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On some mixed integral inequalities and its applications

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Abstract

In this paper, we establish some mixed integral and integro-differential inequalities which can be used as handy tools to study properties of solutions of a certain mixed integral and differential equations.

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1 Introduction

In the theory of differential, integral and integro-differential equations one often has to deal with certain differential and integro-differential inequalities. In the last few years with the development of the theory of nonlinear differential and integral equations, many authors have established several integral

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and integro-differential inequalities, see [1, 2, 4, 6, 7, 8, 9, 10, 11, 12]. These inequalities play an important role in the study of some properties of differential, integral and integro-differential equations. Existence of solutions of a certain mixed integral and integro-differential equations were studied in [3, 13] by M. B. Dhakne and H. L. Tidke.

In this paper, we establish mixed integral and integro-differential inequalities which provide an explicit bound on unknown function. In particular we extend the result established by B. G. Pachpatte in [12]. Some applications are also given to convey the importance of our results.

2 Preliminary

Before proceeding to the statement of our main result, we state some important integral inequalities that will be used in further discussion.

Lemma 2.1 (Fangcui Jiang and Fanwei Meng [5]). *Assume that* $a \geq 0, p \geq q \geq 0,$ *and* $p \neq 0,$ *then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}}, \quad \textit{for any} \quad k > 0. \tag{2.1}$$

Theorem 2.2 (Pachpatte [12]). *Let* $u(t), a(t), b(t), c(t) \in C(I = [\alpha, \beta], R_+),$ $a(t)$ *be continuously differentiable on* $I,$ $a'(t) \geq 0$ *and*

$$u(t) \leq a(t) + \int_\alpha^t b(s)u(s)ds + \int_\alpha^\beta c(s)u(s)ds, \qquad t \in I. \tag{2.2}$$

If $p = \displaystyle\int_\alpha^\beta c(s) \exp\left(\int_\alpha^s b(\sigma)d\sigma\right) ds < 1,$ *then*

$$u(t) \leq M \exp\left(\int_\alpha^t b(s)ds\right) + \int_\alpha^t a'(s) \exp\left(\int_\alpha^s b(\sigma)d\sigma\right) ds, \quad t \in I, \tag{2.3}$$

where

$$M = \frac{1}{1-p}\left[a(\alpha) + \int_\alpha^\beta c(s) \int_\alpha^s a'(\tau) \exp\left(\int_\tau^s b(\sigma)d\sigma\right) d\tau ds\right], \quad t \in I. \tag{2.4}$$

3 Main Results

In this section, we state and prove mixed nonlinear integral inequalities to obtain explicit bound on solutions of a certain mixed integral equations.

Theorem 3.1. *Let* $u(t), f(t), g(t), c(t), c'(t) \in C(I = [\alpha, \beta], \mathbb{R}_+)$ *and* $p \geq q \geq 0, p \neq 0$ *be constants. If*

$$u^p(t) \leq c(t) + \int_\alpha^t f(s)u^q(s)ds + \int_\alpha^\beta g(s)u^p(s)ds$$

and $Q = \displaystyle\int_\alpha^\beta g(s)\exp\left(\int_\alpha^s n_1 f(\sigma)d\sigma\right)ds < 1$, *then*

$$u^p(t) \leq \frac{c(\alpha) + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2 f(\tau)]\exp\left(\int_\tau^s n_1 f(\sigma)d\sigma\right)d\tau\right)ds}{1-Q} \times \exp\left(\int_\alpha^t mf(s)ds\right) + \int_\alpha^t [c'(s) + n_2 f(s)]\exp\left(\int_s^t n_1 f(\sigma)d\sigma\right)ds, \tag{3.1}$$

where $k > 0$, $n_1 = \dfrac{q}{p}k^{\frac{q-p}{p}}$ *and* $n_2 = \dfrac{p-q}{p}k^{\frac{q}{p}}$.

