from X to Z . If $\inf_{x \neq 0} \frac{\|Tx\| + k\|Ax\|}{\|x\|} = \gamma > 0$, for some $k > 0$, then T is a homeomorphism of X with $T(X)$, and hence $T(X)$ is closed in Y .*

Proof. If $T^{-1} : T(X) \rightarrow X$ is not continuous, then there is a sequence $(x_n)_{n=1}^\infty$ in X such that $\|x_n\| = 1$ and $Tx_n \rightarrow 0$. Then there is a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $Ay_n \rightarrow y$ for some $y \in Z$.

Then $0 \leq \gamma\|y_n - y_m\| \leq \|Ty_n - Ty_m\| + k\|Ay_n - Ay_m\| \rightarrow 0$, as $n, m \rightarrow \infty$. Therefore $(y_n)_{n=1}^\infty$ is a Cauchy sequence. Let $y_n \rightarrow x$ in X . Since $\|y_n\| = 1$ for all n , $\|x\| = 1$ and $x \neq 0$. But $Ty_n \rightarrow 0 = Tx$. Since T is one-to-one, $x = 0$. This is a contradiction. Therefore $T^{-1} : T(X) \rightarrow X$ is continuous. \square

Theorem 2.7. *Let $S, T \in \mathcal{B}(X, Y)$. Then $k_1|f(Tx)| \leq |f(Sx)| \leq k_2|f(Tx)|$, for all $x \in X$, for all $f \in Y^*$ and for some $k_1 > 0, k_2 > 0$ if and only if S is a constant multiple of T , $S = \alpha T$, for some scalar α .*

Proof. The first part is trivial by taking $k_1 = k_2 = \frac{1}{|\alpha|}$.

To prove the second part, without loss of generality assume that S and T are one-to-one, by passing to $X/N(S)$. Suppose $x \neq 0$ so that $Sx \neq 0 \neq Tx$. If Sx and Tx are linearly independent, then to each positive integer n , there is a $f_n \in Y^*$ such that $f_n(Sx) = n$ and $f_n(Tx) = 0$. This is impossible, because $|f_n(Sx)| \leq k|f_n(Tx)|$. Write $Sx = \alpha_x Tx$. For $y \neq 0$ such that x and y are linearly independent, we have $S(x+y) = \alpha_{x+y}T(x+y) = \alpha_x Tx + \alpha_y Ty$. Since Tx and Ty are linearly independent, $\alpha_{x+y} = \alpha_x = \alpha_y$. Thus S is a constant multiple of T . \square

Corollary 2.8. *Let $S, T \in \mathcal{B}(X, Y)$. If $k_1|f(Tx)| \leq |f(Sx)| \leq k_2|f(Tx)|$, for all $x \in X$, for all $f \in Y^*$ and for some $k_1 > 0, k_2 > 0$, then S has closed range if and only if T has closed range.*

Theorem 2.9. *Let S and T be continuous linear operators from a Hilbert space X into a normed space Y . Let P be the projection of X onto $N(T)^\perp$. If there is some $k_1 > 0$ such that $k_1|f(Tx)| \leq |f(Sx)|$, for all $x \in X$, for all $f \in Y^*$, then $T = \alpha SP$, for some scalar α .*

Proof. Let us first establish that $T = \alpha S$ on $N(T)^\perp$. Note that $k_1\|Tx\| \leq \|Sx\|$, for all $x \in X$, and $N(S) \subseteq N(T)$. S and T are one-to-one on $N(T)^\perp$. Suppose $x \neq 0$ in $N(T)^\perp$ so that $Sx \neq 0 \neq Tx$.

If Sx and Tx are linearly independent, then to each positive integer n , there is a $f_n \in Y^*$ such that $f_n(Tx) = n$ and $f_n(Sx) = 0$. This is impossible. Then $\alpha Sx = Tx$, for all $x \in N(T)^\perp$ as it was established in the proof of the previous theorem. Thus $\alpha S = T$ on $N(T)^\perp$.

