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Revisiting the Kurtosis of Stationary Processes with Applications to Volatility Models

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Abstract

This paper establishes a number of new results on kurtosis of stationary processes as they play important roles in modelling and applications in financial time series. Some examples from ARCH and GARCH models are added to illustrate the usefulness and applicability of these new results.

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Key words: Time series, Autoregression, Serial Correlation, Kurtosis, Moments, ARCH, GARCH, Stationarity, ARMA, Volatility, Heteroscedasticity.

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1 Introduction

Recent growing interest in financial econometrics has placed a strong emphasis on modelling financial volatilities using both linear and non-linear time series models. The family of autoregressive conditional heteroscedastic (ARCH) models ([5]) and its generalization to GARCH ([2]) have been developed due to flexibility in various applications.

In certain applications of GARCH models to financial data, there is the assumption that the series are conditionally normally distributed, perhaps because the Gaussian (pseudo) likelihood approach is convenient in volatility estimation. However, empirical evidence suggests that the normal GARCH model is not always valid in practice since realizations of several financial time series, including rates of foreign exchange or natural or log returns on stocks, are heavy-tailed with significant leptokurtosis and sometimes time-varying volatilities. Therefore kurtosis plays an important role in modelling such financial data as heavy tailed distributions often can be seen in practice. [6] consider the fourth moment for a family of GARCH (p, q) processes and [1] extend these calculations to include stochastic volatility models.

Therefore, in section 2, we review a number of important results related to kurtosis for later reference.

2 A Review of Kurtosis

Suppose that $\{X_t\}$ is a m^{th} order stationary time series with mean μ and the r^{th} central moment $\mu_r = E[(X_t - \mu)^r]$. The kurtosis $\kappa^{(X)}$ of X_t is given by

$$\kappa^{(X)} = \frac{\mu_4}{\mu_2^2}.$$

It is known that when the distribution of X_t is normal, then $\kappa^{(X)} = 3$. A distribution with $\kappa > 3$ is called heavy tailed or leptokurtic. It has been identified that many financial time series are leptokurtic and the knowledge of the excess kurtosis, $K = \kappa^{(X)} - 3$ is important for further analysis.

Based on a sample of n observations, an estimator of $\kappa^{(X)}$ is given by

$$\hat{\kappa} = \frac{\sum_{t=1}^{n}(X_t - \bar{X})^4/n}{[\sum_{t=1}^{n}(X_t - \bar{X})^2/n]^2},$$

where $\bar{X}$ is the sample mean.

When X_t are independent and identically distributed (iid) normal random variables, [7] showed that

$$\sqrt{n}(\hat{\kappa} - 3) \to N(0, 24).$$

Now we state the kurtosis for four popular models used in many applications of time series. Assume that the time series $\{X_t\}$ is generated by a 4^{th} order stationary, independent noise process $\{\varepsilon_t\}$ with mean zero, variance σ^2 and kurtosis $\kappa^{(\varepsilon)}$.

- The kurtosis for an AR(1) process generated by $X_t = \phi X_{t-1} + \varepsilon_t$ is given by

$$\kappa^{(X)} = \frac{\kappa^{(\varepsilon)}(1 - \phi^2) + 6\phi^2}{1 + \phi^2}. \tag{2.1}$$

- The kurtosis of an MA(1) process given by $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ is

$$\kappa^{(X)} = \frac{\kappa^{(\varepsilon)}(1 + \theta^4) + 6\theta^2}{(1 + \theta^2)^2}. \tag{2.2}$$

  When ε_t is normal in both AR(1) and MA(1) models, it is easy to see that $\kappa^{(X)} = 3$.

- For an ARCH(1) process satisfying $X_t = \sqrt{h_t}\varepsilon_t;\ h_t = \omega + \alpha X_{t-1}^2$, the corresponding kurtosis is

$$\kappa^{(X)} = \frac{\kappa^{(\varepsilon)}(1 - \alpha^2)}{1 - \kappa^{(\varepsilon)}\alpha^2}. \tag{2.3}$$

- For a GARCH(1,1) process given by $X_t = \sqrt{h_t}\varepsilon_t$ and $h_t = \omega + \alpha X_{t-1}^2 + \beta h_{t-1}$, we have
$$\kappa^{(X)} = \frac{\kappa^{(\varepsilon)}[1-(\alpha+\beta)^2]}{1-\kappa^{(\varepsilon)}\alpha^2 - 2\alpha\beta - \beta^2}. \tag{2.4}$$

Even when ε_t is normal with $\kappa^{(\varepsilon)} = 3$, it is easy to verify that $\kappa^{(X)} > 3$ in both the ARCH(1) and GARCH(1,1) models justifying the heavy tail behavior.

