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On the steady MHD couette flow between two infinite parallel plates in an uniform transverse magnetic field

D.W. Kiema[1], W.A. Manyonge[2], J.K. Bitok[3], R.K. Adenyah[4] and J.S. Barasa[5]

Abstract

In this paper we consider laminar viscous incompressible fluid between two infinite parallel plates when the upper plate is moving with constant velocity U_o and the lower plate is held stationary under the influence of inclined magnetic field. The resulting governing partial differential equation is solved by Sumudu Transform and the solution expressed in terms of Hartmann number. The analysis of this shows that, the velocity profile will decrease as the Hartmann number and magnetic inclination increases. This approach can be used to obtain solutions of ordinary differential equations in astronomy, Physics and in controlling engineering problems.

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[1] Department of Mathematics and Physical Sciences Pwani University, P.O. Box 195 - 80108 Kilifi, Kenya. E-mail: dwakstan79@yahoo.com
[2] Centre for Research for New and Renewable Energies Maseno University, P.O. Box 333, Maseno, Kenya. E-mail:wmanyonge@gmail.com
[3] Department of Mathematics and Computer Science, University of Eldoret, P.O. Box 1125, Eldoret, Kenya. E-mail: jacobbitok@yahoo.com
[4] E-mail: adenyarodgers@yahoo.com
[5] E-mail: joel_barasa@yahoo.com

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1 Introduction

The theoretical study of Magnetohydrodynamics (MHD) has been on recent years of great interest due to its wide spread applications on purification of crude oil in petroleum industry, separation of matter from fluids, MHD power generation, aerodynamics and designing of cooling systems with liquid metals among many applications. When an electrically conducting fluid is placed in a constant magnetic field, the motion of the fluid induces currents which create forces on the fluid. The production of these currents has led to the design of among other devices the MHD generators for electricity production. The equations which describe MHD flow are a combination of continuity equation and Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The resultant governing differential equations have to be solved either analytically or numerically.

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An exact solution of Navier-Stokes equation for plane couette flow between two parallel plates without suction was well discussed by Schlichting [4]. Sinha and Choudhary [6] analysed flow of viscous incompressible fluid between two parallel plates, one in uniform motion and the other at rest with suction at the stationary Plate. Singh and Ram [8] studied laminar flow of an electrically conducting fluid through a channel in the presence of a transverse magnetic field under the influence of a periodic pressure gradient and solved the resulting differential equation by the method of Laplace transform. Ram et.al. [9] have discussed Hall effects on heat and mass transfer flow through porous media. Kazuyuki [10] analysed inertia effects in two dimensional MHD channel flow while Al-Hadhrami [12] considered flow of fluids through horizontal channels of porous materials and obtained velocity expressions in terms of the Reynolds number. Ganesh [13] studied unsteady MHD Stokes flow of a viscous fluid between two parallel porous plates. They considered fluid being withdrawn through both walls of the channel at the same rate. Singh et.al. [14] studied Couette laminar viscous MHD fluid flow between two parallel infinite plates in presence of transverse magnetic field and solved this by Laplace Transform. Manyonge et al. [16] considered steady MHD flow between two infinite parallel porous plates in an inclined magnetic field and the resultant differential equation solved analytically by analytical method.

There are numerous integral transforms in literature used to solve differential equations. In Kihcman et al. [15] applied integral transform to partial differential equations with non-homogeneous forcing term and having singular variable data. In 1993, Watugala introduced a new transform which he named it as Sumudu Transform which he defined as

$$F(u) = S[f(t;u)] = \frac{1}{u}\int_0^\infty e^{-(\frac{t}{u})} f(t)\,dt, \quad \in u \in (-\tau_1, \tau_2)$$

and applied it to the solution of ordinary differential equations, see [11].

In this paper we consider two dimensional steady couette flow of an electrically conducting fluid between two infinite parallel plates under the influence of uniform transverse magnetic field. The resulting differential equation is solved by application of Sumudu Transform.

