Journal of Applied Mathematics & Bioinformatics, vol.5, no.1, 2015, 45-51
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2015

Existence and multiplicity of solutions to boundary value problems for nonlinear high-order differential equations

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Abstract

In this paper, we investigate the multiple positive solutions for the boundary value problem of nonlinear differential equations. The arguments are based upon the fixed point theorem of cone expansion and compression with norm type.

Mathematics Subject Classification: O175

Keywords: Higher order boundary value problem；Positive solution；cone fixed point theorem；multiple solutions

1 Introduction

In recent years, many researchers study the existence and multiplicity of

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Article Info: *Received* : November 1, 2014. *Revised* : December 9, 2014.
Published online : January 15, 2015.

solutions to boundary value problems for nonlinear high-order ordinary differential equations, especially for even number order equation. They have obtained a lot of satisfactory results and most authors study the problems under the conjugate boundary conditions or simpler boundary conditions [1]-[3]. However,it is not common that the study on the boundary problems have high-order derivative right side boundary conditions. This paper study the boundary value problems, for a class of differential equations with 2m order right-hand side boundary conditions. In our paper, we give some sufficient conditions for the existence of one or two positive solutions, for the boundary value problems using the properties of the corresponding Green's function and cone expansion and compression fixed point theorem.

2 Preliminary Notes

In this paper, we concerned on the following nonlinear higher-order boundary value problem

$$\begin{cases} (-1)^m y^{(2m)}(x) = f(x, y(x)), a.e. x \in (0,1) \\ y^{(i)}(0) = 0, 0 \le i \le m-1 \\ y^{(j)}(1) = 0, m \le j \le 2m-1 \end{cases} \tag{1}$$

We assume that

(H_1) $f(x,y)$ is continuous and nonnegative on $[0,1]\times[0,+\infty)$.

(H_2) $f(x,y)$ is not equal to zero any where for any compact subinterval

in $[0,1]$, and $\int_0^1 x^{m-1} f(x,y)\mathrm{d}x < +\infty$.

Theorem 2.1 Let B be a Banach space and $K \subset B$ a cone in B. Assume that

Ω_1, Ω_2 are open subset of B with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$.

Let $\Phi : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(i) $\|\Phi y\| \le \|y\|, y \in K \cap \partial\mathbf{\Omega}_1, \|\Phi y\| \ge \|y\|, y \in K \cap \partial\mathbf{\Omega}_2$; or

(ii) $\|\Phi y\| \ge \|y\|, y \in K \cap \partial\mathbf{\Omega}_1, \|\Phi y\| \le \|y\|, y \in K \cap \partial\mathbf{\Omega}_2$.

Then, Φ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To obtain positive solutions for the problem (1). We state some properties of Green's function for (1).

As shown in [5], the problem (1) is equivalent to the integral equation

$$y(x) = \int_0^1 G(x,s) f(s, y(s))\mathrm{d}s$$

where

$$G(x,s) = \begin{cases} \dfrac{1}{[(m-1)!]^2}\displaystyle\int_0^s u^{m-1}(u+x-s)^{m-1}\mathrm{d}u, 0 \le s \le x \le 1, \\ \dfrac{1}{[(m-1)!]^2}\displaystyle\int_0^x u^{m-1}(u+s-x)^{m-1}\mathrm{d}u, 0 \le x \le s \le 1. \end{cases} \tag{2}$$

Moreover, the following results have been recently offered by [5].

Lemma 2.1 For any $x, s \in [0,1]$, we have

$$\alpha(x)\, g(s) \le G(x,s) \le \beta(x) g(s), \quad |\frac{\partial G(x,s)}{\partial x}| \le cs^{m-1},$$

Where

$$c = \frac{2^{m-1}}{[(m-1)!]^2}; \; \alpha(x) = \frac{x^m}{2m-1}; \quad \beta(x) = \frac{x^{m-1}}{m}; \quad g(s) = \frac{1}{[(m-1)!]^2} s^m \quad .$$

Let $\|y\| = \sup_{0 \le y \le 1} |y(x)|$, $\alpha = \min_{\frac{1}{4} \le x \le \frac{3}{4}} \alpha(x)$, $\|\beta\| = \max_{0 \le x \le 1} \beta(x)$. Define the cone K in

Banach

space $C[0,1]$, given by

$$K=\{y\in C[0,1]: y(x)\geq 0, \min_{\frac{1}{4}\leq x\leq\frac{3}{4}} y(x)\geq \alpha\|y\|/\|\beta\|\},$$

We define the operator $\Phi:K\to K$ by

$$(\Phi y)(x)=\int_0^1 G(x,s)f(s,y(s))\mathrm{d}s.$$

For any $y\in K$,we have

$$\min_{\frac{1}{4}\leq x\leq\frac{3}{4}}\Phi y(x) \geq \min_{\frac{1}{4}\leq x\leq\frac{3}{4}}\int_0^1 \alpha(x)\mathrm{g}(\mathrm{s})f(s,y(s))\mathrm{d}s$$

$$\geq \frac{\alpha}{\|\beta\|}\max_{0\leq x\leq 1}\int_0^1 \mathrm{G}(x,\mathrm{s})f(s,y(s))\mathrm{d}s=\frac{\alpha}{\|\beta\|}\|\Phi y\|$$

This implies $\Phi(K)\subset K$.