Let $x \in H$. Write $x = x' + x''$ with $x' \in N(T)$ and $x'' \in N(T)^\perp$. Then $Tx'' = \alpha Sx'' = \alpha SPx$ and $Tx'' = Tx' + Tx'' = Tx$ so that $Tx = \alpha SPx$. Thus $T = \alpha SP$ on X . \square

Corollary 2.10. *Let S and T be continuous linear operators from a Hilbert space X into a normed space Y . If S has closed range in Y and $k_1|f(Tx)| \leq |f(Sx)|$, for all $x \in X$, for all $f \in Y^*$ and for some $k_1 > 0$, then T has closed range in Y .*

Proof. $S^{-1}(0) + R(P) = N(S) + N(T)^\perp$ is closed in X , because $N(S) \subseteq N(T)$. Moreover S has closed range in Y . Therefore SP has closed range in Y . Thus T has closed range in Y . \square

Corollary 2.11. *Let S and T be continuous linear operators from a Hilbert space X into a normed space Y . If $k_1|f(Tx)| \leq |f(Sx)|$, for all $x \in X, f \in Y^*$ and for some $k_1 > 0$, then $k_1\|Tx\| \leq \|Sx\|$, for all $x \in X$. Hence if T has closed range in Y and $S(N(T)) \subseteq S(N(T)^\perp)$, then S has closed range in Y .*

Proof. It follows from Lemma 2.5. \square

3 Topologically Dominated Operators

A characterization for closed range operators between F -spaces has given in [3]. Using this characterization, we compare two continuous linear operators S and T between F -spaces such that closedness of $R(S)$ implies the closedness of $R(T)$. An F -space is a complete metrizable topological vector space and *Fréchet space* is a locally convex F -space [4].

Theorem 3.1. [3] *Let T be a continuous linear operator from an F -space X into an F -space Y . Then T has closed range in Y if and only if for every sequence $(y_n)_{n=1}^{\infty}$ in $T(X)$ which converges to 0, there is a sequence $(x_n)_{n=1}^{\infty}$ in X which also converges to 0 such that $Tx_n = y_n$ for every n .*

Definition 3.2. *Let X and Y be topological vector spaces. Let S and T be linear operators from X into Y . We say that S is topologically dominated by T if the following condition holds : For given neighborhood U of 0 in Y , there is a neighborhood V of 0 in Y such that $T^{-1}(V) \subseteq S^{-1}(U)$.*

We say that S is topologically equivalent to T if S is topologically dominated by T and T is topologically dominated by S .

Lemma 3.3. *Let X and Y be F -spaces. Let S and T be continuous linear operators from X into Y such that $N(S) = N(T)$. If S has closed range and S is topologically dominated by T , then T has closed range.*

Proof. Fix a neighborhood U' of 0 in X . Find a neighborhood U of 0 in Y such that for given $Sx \in U$, there is an element $x' \in U'$ such that $Sx = Sx'$. For this U , find a neighborhood V of 0 in Y such that $T^{-1}(V) \subseteq S^{-1}(U)$. For a given $Tx \in V$, we have $Sx \in U$, and there is an $x' \in U'$ such that $Sx = Sx'$. Then $x - x' \in N(S)$ and hence $x - x' \in N(T)$ so that $Tx = Tx'$ with $x' \in U'$. This proves that T has closed range in Y . \square

Remark 3.4. If S and T are topologically equivalent, then $N(T) = N(S)$ and hence S has closed range if and only if T has closed range.

