Theoretical Mathematics & Applications, vol.5, no.1, 2015, 33-38
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2015

On Extension of Borel-Caratheodory theorem

Deborah Olufunmilayo Makinde[1] and S. Adewale[2]

Abstract

In this paper, we investigated the closure of addition and multiplication of functions in the Borel-Caratheodory theorem.

Mathematics Subject Classification: 30C45
Keywords: Analytic; Borel-Caratheodory; Maximum modulus principle

1 Introduction

A function of complex variable is said to be analytic at a point if the derivative exist not only at a point but also at every point in the neighbourhood of that point.

Let $x + iy$ be a complex number then the real part of the complex number is x. that is $R(x + iy) = x$.
Let $w = f(z)$ be a complex function, where $z = x + iy$ and $w = u + iv$ then $u = u(x, y)$ and $v = v(x, y)$. Hence $w = f(z) = u(x, y) + iv(x, y)$ and the function $f(z)$ is said to have a real u, denoted by $u = \text{Re } f$.

[1] Department of Mathematics, Obafemi Awolowo University, Ile-Ife 220005, Nigeria.
E-mail: domakinde.comp@gmail.com; dmakinde@oauife.edu.ng
[2] Department of Mathematics, Obafemi Awolowo University, Ile-Ife 220005, Nigeria.

Article Info: *Received* : August 23, 2014. *Revised* : October 9, 2014.
Published online : January 10, 2015.

Lemma 1 [1] *Let a function be analytic on a closed disc of radius R centered at the origin. Suppose that $r < R$. Then we have the following inequality;*

$$\|f\|_r \leq \frac{2r}{R-r} \quad sup_{|z|\leq r}\ Re\ f(z) + \frac{R+r}{R-r}|f(0)|$$

Here, the norm on the left hand side denotes the maximum value of f in the closed disc;

$$\|f\|_r = max_{|z|\leq r} \quad |f(z)| = max_{|z|=r} \quad |f(z)|,$$

where the last equality is due to the maximum modulus principle.

2 Main Results

Theorem 2.1. *Let functions f_1 and f_2 be analytic on a closed disc of radius R centered at the origin. Suppose that $r < R$. Then, we have the following inequality;*

$$|f_1(z) + f_2(z)| \leq \frac{R+r}{R-r}\bigg(f_1(0) + f_2(0)\bigg) + \frac{2r}{R-r}A_1 + \frac{2r}{R-r}A_2$$

Where

$$\|f\|_r = max_{|z|\leq r} \quad |f(z)| = max_{|z|=r} \quad |f(z)|$$

Proof. Define A_1 and A_2 by

$$A_1 = sup_{|z|\leq R} \ \text{Re}\ f_1(z)$$

$$A_2 = sup_{|z|\leq R} \ \text{Re}\ f_2(z)$$

First, assume that $f(0) = 0$
Let $f_1(z) + f_2(z) = F(z) \quad and \quad A_1 + A_2 = A'$
Define the function $g(z)$ by

$$g(z) = \frac{f_1(z) + f_2(z)}{z[2(A_1 + A_2)] - \{f_1(z) + f_2(z)\}}$$

$$g(z) = \frac{F(z)}{z[2A' - F(z)]}$$

This function has a removable singularity at $z = 0$. The factor $2A' - F(z) \neq 0$ because

$$\text{Re } \{2A' - F(z)\} = 2A' - \text{Re } F(z) \geq A'$$

Therefore, we have that g is analytic in the disc $\{z \in C; |z| \leq R\}$. If z is on the boundary of the disc, then

$$|g(z)| = \left|\frac{F(z)}{z(2A' - F(z))}\right|$$

$$|g(z)| = \left|\frac{1}{z}\right| \; \left|\frac{F(z)}{(2A' - F(z))}\right| \tag{1}$$

But

$$|2A' - F(z)| = |F(z) - 2A'| \geq |F(z)| \tag{2}$$

Using (2) in (1) we obtain

$$|g(z)| = \left|\frac{1}{z}\right| \; \left|\frac{F(z)}{F(z)}\right|$$

using the principle of maximum modulus, we have

$$|g(z)| \leq \frac{1}{R}$$

for any complex number w with $|w| = r$

$$\begin{aligned}
\Big|g(w)\Big| &= \left|\frac{F(w)}{w(2A' - F(w))}\right| \leq \frac{1}{R} \\
\Big|g(w)\Big| &= \frac{1}{r}\left|\frac{F(w)}{2A' - F(w)}\right| \leq \frac{1}{R} \\
\frac{|F(w)|}{|2A' - F(w)|}\Bigg| &\leq \frac{r}{R} \\
|F(w)| &\leq \frac{r}{R}|2A' - F(w)| \leq \frac{r}{R}[2A' + |F(w)|] \\
|F(w)| &\leq \frac{r}{R}[2A' + |F(w)|] \\
R|F(w)| &- r|F(w)| \leq 2A'r \\
(R - r)|F(w)| &\leq 2A'r \\
|F(w)| &\leq \frac{2A'r}{R - r} \\
\|F(w)\|r &\leq \frac{2A'r}{R - r}
\end{aligned}$$