Proof. Define a function $z(t)$ by

$$z(t) = c(t) + \int_\alpha^t f(s)u^q(s)ds + \int_\alpha^\beta g(s)u^p(s)ds,$$

then $u(t) \leq z^{\frac{1}{p}}(t)$,

$$z(\alpha) = c(\alpha) + \int_\alpha^\beta g(s)u^p(s)ds \tag{3.2}$$

and

$$z'(t) = c'(t) + f(t)u^q(t) \leq c'(t) + f(t)z^{\frac{q}{p}}(t). \tag{3.3}$$

¿From Lemma 2.1 and (3.3), we have

$$z'(t) \leq c'(t) + n_1 f(t)z(t) + n_2 f(t),$$

or, equivalently,

$$\left[\frac{z(t)}{\exp\left(n_1\int_\alpha^t f(s)ds\right)}\right]' \leq [c'(t) + n_2 f(t)]\exp\left(-n_1\int_\alpha^t f(s)ds\right). \tag{3.4}$$

Setting $t = s$; in (3.4) and integrating with respect to s from α to t, we have

$$z(t) \leq z(\alpha)\exp\left(\int_\alpha^t n_1 f(s)ds\right) + \int_\alpha^t [c'(s) + n_2 f(s)]\exp\left(\int_s^t n_1 f(\sigma)d\sigma\right) ds. \tag{3.5}$$

As $u^p(t) \leq z(t)$ from equation (3.5), we have

$$u^p(t) \leq z(\alpha)\exp\left(\int_\alpha^t n_1 f(s)ds\right) + \int_\alpha^t [c'(s) + n_2 f(s)]\exp\left(\int_s^t n_1 f(\sigma)d\sigma\right) ds. \tag{3.6}$$

Now from (3.2) and (3.6), we have

$$\begin{aligned} z(\alpha) \leq c(\alpha) &+ z(\alpha)\int_\alpha^\beta g(s)\exp\left(\int_\alpha^s n_1 f(\sigma)d\sigma\right) ds \\ &+ \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2 f(\tau)]\exp\left(\int_\tau^s n_1 f(\sigma)d\sigma\right) d\tau\right) ds, \end{aligned}$$

or, equivalently,

$$\begin{aligned} &z(\alpha)\left(1 - \int_\alpha^\beta g(s)\exp\left(\int_\alpha^s n_1 f(\sigma)d\sigma\right) ds\right) \\ &\quad \leq c(\alpha) + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2 f(\tau)]\exp\left(\int_\tau^s n_1 f(\sigma)d\sigma\right) d\tau\right) ds, \end{aligned}$$

$$z(\alpha) \leq \frac{c(\alpha) + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2 f(\tau)]\exp\left(\int_\tau^s n_1 f(\sigma)d\sigma\right) d\tau\right) ds}{1 - Q}. \tag{3.7}$$

Using inequality (3.7) in (3.6), we obtain the desired inequality (3.1). This completes the proof. □

Remark 1. It is interesting to note that when $p = q = 1$ the Theorem 3.1 reduces to the inequality stated in Theorem 2.2 and when $p = 1$ and $g = 0$ it reduces to well known Gronwall-Bellman inequality.

Theorem 3.2. *Let $u(t), f(t), g(t), h(t) \in C(I, \mathbb{R}_+)$ and $c \geq 0$ be a constant. If*

$$u^p(t) \leq c + \int_\alpha^t h(s)\left[u^q(s) + \int_\alpha^s f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)u^p(\sigma)d\sigma\right] ds, \ \textit{for } t \in I,$$

then

$$u^p(t) \leq c \exp\left(\int_\alpha^t n_1 A(\sigma)d\sigma\right) + \int_\alpha^t n_2 B(s) \exp\left(\int_s^t n_1 A(\sigma)d\sigma\right), \tag{3.8}$$

where

$$A(t) = h(t)\left[1 + \int_\alpha^t f(\sigma)d\sigma + \frac{1}{n_1}\int_\alpha^\beta g(\sigma)d\sigma\right], \quad B(t) = h(t)\left[1 + \int_\alpha^t f(\sigma)d\sigma\right],$$

and p, q, n_1, n_2 are as same defined in Theorem 3.1.