2 Mathematical Formulation

The basic concept describing magnetohydrodynamics phenomena can be described by considering an electrically conducting fluid moving with a velocity vector V**.** At right angles to this we apply a magnetic field, **B.** We then assume that steady flow conditions have been attained i.e. flow variables are independent of the time t. This condition is purely for analytic reasons so that no macroscopic charge density is being built up at any place in the system as well as all currents are constant in time. Because of the interaction of two fields, namely, velocity and magnetic fields, an electric field denoted by **E** is induced at right angles to both **V** and **B.** This electric field is given by

$$E_{ind} = V \times B \tag{1}$$

If we assume that the conducting fluid is isotropic in spite of the magnetic field, we can denote its electrical conductivity by the scalar quantity σ**.** By Ohms law, the density of the current induced in the conducting fluid denoted by J_{ind} is given by

$$J_{ind} = \sigma E_{ind} \tag{2}$$

or we can simply write this as

$$\mathrm{J_{ind}} = \sigma\ (\mathrm{V} \times \mathrm{B_{app}}) \tag{3}$$

Simultaneously occurring with the induced current is the induced ponderomotivforce or the Lorentz force F_{ind} which is given by

$$F_{\mathrm{ind}} = J_{ind} \times \mathrm{B_{app}} \tag{4}$$

.The Lorentz force is significant in determining the flow profile based on the dimensionless Hartmann number which is given by the ratio of the magnetic body force and the viscous force Ha = $(\mathrm{N.Re})^{1/2}$, where N $=\mathrm{Ha}^2/\mathrm{Re} = \sigma\ .\mathrm{L.B}^2\ /\ \rho\ \mathrm{U}$ stands for the nondimensional interaction parameter known as Stuart number which is defined as the ratio of electromagnetic to inertial forces, and this gives an estimate of the relative importance of a magnetic field of the flow. It is also relevant for flows of conducting fields e.g in fusion reactors, steel casters or plasmas. On the other hand, Re = UL $/\nu$ is the nondimensional hydrodynamic Reynolds number, so the Hartmann number can be rewritten as Ha= L.B. $(\sigma / \mu)^{1/2}$ where μ is the dynamic viscosity and ν is kinematic viscosity.

The Lorentz force will occur because, as an electric generator, the conducting fluid cuts the lines of the magnetic field. The vector **F** is the vector cross product of both **J** and **B** and is a vector perpendicular to the plane of both **J** and **B**. The induced force is parallel to **V** but in opposite direction. Laminar flow through a channel under uniform transverse magnetic field is important because of the use of MHD generator, MHD pump, crude oil purification and electromagnetic flow meter.

We now consider an electrically conducting, viscous, steady and incompressible fluid moving between two infinite parallel plates both kept at a constant distance h between them. The upper plate is moving with constant velocity U_o while the lower plate is kept stationary. The fluid is acted upon by a constant pressure gradient.

The governing equations for the flow of incompressible Newtonian fluid that we use in this study are the continuity equation and the momentum equations which are:

$$\nabla.V=0 \tag{5}$$

$$\rho[\frac{\partial V}{\partial t}+(V.\nabla)V]=-\nabla P+\mu\nabla^2 u+J\times B \tag{6}$$

Where ρ is the fluid density, P is the fluid pressure function acting on the fluid, μ is the fluid dynamic viscosity and $J\times B$ is the Lorentz force.

3 Non-dimensionalizing of the governing equations

We now consider unidirectional flow, in which we choose the axis of the channel formed by the two plates as the x axis and assume that flow is in this direction.

