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The Three Level Two Point Scheme for the Vibrating Membrane Problem

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Abstract

Partial differential equations $P.D.E$s govern mechanical systems which contain multiple parameters. Linear and certain non-linear $P.D.E$s can be solved using such analytic methods as separation of variables. However, certain $P.D.E$s exist, which cannot be solved analytically. This calls for an alternative method of solution. Finite difference Methods (F.D.Ms) provide a realistic physical approach towards the modeling of these problems. The wave equation can be solved using the explicit and therefore conditionally stable Forward in Time and Centered in Space $F.T.C.S$ F.D.M. It is shown here that the Local Truncation Error (LTE) in the result is relatively negligible. An implicit scheme, which is unconditionally stable, is developed and the conclusion made that the scheme can be used to solve other non-linear P.D.Es with a higher degree of stability.

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1 Introduction

The finite difference method was developed by Thom and Apelt [6] in the 1920s, and was used to solve non-linear hydrodynamic equations. Much of the work on finite difference schemes is presented in Jain [1], Rahman [4] and Morton et al [3]. In Vrushali et al [7], the numerical solution to a P.D.E is an approximation to the exact solution, and the LTE is that difference which results when the exact solution is substituted into the finite difference formula.

1.1 The model: 2-Dimensional wave equation

The solution $Z(x, y, t)$ of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right], 0 < x \leq L, 0 < y \leq H, t > 0 \tag{1}$$

represents the approximate displacement of a point (x,y) on the membrane at time t from rest.

1.2 Methodology

This model first considers the analytic solution of the wave equation. The FTCS, which is explicit, is considered in Ronoh et al [5]. Here, an implicit Scheme is developed and the LTE associated with it compared with that of the FTCS.

1.3 The Analytic Solution

It can be shown that the solution $z(x, y, t)$ of the wave equation is given by

$$z(x,y,t) = \left[A_1 \cos(a\sqrt{\frac{n^2}{L^2}+\frac{m^2}{H^2}})\pi t + A_2 \sin(a\sqrt{\frac{n^2}{L^2}+\frac{m^2}{H^2}})\pi t\right]\left[(\sin\frac{n\pi x}{L})(\sin\frac{m\pi y}{H})\right] \tag{2}$$

Here, L and H are the respective spatial dimensions in the $x-$ and $y-$ directions.

2 The Numerical Solution

It is supposed that the weight of the membrane, after it is stretched, is a known function $w(L, H)$ (density ρ for this case), and $(L \times H)$ is a property which is proportional to the area of the membrane. The change in mass,ΔM, is given by

$$\Delta M = \frac{w(L,H)\Delta x \Delta y}{g} \tag{3}$$

The acceleration produced in ΔM by these forces and by the portion of the distributed load is approximately

$$\frac{\partial^2 z}{\partial t^2} = \frac{Tg}{w(L,H)}\left[\frac{\partial}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial}{\partial y}\frac{\partial z}{\partial y}\right] \tag{4}$$

Therefore

$$\frac{\partial^2 z}{\partial t^2} = \frac{Tg}{w(L,H)}\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right] \tag{5}$$

which gives an expression that determines a in equation (1). Much of this is explained in Ronoh et al [5].

2.1 The Forward in Time Centered in Space Scheme

The *F.T.C.S* is obtained by expressing equation 1 as

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) = a\frac{\partial}{\partial x}\left(a\frac{\partial z}{\partial x}\right) + a\frac{\partial z}{\partial y}\left(a\frac{\partial z}{\partial y}\right) \tag{6}$$

Further simplification with $S = \frac{\partial z}{\partial t}, R = a\frac{\partial z}{\partial x}$ and $Q = a\frac{\partial z}{\partial y}$ leads to

$$\frac{\partial S}{\partial t} = a\frac{\partial R}{\partial x} + a\frac{\partial Q}{\partial y} \tag{7}$$