Proof. Denoting a function $z(t)$ by

$$z(t) = c + \int_\alpha^t h(s)\left[u^q(s) + \int_\alpha^s f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)u^p(\sigma)d\sigma\right]ds,$$

we have $u(t) \leq z^{\frac{1}{p}}(t)$, $z(\alpha) = c$, $z(t)$ is a nondecreasing and

$$\begin{aligned} z'(t) &= h(t)\left[u^q(t) + \int_\alpha^t f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)u^p(\sigma)d\sigma\right] \\ &\leq h(t)\left[z^{\frac{q}{p}}(t) + \int_\alpha^t f(\sigma)z^{\frac{q}{p}}(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)z(\sigma)d\sigma\right] \\ &\leq h(t)\left\{\left[1 + \int_\alpha^t f(\sigma)d\sigma\right]z^{\frac{q}{p}}(t) + \left[\int_\alpha^\beta g(\sigma)d\sigma\right]z(t)\right\}. \end{aligned} \tag{3.9}$$

An application of Lemma 2.1 to (3.9), we have

$$\begin{aligned} z'(t) &\leq h(t)\left\{\left[1 + \int_\alpha^t f(\sigma)d\sigma\right][n_1 z(t) + n_2] + \left[\int_\alpha^\beta g(\sigma)d\sigma\right]z(t)\right\} \\ &= h(t)\left\{n_1\left[1 + \int_\alpha^t f(\sigma)d\sigma + \frac{1}{n_1}\int_\alpha^\beta g(\sigma)d\sigma\right]z(t) + n_2\left[1 + \int_\alpha^t f(\sigma)d\sigma\right]\right\} \\ &= n_1 A(t)z(t) + n_2 B(t), \end{aligned}$$

or, equivalently,

$$\left[\frac{z(t)}{\exp\left(\int_\alpha^t n_1 A(\sigma)d\sigma\right)}\right]' \leq n_2 B(t) \exp\left(-\int_\alpha^t n_1 A(\sigma)d\sigma\right). \tag{3.10}$$

By integrating (3.10), we get

$$z(t) \leq z(\alpha) \exp\left(\int_\alpha^t n_1 A(\sigma)d\sigma\right) + \int_\alpha^t n_2 B(s) \exp\left(\int_s^t n_1 A(\sigma)d\sigma\right)ds. \tag{3.11}$$

As $u^p(t) \le z(t)$ from (3.11), we obtain

$$u^p(t) \le z(\alpha)\exp\left(\int_\alpha^t n_1 A(\sigma)d\sigma\right) + \int_\alpha^t n_2 B(s)\exp\left(\int_s^t n_1 A(\sigma)d\sigma\right)ds. \tag{3.12}$$

Using $z(\alpha) = c$ in (3.12), we get the desired inequality (3.8) and hence the proof. □

Remark 2. Note that when $p = q = 1$ the Theorem 3.2 reduces to the inequality established by B.G. Pachpatte in ([12] page no.40).

Corollary 3.3. *Let $u(t), f(t), g(t), h(t)$ be as same defined in Theorem 3.2 and $1 \le c(t)$ be a nondecreasing. If*

$$u^p(t) \le c(t) + \int_\alpha^t h(s)\left[u^q(s) + \int_\alpha^s f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)u^p(\sigma)d\sigma\right]ds, \ \text{for } t \in I, \tag{3.13}$$

then

$$u^p(t) \le c^p(t)\exp\left(\int_\alpha^t n_1 A(\sigma)d\sigma\right) + c^p(t)\int_\alpha^t n_2 B(s)\exp\left(\int_s^t n_1 A(\sigma)d\sigma\right), \tag{3.14}$$

where $p \ge q \ge 1, A(t), B(t)$ and n_1, n_2 are as same defined in Theorem 3.2 and Theorem 3.1 respectively.