If V = u' (x$'$,y', z') **i** +v'(x$'$,y', z') **j** +w' (x$'$,y', z') **k** in which u$'$,v' *and* w' *are* the components of the velocity in $x-$, $y-$ and $z-$ directions respectively and primes denote dimensionless quantities. This now implies that $v' = w' = 0$ and that $u' \neq 0$. Then the continuity equation (5) yields $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=0$. But $\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=0$ so that $\frac{\partial u'}{\partial x'}=0$ from which we infer that u' is independent of x'. This makes the non- linear term [(**V**.∇)] **V** in the Navier-Stokes equations equal to zero because of the unidirectional flow assumption. Since we had assumed a steady flow, the flow variable does not depend on time. By assuming that the flow is two dimensional i.e. the flow variables are independent of $z-$ direction, then this means that , by choosing the axis of the channel as the x–axis, the governing equations of motion for two dimensional steady flow are

$$0=-\frac{1}{\rho}\frac{\partial p'}{\partial x'}+\nu\frac{\partial^2 u'}{\partial y'^2}+\frac{F_x}{\rho} \tag{7}$$

$$0=-\frac{1}{\rho}\frac{\partial p'}{\partial y'} \tag{8}$$

where $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity and F_x is the component of the magnetic force in the x - direction. We note that p is a function of x only and that pressure does not depend on y from equation (8). Also, assuming unidirectional flow $v' = w' = 0$ and $B_x = B_z = 0$ so that V = u' **i** and B = B_0 **j** where B_0 is the magnetic field strength component assumed to be applied to a direction perpendicular to fluid motion (y -direction). Therefore, using the cross product of (7) we obtain

$$F_x=\sigma\,[(u'\,i\times j\;B_o)\times j\;B_o] \tag{9}$$

This is easily shown to reduce to

$$\frac{F_x}{\rho} = -\frac{\sigma}{\rho} B_o^2 u' \tag{10}$$

Using equation (10) into equation (7) we obtain

$$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial x'} + \nu \frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma}{\rho} B_o^2 u' \tag{11}$$

or

$$\frac{d^2 u'}{dy'^2} - \frac{\sigma}{\mu} B_o^2 u' = \frac{1}{\mu}\frac{dp'}{dx'} \tag{12}$$

or

$$\frac{d^2 u'}{dy'^2} - \frac{\sigma}{\mu} B_o^2 \sin(\alpha) u' = \frac{1}{\mu}\frac{dp'}{dx'} \tag{13}$$

where we have taken ordinary derivatives instead of partial derivatives and α is the angle between **V** and **B**. Equation (13) is a general equation in that the two fields can be assessed at any angle $0 \le \alpha \le \pi$ and is solved subject to boundary conditions :-

$$u = 0 \quad y = -1 \qquad u = U_o \quad y = +1 \tag{BC}$$

We can now drop the primes for convenience in (12) ,(13) and (8) to have

$$\frac{d^2 u}{dy^2} - \frac{\sigma}{\mu} B_o^2 u = \frac{1}{\mu}\frac{dp}{dx} \tag{14}$$

or

$$\frac{d^2 u}{dy^2} - \frac{\sigma}{\mu} B_o^2 \sin(\alpha) u = \frac{1}{\mu}\frac{dp}{dx} \tag{15}$$

and

$$0 = -\frac{1}{\rho}\frac{dp}{dy} \tag{16}$$

Since the flow is couette then we shall have pressure gradient taken to be zero i.e. $\frac{dp}{dx}=0$ in (15). If we let l be the characteristic length, the dimensionless equation (15) reverts back to the non-dimensionless form and we can now define the dimensionless quantities as

$$x=\frac{x^{'}}{l},\quad y=\frac{y^{'}}{l},\quad p=\frac{p^{'}l^{2}}{\rho\nu^{2}},\quad u=\frac{u^{'}l}{\nu} \tag{17}$$