The scheme is then derived as

$$\frac{1}{k}\Delta_t(S^n_{m,l}) = \frac{a}{h_1}\mu\delta_x(R^n_{m,l}) + \frac{a}{h_2}\mu\delta_y(Q^n_{m,l}) \tag{8}$$

where k is the time step, and h_1 and h_2 are the spatial dimensions in the $x-$ and $y-$ directions respectively. $\mu\delta_x$ and $\mu\delta_y$ are the respective averaging operators. To evaluate (See Ronoh et al), and for the special case where the spatial dimensions are equal, one sets $\zeta = \frac{a^2k^2}{h^2}$. The FTCS is thus obtained as

$$Z^{n+2}_{m,l} = 2Z^{n+1}_{m,l} - (1+\zeta)Z^n_{m,l} + \frac{\zeta}{4}\left(Z^n_{m+2,l} + z^n_{m-2,l} + z^n_{m,l+2} + Z^n_{m,l-2}\right) \tag{9}$$

Equation (9) approximates the displacement of a single point $Z^{n+2}_{m,l}$ from the origin given the six points $Z^{n+1}_{m,l}, Z^n_{m,l}, Z^n_{m+2,l}, Z^n_{m-2,l}, Z^n_{m,l+2}$ and $Z^n_{m,l-2}$. Being explicit, the scheme is known to be conditionally stable, hence the call for a similar scheme, but which is implicit in nature.

2.2 The Implicit Three Level Two Point Scheme

This scheme, based on the Crank Nicolson, is obtained for equation (1) by approximating $z^n_{.,.}$ with $\frac{1}{2}(Z^n_{.,.} + Z^{n+1}_{.,.})$, and the derivative terms $R^n_{.,.}$ and $Q^n_{.,.}$ in equation (7) with the averages $\frac{1}{2}(R^n_{.,.} + R^{n+1}_{.,.})$ and $\frac{1}{2}(Q^n_{.,.} + Q^{n+1}_{.,.})$ respectively. We have the following result;

$$\begin{aligned} S^{n+1}_{m,l} &= S^n_{m,l} + \frac{ak}{4h_1}\left((R^{n+1}_{m+1,l} + R^n_{m+1,l}) - (R^{n+1}_{m-1,l} + R^n_{m-1,l})\right) \\ &\quad + \frac{ak}{4h_2}\left((Q^{n+1}_{m,l+1} + Q^n_{m,l+1}) - (Q^{n+1}_{m,l-1} + Q^n_{m,l-1})\right) \end{aligned} \tag{10}$$

Now

$$R^n_{m,l} = \frac{a}{2h_1}\mu\delta_x\left(Z^n_{m,l} + Z^{n+1}_{m,l}\right) \tag{11}$$

and

$$Q^n_{m,l} = \frac{a}{2h_2}\mu\delta_y\left(Z^n_{m,l} + Z^{n+1}_{m,l}\right) \tag{12}$$

from which we obtain

$$R^n_{m,l} = \frac{a}{2h_1}\left((Z^n_{m+1,l} + Z^n_{m-1,l}) + (Z^{n+1}_{m+1,l} + Z^{n+1}_{m-1,l})\right) \tag{13}$$

and

$$Q^n_{m,l} = \frac{a}{2h_2}\left((Z^n_{m+1,l} + Z^n_{m-1,l}) + (Z^{n+1}_{m+1,l} + Z^{n+1}_{m-1,l})\right) \tag{14}$$

respectively. We then compute $R^n_{m+1,l}, R^n_{m-1,l}, Q^n_{m,l-1}$ and $Q^n_{m,l-1}$ for the n^{th} time level, and $R^{n+1}_{m+1,l}, R^{n+1}_{m-1,l}, Q^{n+1}_{m,l-1}$ and $Q^{n+1}_{m,l-1}$ for the succeeding time level $n+1$. On substitution, and for the special case where $h_1 = h_2 = h$, we obtain