Lemma 3.5. *Let S and T be continuous linear operators from a Hilbert space X into an F -space Y . Let S be topologically dominated by T . If S has closed range in Y with $T(N(S)) \subseteq T(N(S)^\perp)$, then T has closed range in Y .*

Proof. Note that $N(T) \subseteq N(S)$. S and T are one-to-one on $N(S)^\perp$. Let U' be an open ball with center 0 in X . Find a neighborhood U of 0 in Y such that for given $Sx \in U$, there is some $x_1 \in U'$ such that $Sx_1 = Sx$. Find a neighborhood V of 0 in Y such that $T^{-1}(V) \subseteq S^{-1}(U)$.

Fix $Tx \in V$ with $x \in N(S)^\perp$. Then $Sx \in U$ and $x \in N(S)^\perp$ and hence $x \in U'$, because S is one-to-one on $N(S)^\perp$. Therefore $T(N(S)^\perp)$ is closed in Y and hence $T(X) = T(N(S)^\perp)$ is closed. \square

Lemma 3.6. *Let X be a Banach space, and Y, Z be Fréchet spaces. Let $S : X \rightarrow Y$ be an injective continuous linear operator and $B : X \rightarrow Z$ be a compact operator. Let $(p_n)_{n=1}^\infty$ be a family of seminorms which define the topology on Y . Let $(q_m)_{m=1}^\infty$ be a family of seminorms which define the topology on Z .*

Suppose $\inf_{x \neq 0} \frac{p_n(Sx) + kq_m(Bx)}{\|x\|} = \gamma > 0$, for some $k > 0$, for some p_n and for some q_m . Then S is a homeomorphism of X with $S(X)$ and hence $S(X)$ is closed in Y .

Proof. If $S^{-1} : S(X) \rightarrow X$ is not continuous, then there is a sequence $(x_n)_{n=1}^\infty$ in X such that $\|x_n\| = 1$ and $Sx_n \rightarrow 0$. Find a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $By_n \rightarrow y$ for some $y \in Y$.

Then $(y_n)_{n=1}^\infty$ is Cauchy and let $y_n \rightarrow x$ in X . Then $\|x\| = 1$ so that $x \neq 0$. But $Sy_n \rightarrow Sx = 0$ so that $x = 0$, because S is one-to-one. This contradiction proves that S^{-1} is continuous on $S(X)$. \square

Theorem 3.7. *Let X be an F -space and Y be a Fréchet space and S, T be continuous linear operators from X to Y . Then*

$$k_1|f(Tx)| \leq |f(Sx)| \leq k_2|f(Tx)|,$$

for all $f \in Y^$, for all $x \in X$ and for some $k_1 > 0$, $k_2 > 0$ if and only if S is a constant multiple of T .*

Moreover S has closed range if and only if T has closed range.

Proof. The proof follows from the arguments of the proof of Theorem 2.7 and Corollary 2.8. \square

Theorem 3.8. *Let X be a Hilbert space and Y be a Fréchet space. Let S, T be continuous linear operators from X into Y and for some $k_1 > 0$,*

$$k_1|f(Tx)| \leq |f(Sx)|,$$

for all $x \in X$, for all $f \in Y^$. Let P be the projection of X onto $N(T)^\perp$. Then*

$$T = \alpha SP,$$

for some scalar α .

If S has closed range in Y , then T has closed range in Y .

Moreover $kp(Tx) \leq p(Sx)$, for all $x \in X$, for any continuous seminorm p on X .

Proof. The proof follows from the arguments of the proofs of Theorem 2.9 and Corollary 2.10. \square

Corollary 3.9. *Let X be a Hilbert space, $(Y, (p_n)_{n=1}^\infty)$ be a Fréchet space and S, T be continuous linear operators from X into Y . Let $k_1|f(Tx)| \leq |f(Sx)|$, for all $x \in X$, for all $f \in Y^*$ and for some $k_1 > 0$. Then $k_1p_n(Tx) \leq p_n(Sx)$, for all $x \in X$, for the seminorms p_n on Y , $n = 1, 2, \dots$. Hence if T has closed range in Y and $S(N(T)) \subseteq S(N(T)^\perp)$, then S has closed range in Y .*

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