Proof. Since $1 \le c(t)$ and nondecreasing, from (3.13), we have

$$\left(\frac{u(t)}{c(t)}\right)^p \le 1 + \int_\alpha^t h(s)\left[\left(\frac{u(s)}{c(s)}\right)^q(s) + \int_\alpha^s f(\sigma)\left(\frac{u(\sigma)}{c(\sigma)}\right)^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)\left(\frac{u(\sigma)}{c(\sigma)}\right)^p(\sigma)d\sigma\right]ds, \tag{3.15}$$

for $t \in I$. Applying Theorem 3.2 to (3.15), we get (3.14). This completes the proof. □

Theorem 3.4. *Let $u(t), u'(t), f(t), g(t), c(t), c'(t) \in C(I, \mathbb{R}_+)$ and $u(\alpha) = 0$. If*

$$[u'(t)]^p \le c(t) + \int_\alpha^t f(s)u^q(s)ds + \int_\alpha^\beta g(s)[u'(s)]^p ds \tag{3.16}$$

and $Q_1 = \int_\alpha^\beta g(s) \exp\left(\int_\alpha^s (\sigma-\alpha)^q n_1 f(\sigma) d\sigma\right) ds < 1,$ *then*

$$[u'(t)]^p \leq \frac{c(\alpha) + \int_\alpha^\beta g(s) \left(\int_\alpha^s [c'(\tau) + (\tau-\alpha)^q n_2 f(\tau)] \exp\left(n_1 \int_\tau^s (\sigma-\alpha)^q f(\sigma) d\sigma\right) d\tau\right) ds}{1-Q_1}$$
$$\times \exp\left(n_1 \int_\alpha^t (s-\alpha)^q f(s) ds\right)$$
$$+ \int_\alpha^t [c'(s) + (s-\alpha)^q n_2 f(s)] \exp\left(n_1 \int_s^t (s-\alpha)^q f(s) ds\right) ds, \qquad (3.17)$$

p, q, n_1, n_2 *are as same defined in Theorem 3.1.*

Proof. Define a function $z(t)$ by

$$z(t) = c(t) + \int_\alpha^t f(s) u^q(s) ds + \int_\alpha^\beta g(s) [u'(s)]^p ds,$$

then $u(t) \leq \int_\alpha^t z^{\frac{1}{p}}(s),\ z(t)$ is a nondecreasing,

$$z(\alpha) = c(\alpha) + \int_\alpha^\beta g(s) [u'(s)]^p(s) ds \qquad (3.18)$$

and

$$z'(t) \leq c'(t) + (t-\alpha)^q f(t) z^{\frac{q}{p}}(t),. \qquad (3.19)$$

Applying Lemma 2.1 to (3.19), we have

$$\begin{aligned} z'(t) &\leq c'(t) + (t-\alpha)^q f(t)[n_1 z(t) + n_2] \\ &= c'(t) + n_1 (t-\alpha)^q f(t) z(t) + n_2 (t-\alpha)^q f(t) \\ &= c'(t) + n_1 (t-\alpha)^q f(t) z(t) + n_2 (t-\alpha)^q f(t), \end{aligned}$$

or, equivalently,

$$\left[\frac{z(t)}{\exp\left(n_1 \int_\alpha^t (s-\alpha)^q f(s) ds\right)}\right]' \leq [c'(t) + n_2 (t-\alpha)^q f(t)] \exp\left(-n_1 \int_\alpha^t (s-\alpha)^q f(s) ds\right). \qquad (3.20)$$

By integrating (3.20), we get

$$\frac{z(t)}{\exp\left(n_1 \int_\alpha^t (s-\alpha)^q f(s) ds\right)} \leq z(\alpha) + \int_\alpha^t [c'(s) + n_2 (s-\alpha)^q f(s)]$$
$$\times \exp\left(-n_1 \int_\alpha^s (s-\alpha)^q f(s) ds\right) ds,$$

$$z(t) \leq z(\alpha) \exp\left(\int_\alpha^t n_1(s-\alpha)^q f(s)ds\right) + \int_\alpha^t [c'(s) + n_2(s-\alpha)^q f(s)] \times \exp\left(\int_s^t n_1(\sigma-\alpha)^q f(\sigma)d\sigma\right) ds. \tag{3.21}$$