$$\begin{aligned} Z^{n+2}_{m,l} &- \frac{\xi^2}{4}\left(Z^{n+2}_{m+2,l} + Z^{n+2}_{m-2,l} + Z^{n+2}_{m,l+2} + z^{n+2}_{m,l-2}\right) \\ &= 2(1-\xi^2)Z^{n+1}_{m,l} + \xi^2\left(Z^{n+1}_{m+2,l} + Z^{n+1}_{m-2,l} + z^{n+1}_{m,l+2} + z^{n+1}_{m,l-2}\right) \\ &\quad - (1+\xi^2)Z^n_{m,l} + \frac{\xi^2}{4}\left(Z^n_{m+2,l} + z^n_{m-2,l} + Z^n_{m,l+2} + Z^n_{m,l-2}\right) \end{aligned} \tag{15}$$

where $\xi = \frac{ak}{2h}$. Implicit schemes, which include the Crank Nicolson's, are known to be unconditionally stable. The Scheme in equation (15) is implicit in that the five points $Z^{n+2}_{.,.}$ at the time level $t+2$ are determined simultaneously given approximate values of $Z^{n+1}_{.,.}$ and $Z^n_{.,.}$, at the time levels $t+1$ and t respectively. The scheme (15) is therefore characteristically stable as desired.

3 Case Study - Flow in a river

The case of a river with the average velocity of water as $6m/s$ is considered. The dimensions of the rectangular membrane (Table 1) are set to $(60\times 80)mm^2$ and with $n \geq 1, h = 5mm$, the behavior of the schemes (9) and (15) is examined relative to equation (2) .

3.1 Notes

1. For this study, a is set such that $a^2 = 6m/s$ and $h = 0.005m$, which yields $k = 0.000833333s$.

2. Outside the medium it is assumed that the displacement relative to the vibrating membrane is zero, so that for $n = -1$, equations (9) and (15) yield

$$Z^1_{m,l} = 2Z^0_{m,l} \tag{16}$$

3. For the initial condition $n = 0$, we take $k = 0$, which leads to

$$Z_{m,l}^{2} = 2Z_{m,l}^{1} - Z_{m,l}^{0} \tag{17}$$

4. The two preceding equations (above) lead to

$$Z_{m,l}^{2} = 3Z_{m,l}^{0} \tag{18}$$

In Table 1, there are 12 5mm steps $x_0, x_1, ..., x_{12}$ on the $x - axis$ and 18 5mm steps $y_0, y_1, ..., y_{15}$ on the $y - axis$. Thus, m=2,...,10 and l=2,...,14.

3.2 Discretization using the FTCS

From the notes, equation (9) leads to

$$Z_{m,l}^{n+2} = 2Z_{m,l}^{n+1} - pZ_{m,l}^{n} + q\left(Z_{m+2,l}^{n} + z_{m-2,l}^{n} + z_{m,l+2}^{n} + Z_{m,l-2}^{n}\right) \tag{19}$$

where $p = 1.1666666666667$ and $q = 0.0416666666667$.

3.3 Discretization using the Three Level two point scheme

From the notes, equation (15) leads to

$$\begin{aligned} &rZ_{m,l}^{n+2} - s\left(Z_{m+2,l}^{n+2} + Z_{m-2,l}^{n+2} + Z_{m,l+2}^{n+2} + z_{m,l-2}^{n+2}\right) \\ &\quad = uZ_{m,l}^{n+1} + v\left(Z_{m+2,l}^{n+1} + Z_{m-2,l}^{n+1} + z_{m,l+2}^{n+1} + z_{m,l-2}^{n+1}\right) \\ &\qquad - rZ_{m,l}^{n} + s\left(Z_{m+2,l}^{n} + z_{m-2,l}^{n} + Z_{m,l+2}^{n} + Z_{m,l-2}^{n}\right) \end{aligned} \tag{20}$$

where $r = 1.0416666666634$, $s = 0.0104166666658$, $u = 1.9166666666733$ and $v = 0.0416666666633$

3.4 The Local Truncation Error (LTE)

The LTE, $T_{m,l}^{n}$ will be obtained as

$$T_{m,l}^{n} = \left|z_{m,l}^{n} - Z_{m,l}^{n}\right| \tag{21}$$

where $z_{m,l}^{n}$ is the analytic solution. Table 1 and Table 2 give the LTEs as obtained for the FTCS and the Three Level Two Point schemes respectively.