We can easily obtain it, $\Phi:K\to K$ is a completely continuous mapping.

3 Main Results

Let $A=\int_0^1 g(s)\mathrm{d}s, B=\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)\mathrm{d}s$, we denote $\Omega_{\mathrm{r}}=\{y\in K,\|y\|<r\}$ ($r>0$),

then $\partial\Omega_{\mathrm{r}}=\{y\in K,\|y\|=r\}$.

Theorem 3.1 Assume that $(H_1),(H_2)$ hold, if there exist two positive numbers m_1,m_2 and $m_1\neq m_2$, let $0<m_1,m_2<+\infty$ such that

（Ⅰ） $\forall x\in[0,1], y\in[0,m_1]$, $f(x,y)\leq\frac{m_1}{\|\beta\|}A^{-1}$;

（II） $\forall x \in [\frac{1}{4}, \frac{3}{4}], y \in [\frac{\alpha}{\|\beta\|} m_2, m_2], f(x,y) \geq \frac{m_2}{\alpha} B^{-1}$.

then the problem(1) has at least one positive solution, satisfying

$$\min\{m_1, m_2\} \leq \|y\| \leq \max\{m_1, m_2\}.$$

Proof. Let $m_1 < m_2$, then $\Omega_{m_1} = \{y \in K, \|y\| < m_1\}$ and $\Omega_{m_2} = \{y \in K, \|y\| < m_2\}$

$$\forall y \in \partial\Omega_{m_1}, \ \|\Phi y(x)\| \leq \max_{0 \leq x \leq 1} \int_0^1 \beta(x) g(s) f(s, y(s)) ds \leq m_1 A^{-1} \int_0^1 g(s) ds = \|y\|$$

$$\forall y \in \partial\Omega_{m_2}, \text{ then } \min_{\frac{1}{4} \leq x \leq \frac{3}{4}} y(x) \geq \alpha \|y\| / \|\beta\| = \frac{\alpha m_2}{\|\beta\|}.$$

It is easy to see

$$\|\Phi y(x)\| \geq \min_{\frac{1}{4} \leq x \leq \frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \alpha(x) g(s) f(s, y(s)) ds \geq m_2 B^{-1} \int_{\frac{1}{4}}^{\frac{3}{4}} g(s) ds = \|y\|.$$

In view of Theorem 2.1, we know that Φ has a fixed point $y \in \overline{\Omega}_{m_2} \setminus \Omega_{m_1}$ .That is to say, y is a positive solution of (1)，and $m_1 \leq \|y\| \leq m_2$.

Inference 3.1 Assume that $(H_1), (H_2)$ hold. Our assumptions throughout are，

(III) $\lim_{y \to 0^+} \max_{x \in [0,1]} \frac{f(x,y)}{y} < \frac{A^{-1}}{\|\beta\|}$ and $\lim_{y \to +\infty} \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(x,y)}{y} > \frac{B^{-1}}{\alpha^2} \|\beta\|$,

then the problem(1) has at least one positive solution.

Proof. Using (III) we have, for any $x \in [0,1]$, there exists a sufficiently small $m_1 > 0$, such that

$$f(x,y) \leq \frac{A^{-1}}{\|\beta\|} m_1, \quad y \in (0, m_1] \tag{3}$$

Following from $\lim_{y \to +\infty} \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(x,y)}{y} > \frac{B^{-1}}{\alpha^2} \|\beta\|$, that there exists a sufficiently large

$m_2 > 0$, for any $x \in [\frac{1}{4}, \frac{3}{4}]$ such that,

$$f(x,y) \geq \frac{B^{-1}}{\alpha^2} \|\beta\| y \geq \frac{m_2}{\alpha} B^{-1}, \quad y \in [\frac{\alpha}{\|\beta\|} m_2, m_2] \tag{4}$$

Thus, theorem 3.1 implies，then the problem(1) has at least one positive solution.

Inference 3.2 Assume that $(H_1),(H_2)$ hold，Our assumptions throughout are，

$$\text{(IV)} \quad \lim_{y\to 0^+} \min_{x\in[\frac{1}{4},\frac{3}{4}]} \frac{f(x,y)}{y} > \frac{B^{-1}}{\alpha^2} \|\beta\|, \quad \text{and} \quad \lim_{y\to +\infty} \max_{x\in[0,1]} \frac{f(x,y)}{y} < \frac{A^{-1}}{\|\beta\|},$$

then the problem (1) has at least one positive solution.