Although it is easy to calculate the kurtosis for lower order models, the algebraic complexity will occur for higher order models. Therefore, section 3 is devoted to establish a number of new results for the kurtosis of 4^{th} order stationary general linear processes, which is of considerable interest in its own right in many applications of time series analysis and related problems.

3 Main Results

Let $\{X_t\}$ be a general linear process (glp) given by
$$X_t - \mu = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \tag{3.1}$$
where $\{\varepsilon_t\}$ is a sequence of 4^{th} order stationary, independent, identically distributed random variables with $E(\varepsilon_t) = 0, E(\varepsilon_t^2) = \sigma^2$ and the kurtosis $\kappa^{(\varepsilon)}$ for all t. Further, assume $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ and $\sum_{j=0}^{\infty} \psi_j^4 < \infty$.

In many applications of time series, $\{X_t\}$ can be approximated by a finite parameter ARMA type model of the form
$$\phi(B)X_t = C + \theta(B)\varepsilon_t,$$
where C is a constant, B is the backshift operator and the polynomials $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p;\ \theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q$ have no common zeros and the zeros lie outside the unit circle (known as AR and MA regularity conditions).

Note that these AR and MA regularity conditions ensure the existence of 4^{th} order stationarity and invertibility of the ARMA representation. In addition,

- when $\phi(1) \neq 1,\ \mu = E(X_t) = \frac{C}{\phi(1)}$
- $X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$, where $\psi(B) = [\phi(B)]^{-1}[\theta(B)] = \sum_{j=0}^{\infty} \psi_j B^j$ satisfy the conditions $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ and $\sum_{j=0}^{\infty} \psi_j^m < \infty$. See, for example, Wold (1954).
- when $\theta(1) \neq 1,\ \pi(B) = [\theta(B)]^{-1}[\phi(B)] = \sum_{j=0}^{\infty} \pi_j B^j$ and satisfy $\varepsilon_t + \mu^* = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ such that $\sum_{j=0}^{\infty} \pi_j^2 < \infty$ and $\sum_{j=0}^{\infty} \pi_j^m < \infty$, where $\mu^* = \frac{C}{\theta(1)}$.

Now we state and prove the theorem below which establishes the kurtosis of a finite linear combination of $\{\varepsilon_i\}$.

Theorem 3.1 *Suppose that* $\{\varepsilon_i\}; i = 0, 1, \ldots, \ell$ *is a sequence of independent and non-identically distributed random variables each with* $E(\varepsilon_i) = 0,\ E(\varepsilon_i^2) = \sigma_i^2$ *and kurtosis* $\kappa^{(\varepsilon_i)}$ *for each* i*. Then the kurtosis of* $Y = \sum_{i=0}^{\ell} a_i \varepsilon_i$ $(a_i; i = 0, 1, \ldots, \ell$ *are finite real numbers) can be expressed as*

$$\kappa^{(Y)} = \frac{\sum_{i=0}^{\ell} a_i^4 \sigma_i^4 \kappa^{(\varepsilon_i)} + 6 \sum_{i=0}^{\ell-1} \sum_{j>i}^{\ell} a_i^2 a_j^2 \sigma_i^2 \sigma_j^2}{(\sum_{i=0}^{\ell} a_i^2 \sigma_i^2)^2}. \tag{3.2}$$

Proof: For any k such that $0 < k < \ell$, let $Y = A + B$, where $A = \sum_{i=0}^{k} a_i \varepsilon_i$ and $B = \sum_{i=k+1}^{\ell} a_i \varepsilon_i$. Then $E(A) = E(B) = 0; A$ and B are independent and $E[f(A)g(B)] = E[f(A)]E[g(B)]$, where $f()$ and $g()$ are any two bounded, measurable functions of A and B respectively. Therefore,

$$E(Y^4) = E(A^4) + 4E(A^3B) + 6E(A^2B^2) + 4E(AB^3) + E(B^4)$$

reduces to

$$E(Y^4) = E(A^4) + 6E(A^2B^2) + E(B^4).$$

Using the multinomial expansion of each term on right hand side (rhs) and noting that $E(\varepsilon_i \varepsilon_j^3) = 0,\ j \neq i$, it is easy to evaluate the expected values

$E(A^4)+E(B^4)$ and $E(A^2B^2)$ with terms involving even powers of ε_i^4 and $\varepsilon_i^2\varepsilon_j^2,\ j\neq i$ and gives the numerator of (3.2)

$$E(Y^4)=E(A^4)+6E(A^2B^2)+E(B^4)=\sum_{i=0}^{\ell}a_i^4\sigma_i^4\kappa^{(\varepsilon_i)}+6\sum_{i=0}^{\ell-1}\sum_{j>i}^{\ell}a_i^2a_j^2\sigma_i^2\sigma_j^2.$$

Since $Var(Y)=\sum_{i=0}^{\ell}a_i^2\sigma_i^2$, (3.2) follows.