Table 1: The LTE for the FTCS

$T^n_{3,l}$	$n=1$	$n=2$	$n=3$	$n=4$
$T^n_{3,0}$	0	0	0	0
$T^n_{3,1}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T^n_{3,2}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T^n_{3,3}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T^n_{3,4}$	0.0000000016314	0.0000000081558	0.792893196164	2.7751262175664
$T^n_{3,5}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T^n_{3,6}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T^n_{3,7}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T^n_{3,8}$	0	0	0	0
$T^n_{3,9}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T^n_{3,10}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T^n_{3,11}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T^n_{3,12}$	0.0000000016314	0.0000000081558	0.792893196164	2.7751262175664
$T^n_{3,13}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T^n_{3,14}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T^n_{3,15}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T^n_{3,16}$	0	0	0	0

4 Conclusion and Recommendation

The analysis of the local truncation error (LTE) shows that the implicit Three Level Two Point Scheme is more stable than the explicit forward time centred space (FTCS). The Three Level Two Point Scheme was obtained by approximating $z^{n+2}_{m,l}$ with the average $\frac{1}{2}(Z^n_{m,l} + Z^{n+1}_{m,l})$. Studies can be made on the accuracy and stability of a scheme that uses the average $\frac{1}{3}(Z^{n-1}_{m,l} + Z^n_{m,l} + Z^{n+1}_{m,l})$ in the approximation of $z^{n+2}_{m,l}$. Generalization can be made on the use of a finite average in the approximation of $z^{n+2}_{m,l}$, and further on the use and application of infinite averages with compact support.

Table 2: The LTE for the Three Level Two point Scheme

$T^n_{3,l}$	$n=1$	$n=2$	$n=3$	$n=4$
$T^n_{3,0}$	0	0	0.16863329379364	1.31996125103379
$T^n_{3,1}$	0.0000000006243	0.0000000031211	0.3121403719816	0.9346904347001
$T^n_{3,2}$	0.0000000011536	0.000000005767	0.1919367550859	0.715426716632
$T^n_{3,3}$	0.0000000015072	0.000000007535	0.2487875463199	0.9273286457012
$T^n_{3,4}$	0.0000000016314	0.0000000081558	0.2739144560973	1.0450498566985
$T^n_{3,5}$	0.0000000015072	0.000000007535	0.2529574895238	0.9273286457012
$T^n_{3,6}$	0.0000000011536	0.000000005767	0.1978415571988	0.7632235406071
$T^n_{3,7}$	0.0000000006243	0.0000000031211	0.1048022435579	0.3997790489252
$T^n_{3,8}$	0	0	0.00000000000094	0.00000000032239
$T^n_{3,9}$	0.0000000006243	0.0000000031211	0.104802243483	0.3997790138485
$T^n_{3,10}$	0.0000000011536	0.000000005767	0.1978415571145	0.7632235587443
$T^n_{3,11}$	0.0000000015072	0.000000007535	0.2529574831738	0.9644889923732
$T^n_{3,12}$	0.0000000016314	0.0000000081558	0.2739144486397	1.0450506374673
$T^n_{3,13}$	0.0000000015072	0.000000007535	0.248787015191	0.9269768779044
$T^n_{3,14}$	0.0000000011536	0.000000005767	0.1919361032533	0.71543940953927
$T^n_{3,15}$	0.0000000006243	0.0000000031211	0.3121832606954	0.9349673065078
$T^n_{3,16}$	0	0	0.16868936322900	1.32144063522519

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