Proof. Follows from (IV), there exists $0 < m_1 < m_2$, for any $x \in [\frac{1}{4}, \frac{3}{4}]$, such that

$$\frac{f(x,y)}{y} \geq \frac{B^{-1}}{\alpha^2} \|\beta\|, \quad y \in [0, m_1]$$

Consequently, we have for any $x \in [\frac{1}{4}, \frac{3}{4}]$,

$$f(x,y) \geq \frac{B^{-1}}{\alpha^2} \|\beta\| y \geq \frac{m_1}{\alpha} B^{-1}, \quad y \in [\frac{\alpha}{\|\beta\|} m_1, m_1] \tag{5}$$

Following from $\lim_{y\to +\infty} \max_{x\in[0,1]} \frac{f(x,y)}{y} < \frac{A^{-1}}{\|\beta\|}$, there exists $\varepsilon_1 > 0$, such that $\frac{A^{-1}}{\|\beta\|} - \varepsilon_1 > 0$, and

$$f(x,y) \leq (\frac{A^{-1}}{\|\beta\|} - \varepsilon_1) y, \quad y \geq m_2 .$$

Let $c = \max\{f(x,y) : 0 \leq y \leq m_2\}$, then it is obvious to see

$$f(x,y) \leq c + (\frac{A^{-1}}{\|\beta\|} - \varepsilon_1) y, y \in [0, +\infty) .$$

Choose $r > \max\{m_2, c(\varepsilon_1 \|\beta\|)^{-1}\}$ .Let $\Omega_r = \{y \in C[0,1] : \|y\| < r\}$. then for

for any $y \in \partial\Omega_r$, we have

$$f(x,y) \le c + (\frac{A^{-1}}{\|\beta\|} - \varepsilon_1)\|y\| \le \frac{A^{-1}}{\|\beta\|} r . \tag{6}$$

Thus, we obtain there exist $x^{'} \in [0,1]$ and $r \ge m_2$, for all $x \in [0,1]$ and $y \in [0,r]$,

$f(x,y) \le \frac{A^{-1}}{\|\beta\|} r$. By Theorem 3.1 we know, then the problem(1) has at least one positive solution.

Theorem 3.2 Assume that $(H_1),(H_2)$ hold，for all $m_2 > 0$，satisfy the (Ⅱ). Let

$$\lim_{y\to 0^+} \max_{x\in[0,1]} \frac{f(x,y)}{y} < \frac{A^{-1}}{\|\beta\|}, \lim_{y\to +\infty} \max_{x\in[0,1]} \frac{f(x,y)}{y} < \frac{A^{-1}}{\|\beta\|},$$

Then the problem (1) has two positive solutions.

Proof. We can prove as the same as Corollary 1 and 2.Following from there exist $0 < m_1 < m_2 < r_1$， for any $x \in [0,1]$,

$$f(x,y) \le \frac{A^{-1}}{\|\beta\|} m_1, \quad y \in (0, m_1].$$

and exist $x^{'} \in [0,1]$ and $r_1 \ge m_2$, such that

$$f(x,y) \le \frac{A^{-1}}{\|\beta\|} r_1, \quad \text{when } x \in [0,1] \text{ and } y \in [0, r_1]$$

Then by (IV)，we know form Theorem 3.1，Then the problem (1) has two positive solutions y_1, y_2, and satisfy $0 < m_1 < \|y_1\| < m_2 < \|y_2\| < r_1$.

Theorem 3.3 Assume that $(H_1),(H_2)$ hold，for all $m_1 > 0$，satisfy the (IV). Let

$$\lim_{y\to 0^+}\min_{x\in[\frac{1}{4},\frac{3}{4}]}\frac{f(x,y)}{y}>\frac{\|\beta\|}{\alpha^2}B^{-1}, \quad \text{and} \quad \lim_{y\to +\infty}\min_{x\in[\frac{1}{4},\frac{3}{4}]}\frac{f(x,y)}{y}>\frac{\|\beta\|}{\alpha^2}B^{-1}.$$

Then the problem (1) has two positive solutions y_1, y_2, and satisfy

$$0<m_1^{''}<\|y_1\|<m_2<\|y_2\|<m_2^{''}.$$

Proof. There exist $0<m_1^{'}<m_1<m_2^{''}$, for any $x\in[\frac{1}{4},\frac{3}{4}], y\in[\frac{\alpha}{\|\beta\|}m_1^{'},m_1^{'}]$, we have

$$\min_{x\in[\frac{1}{4},\frac{3}{4}]} f(x,y)\geq\frac{B^{-1}}{\alpha^2}\|\beta\|y\geq\frac{m_1^{'}}{\alpha}B^{-1}.$$

Hence we obtain for any

$$\min_{x\in[\frac{1}{4},\frac{3}{4}]} f(x,y)\geq\frac{B^{-1}}{\alpha^2}\|\beta\|y\geq\frac{m_2^{''}}{\alpha}B^{-1}, \quad x\in[\frac{1}{4},\frac{3}{4}], y\in[\frac{\alpha}{\|\beta\|}m_2^{''},m_2^{''}].$$

Then by (IV), we know from Theorem 3.1. Then the problem (1) has two positive solutions y_1, y_2, satisfy $0<m_1^{'}<\|y_1\|<m_1<\|y_2\|<m_2^{'}$.

4 Conclusion

We study a class of higher order nonlinear differential equation boundary valu e problem and give some sufficient conditions for existence of positive solutions u sing the cone fixed point theorems. It enriches the results of the related literatures further.

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