When ε_i are iid with kurtosis $\kappa^{(\varepsilon)}$, we have the following corollary.

Corollary 3.1 When ε_i are iid with kurtosis $\kappa^{(\varepsilon)}$, then $\kappa^{(Y)}$ in (3.2) reduces to

$$\kappa^{(Y)}=\frac{\kappa^{(\varepsilon)}\sum_{i=0}^{\ell}a_i^4+6\sum_{i=0}^{\ell-1}\sum_{j>i}^{\ell}a_i^2a_j^2}{(\sum_{i=0}^{\ell}a_i^2)^2}. \tag{3.3}$$

The next theorem gives the corresponding result for a glp given in (3.1).

Theorem 3.2 *Let $\{X_t\}$ be a 4^{th} order stationary process satisfying (3.1). Then the kurtosis of $\{X_t\}$ is given by*

$$\kappa^{(X)}=\frac{\kappa^{(\varepsilon)}\sum_{j=0}^{\infty}\psi_j^4+6\sum_{i=0}^{\infty}\sum_{j>i}\psi_i^2\psi_j^2}{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2}. \tag{3.4}$$

Proof: The theorem immediately follows from (3.3) by allowing $\ell\to\infty$.

The section 4 considers some useful examples to illustrate the usefulness of this result given in (3.4).

4 Examples

First consider the family of stationary Gaussian ARMA processes and without loss of generality, assume that $\mu=0$.

4.1 Gaussian ARMA processes

- Using the fact that the noise process $\{\varepsilon_t\}$ is Gaussian, we have $\kappa^{(\varepsilon)} = 3$. Hence the equation (3.4) for the kurtosis of X_t becomes

$$\kappa^{(X)} = \frac{3\left\{\sum_{j=0}^{\infty}\psi_j^4 + 2\sum_{i=0}^{\infty}\sum_{j>i}\psi_i^2\psi_j^2\right\}}{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2} = \frac{3\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2}{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2} = 3,$$

consistent with the knowledge that the process X_t in (3.1) is also Gaussian.

Now we verify the results in (2.1) and (2.2) for non-Gaussian AR(1) and MA(1) processes.

4.2 Non-Gaussian ARMA processes

- ARMA(0,0) process

When $p = 0$ and $q = 0$, we have $\phi(B) \equiv 1$ and $\theta(B) \equiv 1$. Therefore, $\psi_0 = 1$ and $\psi_j = 0$ for all $j \geq 1$ and the result (3.4) reduces to the expected conclusion of $\kappa^{(X)} = \kappa^{(\varepsilon)}$.

- AR(1) process

When $p = 1$ and $q = 0$, $\psi_j = \phi^j, \ j \geq 0$. This gives

$$N = \kappa^{(\varepsilon)}\sum_{j=0}^{\infty}\psi_j^4 + 6\sum_{i=0}^{\infty}\sum_{j>i}\psi_i^2\psi_j^2 = \frac{\kappa^{(\varepsilon)}}{1-\phi^4} + \frac{6\phi^2}{(1-\phi^2)(1-\phi^4)}$$

and

$$D = \left(\sum_{j=0}^{\infty}\psi_j^2\right)^2 = (\frac{1}{1-\phi^2})^2.$$

The ratio $\frac{N}{D}$ reduces to $\kappa^{(X)}$ given in (2.1) for an AR(1) process.

- MA(1) process

  When $p = 0$ and $q = 1,\ \psi_0 = 1,\ \psi_1 = \theta$ and $\psi_j = 0,\ j \geq 2$. Therefore,

$$N = \kappa^{(\varepsilon)} \sum_{j=0}^{\infty} \psi_j^4 + 6 \sum_{i=0}^{\infty} \sum_{j>i} \psi_i^2 \psi_j^2 = \kappa^{(\varepsilon)}(1+\theta^4) + 6\theta^2$$

  and

$$D = \left(\sum_{j=0}^{\infty} \psi_j^2 \right)^2 = (1+\theta^2)^2.$$

  The ratio $\frac{N}{D}$ reduces to $\kappa^{(X)}$ given in (2.2) for a MA(1) process.

- ARMA(1,1)

  When $p = 1$ and $q = 1$, we have the most useful ARMA model in practice given by

$$X_t = \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}.$$

  In this case $\psi_0 = 1,\ \psi_j = \phi^{j-1}(\phi + \theta),\ j \geq 1$ and

$$N = \frac{\kappa^{(\varepsilon)}[(1-\phi^4) + (\phi+\theta)^4]}{(1-\phi^4)} + \frac{6(\phi+\theta)^2[(1-\phi^4) + \phi^2(\phi+\theta)^2]}{(1-\phi^4)(1-\phi^2)},$$

$$D = [\frac{(1-\phi^2) + (\phi+\theta)^2}{(1-\phi^2)}]^2.$$

  The ratio $\frac{N}{D}$ gives the corresponding $\kappa^{(X)}$ for this ARMA(1,1) process.