By substituting the quantities of (17) into (13) we get

$$\frac{d^{2}u}{dy^{2}}-\frac{\sigma}{\mu}B_{o}^{2}l^{2}\sin^{2}(\alpha)u=0 \tag{18}$$

or

$$\frac{d^{2}u}{dy^{2}}-M^{2}u=0 \tag{19}$$

Where $M=M^{*}\sin\alpha$ and $M^{*}=lB_{o}(\frac{\sigma}{\mu})^{\frac{1}{2}}=Ha$ is the Hartmann Number given by $Ha^{2}=\frac{\sigma B_{o}^{2}l^{2}}{\mu}$. We now solve equation (19) by using Sumudu Transform method by first evaluating initial conditions with the help of boundary conditions:-

$$u=0 \quad y=-1 \qquad\qquad u=U_{o} \quad y=+1 \qquad\qquad \text{(BC)}$$

This is a second order differential equation which can be transformed to its Sumudu equivalent as $\frac{G(u)-y(0)}{u^{2}}-\frac{y'(0)}{u}-M^{2}G(u)=0$. Multiplying all through by u^{2} and on rearranging we obtain

$$G(u)-y(0)-u\,y'(0)-M^{2}u^{2}G(u)=0$$

or

$$G(u)[1-M^{2}u^{2}]=y(0)+u\,y'(0).$$

The general Sumudu solution is $G(u)=\frac{y(0)+u\,y'(0)}{1-M^{2}u^{2}}$. Let $y(0)=c_{1}$ and $y'(0)=c_{2}$ so that we have $G(u)=\frac{c_{1}+c_{2}u}{1-M^{2}u^{2}}$. Upon inverting we obtain the general solution as

$$u(y) = c_1 \cosh My + c_2 \frac{\sinh My}{M} \tag{20}$$

Using the boundary conditions (BC) we obtain

$$0 = c_1 \cosh M - c_2 \sinh M \tag{21}$$

$$u_o = c_1 \cosh M + c_2 \sinh M \tag{22}$$

On solving (21) and (22) we obtain

$$c_1 = \frac{U_o}{2\cosh M} \quad \text{and} \quad c_2 = \frac{U_o}{\sinh M}$$

and substituting these to equation **(**20) we obtain

$$u(y) = \frac{U_0}{2}\left[\frac{\cosh My}{\cosh M} + \frac{\sinh My}{\sinh M}\right]$$

or

$$\frac{u\,(y)}{U_0} = \frac{\sinh\,[M\,(1+y)]}{\sinh 2M}, \tag{23}$$

where $M = M^* \sin\alpha$ and $M^* = lB_o\left(\frac{\sigma}{\mu}\right)^{\frac{1}{2}} = Ha$ is the Hartmann Number given by $Ha^2 = \frac{\sigma B_o{}^2 l^2}{\mu}$ and is directly proportional to the magnetic field $B_{0.}$ Flow velocity for Hartmann numbers $Ha = 0.5$, $Ha = 1.5$, $Ha = 2.5$ and angle of inclinations for $\alpha = 15^\circ$, $\alpha = 30^\circ$ and $\alpha = 45^\circ$ are shown in the figures below.

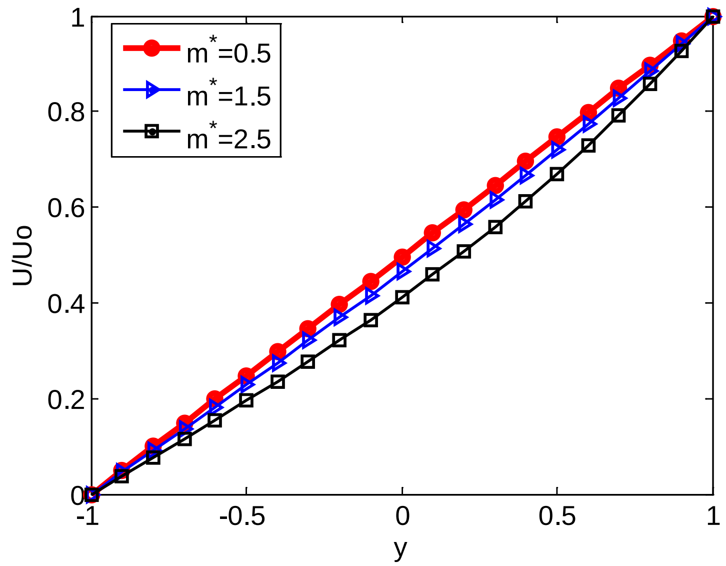

Figure 1: Velocity profiles for various Hartmann numbers for angle of inclination 15^0