As $[u'(t)]^p \leq z(t)$ from (3.21), we have

$$[u'(t)]^p \leq z(\alpha) \exp\left(\int_\alpha^t n_1(s-\alpha)^q f(s)ds\right) + \int_\alpha^t [c'(s) + n_2(s-\alpha)^q f(s)] \exp\left(\int_s^t n_1(\sigma-\alpha)^q f(\sigma)d\sigma\right) ds. \tag{3.22}$$

Now from (3.18) and (3.22), we have

$$\begin{aligned} z(\alpha) &\leq c(\alpha) + z(\alpha)\int_\alpha^\beta g(s) \exp\left(\int_\alpha^s n_1(\sigma-\alpha)^q f(\sigma)d\sigma\right) ds \\ &\quad + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2(\tau-\alpha)^q f(\tau)] \exp\left(\int_\tau^s n_1(\sigma-\alpha)^q f(\sigma)d\sigma\right) d\sigma\right) ds, \\ z(\alpha)&\left(1 - \int_\alpha^\beta g(s) \exp\left(\int_\alpha^s n_1(\sigma-\alpha)^q f(\sigma)d\sigma\right) ds\right) \leq c(\alpha) \\ &\quad + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2(\tau-\alpha)^q f(\tau)] \exp\left(\int_\tau^s n_1(\sigma-\alpha)^q f(\sigma)d\sigma\right) d\tau\right) ds, \\ z(\alpha) &\leq \frac{c(\alpha) + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + n_2(\tau-\alpha)^q f(\tau)] \exp\left(\int_\tau^s n_1(\sigma-\alpha)^q f(\sigma)d\sigma\right) d\tau\right) ds}{1-Q_1}. \end{aligned} \tag{3.23}$$

The required inequality (3.17) follows, from inequalities (3.22) and (3.23). This completes the proof. □

Theorem 3.5. *Let $u(t), f(t), g(t), h(t) \in C(I, \mathbb{R}_+)$ and $c \geq 0$ be a constant. If*

$$[u'(t)]^p \leq c + \int_\alpha^t h(s)\left[u^q(s) + \int_\alpha^s f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)[u'(\sigma)]^p d\sigma\right] ds, \ \textit{for } t \in I,$$

then

$$[u'(t)]^p \leq c \exp\left(\int_\alpha^t n_1 A_1(\sigma)d\sigma\right) + \int_\alpha^t n_2 B_1(s) \exp\left(\int_s^t n_1 A_1(\sigma)d\sigma\right), \tag{3.24}$$

where

$$A_1(t) = h(t)\left[[t-\alpha]^q + \int_\alpha^t [\sigma-\alpha]^q f(\sigma)d\sigma + \frac{1}{n_1}\int_\alpha^\beta g(\sigma)d\sigma\right],$$

$$B_1(t) = h(t)\left[[t-\alpha]^q + \int_\alpha^t [\sigma-\alpha]^q f(\sigma)d\sigma\right]$$

and p, q, n_1, n_2 are as same defined in Theorem 3.1.

Proof. Denoting a function $z(t)$ by

$$z(t) = c + \int_\alpha^t h(s)\left[u^q(s) + \int_\alpha^s f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)[u(\sigma)]^p d\sigma\right]ds,$$

we have $u'(t) \le z^{\frac{1}{p}}(t)$, $z(\alpha) = c$, $z(t)$ is a nondecreasing and

$$\begin{aligned} z'(t) &= h(t)\left[u^q(t) + \int_\alpha^t f(\sigma)u^q(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)[u'(\sigma)]^p d\sigma\right] \\ &\le h(t)\left[[t-\alpha]^q z^{\frac{q}{p}}(t) + \int_\alpha^t [\sigma-\alpha]^q f(\sigma) z^{\frac{q}{p}}(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)z(\sigma)d\sigma\right] \\ &\le h(t)\left\{\left[[t-\alpha]^q + \int_\alpha^t [\sigma-\alpha]^q f(\sigma)d\sigma\right] z^{\frac{q}{p}}(t) + \left[\int_\alpha^\beta g(\sigma)d\sigma\right] z(t)\right\}. \end{aligned} \tag{3.25}$$