In general case, where $f(0)$ does not necessarily vanish, then

$$h_1(z) = f_1(z) - f_1(0) \ \ and \ \ h_2(z) = f_2(z) - f_2(0)$$

by the law of triangular inequality, we have

$$Sup_{|z|\leq R} \text{ Re } h_1(z) \leq Sup_{|z|\leq R} \text{ Re } f_1(z) + |f_1(0)| \tag{3}$$

$$Sup_{|z|\leq R} \text{ Re } h_2(z) \leq Sup_{|z|\leq R} \text{ Re } f_2(z) + |f_2(0)| \tag{4}$$

Adding (3) and (4)

$$Sup_{|z|\leq R} \text{ Re } [h_1(z) + h_2(z)] \leq Sup_{|z|\leq R} \text{ Re } [f_1(z) + f_2(z)] + f_1(0) + f_2(0)$$

Let $h_1(z) + h_2(z) = H(z) = f_1(z) - f_1(0) + f_2(z) - f_2(0) = (f_1 + f_2)(z) - (f_1 + f_2)(0)$

$$Sup_{|z|\leq R} \text{ Re } H(z) \leq Sup_{|z|\leq R} \text{ Re } F(z) + |F(0)|$$

Where $f_1(0) + f_2(0) = F(0)$
Because $H(0) = h_1(0) + h_2(0) = 0$, we can say that

$$|F(z) - F(0)| \leq \frac{2r}{R-r}(A' + |F(0)|).$$

If $|z| \leq r$, furthermore

$$|F(z) - F(0)| \geq |F(z)| - |F(0)|$$

$$|F(z)| - |F(0)| \leq |F(z) - F(0)|$$

$$|F(z)| - |F(0)| \leq \frac{2r}{R-r}(A' + |F(0)|)$$

$$|F(z)| \leq |F(0)| + \frac{2r}{R-r}(A' + |F(0)|)$$

$$|F(z)| \leq \left(1 + \frac{2r}{R-r}\right)|F(0)| + \frac{2r}{R-r}A'$$

simplifying, we obtain

$$|F(z)| \leq \frac{R+r}{R-r}|F(0)| + \frac{2r}{R-r}A'$$

and by hypothesis, we have

$$|F_1(z)| \leq \frac{R+r}{R-r}\left(|f_1(0) + f_2(0)|\right) + \frac{2r}{R-r}A_1 + \frac{2r}{R-r}A_2$$

This completes the proof proof of Theorem 1. $\square$

Theorem 2.2. *Let functions f_1 and f_2 be analytic on a closed disc of radius R centered at the origin. Suppose that $r < R$. Then, we have the following inequality;*

$$\|f_1 f_2\| r \leq \frac{2r}{R-r} A + \frac{R+r}{R-r} |f(0)\ f(0)|$$

Where

$$\|f\| r = max_{|z| \leq r}\ |f(z)| = max_{|z|=r}\ |f(z)|$$

Proof. Let $A = Sup_{|z| \leq R}$ Re $\{(f_1(z)\ f_2(z))\}$
First assume that $f(0) = 0$
We define the function g by

$$g(z) = \frac{f_1(z) f_2(z)}{z[2A - \{f_1(z) f_2(z)\}]}$$

where $K(z) = f_1(z) f_2(z)$
The function has a removable singularity at $z = 0$ then the factor
$2A - K(z) \neq 0$ because
Re $\{2A - K(z)\} = 2A -$ Re $\{K(z)\} \geq A$
Therefore, $2A -$ Re $\{K(z)\} \geq A$
Therefore, we have that g is analytic in the disc $[z \in C : |z| \leq R]$. If z is on the boundary of this disc then

$$|g(z)| = \left| \frac{K(z)}{z[2A - [k(z)]} \right| \tag{5}$$

$$|g(z)| = \left| \frac{1}{z} \right| \left| \frac{K(z)}{[2A - k(z)]} \right|$$

$$|2A - K(z)| = |K(z) - 2A| \geq K(z) \tag{6}$$

Using 5 and 6 we obtain

$$|g(z)| = \left| \frac{1}{z} \right| \frac{K(z)}{K(z)} \right|$$

By the principle of maximum modulus

$$|g(z)| \leq \frac{1}{R}$$

and following the proof of Theorem 1 we obtain

$$|K(z)| \leq \frac{2r}{R-r} A + \frac{R+r}{R-r} |K(0)|$$

$$K(z) = f_1(z)\ f_2(z)$$
$$|f_1(z)\ f_2(z)| \leq \frac{2r}{R-r}A + \frac{R+r}{R-r}|f(0)\ f(0)|$$
$$\|f_1\ f_2\|r \leq \frac{2r}{R-r}A + \frac{R+r}{R-r}|f(0)\ f(0)|$$

this completes the proof of Theorem 2. □

3 Conclusion

We conclude that the Borel-Caratheodory theorem is closed under the operation of addition and multiplication of analytic functions.

Results 1 and 2 generalize the lemma above.

References

[1] Borel-Catheodory theorem-Wikipedia free Encyclopedia.

[2] Mark J.A. and Athanassios S.F., *Complex Variables Introduction and Application*, Cambridge University Press, 1977.