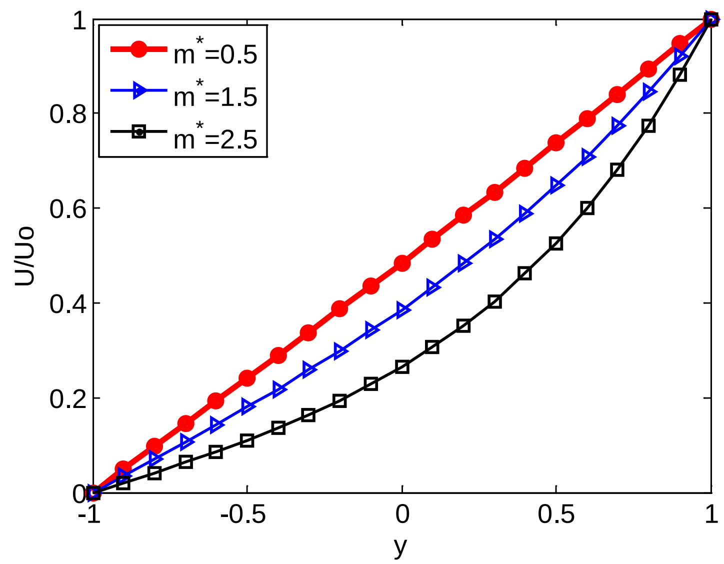

Figure 2: Velocity profiles for various Hartmann numbers for angle of inclination 30^0

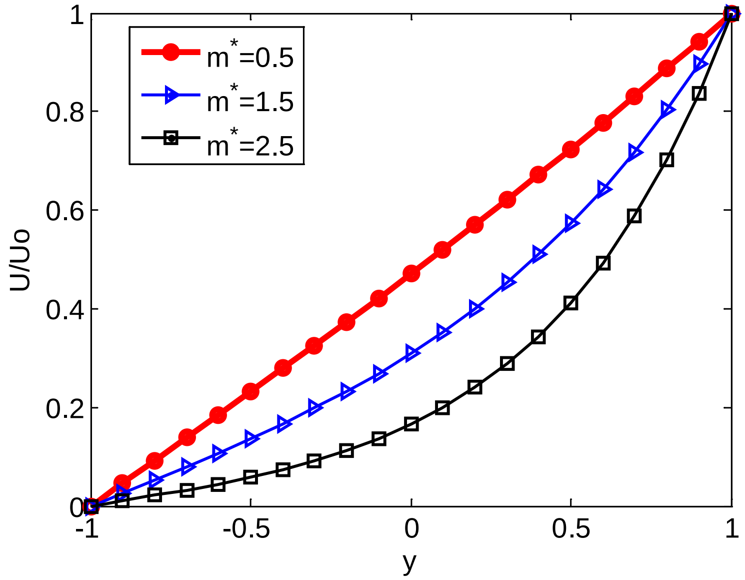

Figure 3: Velocity profiles for various Hartmann numbers for angle of inclination 45^0

4 Discussion of results and Conclusions

Equation (19) has been solved by using Sumudu Transform for the linear differential equation with constant coefficients .We assumed some initial conditions c_1 and c_2 which resulted the expression for the velocity of fluid particles be derived in terms of hyperbolic functions. The velocity profiles drawn for Hartmann numbers $Ha = 0.5$, $Ha = 1.5$, $Ha = 2.5$ and angle of inclinations for $\alpha = 15^\circ$, $\alpha = 30^\circ$ and $\alpha = 45^\circ$ shows that, increase in magnetic field strength and magnetic inclinations results into decreases in the velocity profiles.

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