An application of Lemma 2.1 to (3.25), we have

$$\begin{aligned} z'(t) &\le h(t)\left\{\left[[t-\alpha]^q + \int_\alpha^t [\sigma-\alpha]^q f(\sigma)d\sigma\right][n_1 z(t) + n_2] + \left[\int_\alpha^\beta g(\sigma)d\sigma\right] z(t)\right\} \\ &= h(t)\left\{n_1\left[[\sigma-\alpha]^q + \int_\alpha^t [\sigma-\alpha]^q f(\sigma)d\sigma + \frac{1}{n_1}\int_\alpha^\beta g(\sigma)d\sigma\right] z(t) \right. \\ &\quad \left. + n_2\left[[t-\alpha]^q + \int_\alpha^t [\sigma-\alpha]^q f(\sigma)d\sigma\right]\right\} \\ &= n_1 A_1(t)z(t) + n_2 B_1(t), \end{aligned}$$

or, equivalently,

$$\left[\frac{z(t)}{\exp\left(\int_\alpha^t n_1 A_1(\sigma)d\sigma\right)}\right]' \le n_2 B_1(t)\exp\left(-\int_\alpha^t n_1 A_1(\sigma)d\sigma\right). \tag{3.26}$$

By integrating (3.26), we get

$$z(t) \leq z(\alpha)\exp\left(\int_\alpha^t n_1A_1(\sigma)d\sigma\right) + \int_\alpha^t n_2B_1(s)\exp\left(\int_s^t n_1A_1(\sigma)d\sigma\right)ds. \tag{3.27}$$

As $[u'(t)]^p \leq z(t)$ from (3.27), we obtain

$$[u'(t)]^p \leq z(\alpha)\exp\left(\int_\alpha^t n_1A_1(\sigma)d\sigma\right) + \int_\alpha^t n_2B_1(s)\exp\left(\int_s^t n_1A_1(\sigma)d\sigma\right)ds. \tag{3.28}$$

Using $z(\alpha) = c$ in (3.28), we get the desired inequality (3.24) and hence the proof. □

4 Applications

One of the main motivations for the study of different type inequalities given in the previous sections is to apply them as tools in the study of various classes of integral equations. In the following section we give application of some theorems of previous sections. In fact we discuss the boundedness behavior of solutions of a nonlinear mixed integral equations.

Example 1. Consider the following general mixed nonlinear integral equation

$$y^p(t) = x(t) + \int_\alpha^t F(s, y^q(s))ds + \int_\alpha^\beta G(s, y^p(s))ds, \tag{4.1}$$

for $t \in I$, where $p \geq q \geq 0, p \neq 0, y(t)$ is unknown function, $x \in C(I, \mathbb{R}^n), F, G \in C(I \times \mathbb{R}^n, \mathbb{R}^n)$,
$I = [\alpha, \beta], \mathbb{R}^n$ is n dimensional Euclidean space with norm $|.|$.

Suppose that the functions x, y, F, G in equation (4.1) satisfy the following conditions :

$$|x(t)| \leq c(t), \tag{4.2}$$

$$|F(s, y^q)| \leq g(s)|y|^q, \tag{4.3}$$

$$|G(s, y^p)| \leq f(s)|y|^p, \tag{4.4}$$

where $p \geq q \geq 0, p \neq 0, c, f, g$ are as same defined in Theorem 3.1. If $y(t), t \in I$ is a solution of equation (4.1) and $Q < 1$, then

$$|y(t)|^p \leq \frac{c(\alpha) + \int_\alpha^\beta g(s) \left(\int_\alpha^s [c'(\tau) + n_2 f(\tau)] \exp\left(\int_\tau^s n_1 f(\sigma) d\sigma \right) d\tau \right) ds}{1 - Q} \times \exp\left(\int_\alpha^t m f(s) ds \right) + \int_\alpha^t [c'(s) + n_2 f(s)] \exp\left(\int_s^t n_1 f(\sigma) d\sigma \right) ds, \tag{4.5}$$

where Q, n_1, n_2 are as same defined in Theorem 3.1.

¿From equation (4.1)-(4.4), we obtain

$$|y(t)|^p \leq |x(t)| + \int_\alpha^t f(s)|y(s)|^q ds + \int_\alpha^\beta g(s)|y(s)|^p ds. \tag{4.6}$$

An application of Theorem 3.1, we get (4.6). This show that solution $y(t)$ is bounded.

Example 2. Consider the following general mixed nonlinear integrodifferential equation

$$[y'(t)]^p = x(t) + \int_\alpha^t F\left(s, y^q(s)\right) ds + \int_\alpha^\beta G\left(s, [y'(s)]^p\right) ds, \tag{4.7}$$

for $t \in I$, where $p \geq q \geq 0, p \neq 0, y(t)$ is unknown function, $x \in C(I, \mathbb{R}^n), F, G \in C(I \times \mathbb{R}^n, \mathbb{R}^n)$,
$I = [\alpha, \beta], \mathbb{R}^n$ is n dimensional Euclidean space with norm $|.|$.

Suppose that the functions x, y, F, G in equation (4.7) satisfy the following conditions

$$|x(t)| \leq c(t), \tag{4.8}$$

$$|F(t, s, y^p)| \leq g(t, s) \left|y'\right|^p, \tag{4.9}$$

$$|G(t, s, y^q)| \leq f(t, s)|y|^q, \tag{4.10}$$

where $p \geq q \geq 0, p \neq 0, c, f, g$ are as same defined in Theorem 3.4. If $y(t), t \in I$ is a solution of equation (4.7) and $Q_<1$, then

$$|y(t)|^p \leq \frac{c(\alpha) + \int_\alpha^\beta g(s) \left(\int_\alpha^s [c'(\tau) + (\tau - \alpha)^q n_2 f(\tau)] \exp\left(n_1 \int_\tau^s (\sigma - \alpha)^q f(\sigma) d\sigma \right) d\tau \right) ds}{1 - Q_1} \times \exp\left(n_1 \int_\alpha^t (s - \alpha)^q f(s) ds \right) + \int_\alpha^t [c'(s) + (s - \alpha)^q n_2 f(s)] \exp\left(n_1 \int_s^t (s - \alpha)^q f(s) ds \right) ds, \tag{4.11}$$

where Q_1, n_1, n_2 are as same defined in Theorem 3.4. From equation (4.7)-(4.10), we obtain

$$|y'(t)|^p \leq c(t) + \int_\alpha^t f(s)|y(s)|^q ds + \int_\alpha^\beta g(s)|y'(s)|^p ds. \tag{4.12}$$

Applying Theorem 3.4 to (4.12), we get (4.11).

Example 3. We calculate the explicit bound on the solution of the following nonlinear integral equation of the form:

$$u^3(t) = 6 + \int_0^t \frac{1}{1+s} u(s) ds + \int_0^7 s u^3(s) ds \tag{4.13}$$

where $u(t)$ are defined as in Theorem 3.1 and we assume that every solution $u(t)$ of (4.13) exists on R_+.
By Theorem 3.1, we have $p = 3, q = 2, \quad n_1 = \frac{q}{p} k^{\frac{q-p}{p}} = \frac{2}{3} k^{\frac{-1}{3}}, n_2 = \frac{p-q}{p} k^{\frac{q}{p}} = \frac{1}{3} k^{\frac{2}{3}}, \alpha = 0, \beta = 1, k > 0, f(s) = \frac{1}{1+s}, g(s) = s$ and

$$\begin{aligned} Q &= \int_\alpha^\beta g(s) \exp\left(\int_\alpha^s n_1 f(\sigma) d\sigma \right) ds \\ &= \int_0^1 s \exp\left(\frac{2}{3} \frac{1}{\sqrt[3]{k}} \int_0^s \frac{1}{\sigma+1} d\sigma \right) ds \\ &= \frac{3\left(3\sqrt[3]{k} + 2^{\frac{2}{3\sqrt[3]{k}}+2}\right)\sqrt[3]{k}}{2\left(9k^{2/3} + 9\sqrt[3]{k} + 2\right)} < 1, \text{ for any } \quad k > 0. \end{aligned}$$

Thus all the conditions of the Theorem 3.1 are satisfied, hence we obtain

$$\begin{aligned} u(t) \leq & \left[\int_0^1 \frac{1}{3} s \left(k^{2/3} \int_0^s \frac{\exp\left(\int_\tau^s \frac{2\frac{1}{\sqrt[3]{k}}}{3(\sigma+1)} d\sigma \right)}{\tau+1} d\tau \right) ds + 6 \right. \\ & \left. \times \frac{\exp\left(\frac{2}{3} \frac{1}{\sqrt[3]{k}} \int_0^t \frac{1}{s+1} ds \right)}{1 - \int_0^1 s \exp\left(\frac{2}{3} \frac{1}{\sqrt[3]{k}} \int_0^s \frac{1}{\sigma+1} d\sigma \right) ds} + \int_0^t \frac{k^{2/3} \exp\left(\frac{2}{3} \frac{1}{\sqrt[3]{k}} \int_\tau^t \frac{1}{s+1} ds \right)}{3(\tau+1)} d\tau \right]^{\frac{1}{3}} \\ = & \left[\frac{\left(\frac{k\left(3\left(2^{\frac{2}{3\sqrt[3]{k}}+2} - 3\right)\sqrt[3]{k} - 2\right)}{4\left(9k^{2/3} + 9\sqrt[3]{k} + 2\right)} + 6 \right) (t+1)^{\frac{2}{3\sqrt[3]{k}}}}{1 - \frac{3\left(3\sqrt[3]{k} + 2^{\frac{2}{3\sqrt[3]{k}}+2}\right)\sqrt[3]{k}}{2\left(9k^{2/3} + 9\sqrt[3]{k} + 2\right)}} + \frac{1}{2} k \left((t+1)^{\frac{2}{3\sqrt[3]{k}}} - 1 \right) \right]^{\frac{1}{3}}. \end{aligned}$$

References

[1] A. Abdeldaim and M.Yakout, On some new inequalities of Gronwall-Bellman-Pachpatte type, *Appl. Math. Comput.*, **217**, (2011), 7887-7899.

[2] R. Bellman, The stability of solutions of linear differential equations, *Duke Math. J.*, **10**, (1943), 643 -647.

[3] M.B. Dhakne and G.B. Lamb, On globle existance of abstrct nonlinear integrodifferential equations, *Indian J. of pure and appl. math.*, **33**(5), 665-676.

[4] H. El-Owaidy, A. Ragab and A. Abdeldaim,On some new integral inequalities of Growall Bellman type, *Appl. Math. Comput.*, **106**, (1999), 289-303.

[5] Fangcui Jiang and Fanwei Meng, Explicit bounds on some new nonlinear integral inequality with delay, *J. Comput. Appl. Math.*, **205**, (2007), 479-486.

[6] Fan Wei Meng and Wei Nian Li, On some new integral inequalitise and their applications, *Appl. Math. Comput.*, **148**, (2004), 381-392.

[7] H. T. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Ann. Math.*, **2**, (1918-1919), 292-296.

[8] Lianzhong Li, Fanwei Mengb, Leliang Hea;Some generalized integral inequalities and their applications, *J. Math. Anal. Appl.*, **372**, (2010), 339-349.

[9] B.G. Pachpatte, A note on Gronwall type integral and integro differential inequalities, *Tamkang J. Math.*, **8**, (1977), 53-59.

[10] B.G. Pachpatte, On some fundamental integro differential inequalities for differential equations, *Chinese J. Math.*, **6**(1), (1978), 17-23.

[11] B.G. Pachpatte, Inequalities for Differential and Integral Equations, *Academic Press, New York and London*, 1998.

[12] B.G. Pachpatte, Integral and finite difference inequalities and applications, *North-Holland Mathematics studies*, **205**, 2006.

[13] H.L. Tidke and M.B. Dhakne, Existance of solutions for nolinear mixed type integrodifferential equation of second order, *Surveys in Mathematics and its Applications*, **5**, (2010), 61-72

[14] Young-Ho, Kim, Gronwall-Bellman and Pachpatte type integral inequalities with applications, *Nonlinear Anal.*, **71**, (2009), e2641-e2656.