4.3 An Illustration

As an illustration, we provide three ARMA(1,1) processes generated by the following innovations:

- Gaussian with $\mu = 0$ and $\sigma = 1$:

$$\kappa^{(G)} = 3.$$

- Lognormal with $\mu = 0$ and $\sigma = 1$:

$$\kappa^{(LN)} = 110.94.$$

- t with 5 df:

$$\kappa^{(t5)} = 6.$$

The acf and pacf plots (see the Appendix) show that all three can be considered as ARMA(1,1) processes. However, the time series plots 2 and 3 show that they have higher kurtosis than the time series plot in 1.

Theorem 3.2 is applicable only for glp satisfying the form in (3.1). However, in some applications of financial time series one needs to develop similar expressions for the squared process $\{X_t^2\}$.

5 Evaluating $\kappa^{(X)}$ when X_t^2 follows a stationary ARMA Process

Suppose that X_t follows a zero mean, 4th order stationary process and $\{X_t^2\}$ is generated by an ARMA (r,r) type process given by

$$\Phi(B)X_t^2 = \omega + \beta(B)\eta_t, \tag{5.1}$$

where

- $\omega > 0$ and $\{\eta_t\}$ is a martingale difference sequence with mean zero and variance σ_η^2.
- the polynomials $\Phi(B) = 1-\Phi_1 B-\cdots-\Phi_r B^r,\ \beta(B) = 1-\beta_1 B-\cdots-\beta_r B^r$ (B is the backshift operator) have distinct zeros outside the unit circle and $\Phi(1) > 0$.

The last condition ensures that the process $\{X_t\}$ is stationary up to 4th order and $\{X_t^2\}$ possesses the following Wold representation

$$X_t^2 = \delta + \sum_{j=0}^{\infty} \Psi_j \eta_{t-j}, \tag{5.2}$$

where

- $\Psi_0 = 1$,
- $\delta = E(X_t^2) = \frac{\omega}{1-\Phi_1-\ldots-\Phi_r} = \frac{\omega}{\Phi(1)} > 0$,
- $\sum_{j=0}^{\infty} \Psi_j^2 < \infty$.

This leads to the theorem below.

Theorem 5.1 *The kurtosis of the process $\{X_t\}$ is given by*

$$\kappa^{(X)} = \frac{\delta^2 + \sigma_\eta^2(1+a)}{\delta^2}, \tag{5.3}$$

where $\sigma_\eta^2 = Var(\eta_t)$ and $a = \sum_{j=1}^{\infty} \Psi_j^2$.

Proof: The proof is straight forward and follows from (5.2) since

$$E(X_t^4) = E(\delta + \sum_{j=0}^{\infty} \Psi_j \eta_{t-j})^2$$

and the right hand side reduces to $\delta^2 + \sigma_\eta^2(1 + a)$. Noting the fact that $Var(X_t) = E(X_t^2) = \delta$, the theorem follows.

Below illustrates how this theorem can be used to evaluate the kurtosis of any GARCH process.

5.1 Kurtosis for GARCH Processes

Considers the family of GARCH (l, m) process given by

$$X_t = \sqrt{h_t}\varepsilon_t, \tag{5.4}$$

$$h_t = \omega + \sum_{i=1}^{l} \alpha_i X_{t-i}^2 + \sum_{j=1}^{m} \beta_j h_{t-j}, \;\; \omega > 0, \alpha_i \geq 0, \beta_j \geq 0, \tag{5.5}$$

where the following conditions hold:

(i) ε_t comprises of a stationary sequence of independent and identically distributed random variables with zero mean, unit variance and kurtosis $\kappa^{(\varepsilon)} = E(\varepsilon_t^4) < \infty$ for all t;

(ii) h_t is the conditional variance of X_t given the history $F_{t-1} = \{X_{t-1}, X_{t-2}, \ldots\}$ such that $h_t = \mathrm{Var}(X_t|F_{t-1})$;

(iii) the parameters $\omega, \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m$ appear in (5.5) are real-valued satisfying $h_t > 0$ and $\sum_{i=1}^{l} \alpha_i + \sum_{j=1}^{m} \beta_j < 1$ for all t.

When $r = \max(l, m)$, an alternative equivalent representation of (5.4) and (5.5) is given by

$$\Phi(B)X_t^2 = \omega + \beta(B)\eta_t, \tag{5.6}$$

where $\eta_t = X_t^2 - h_t = (\varepsilon_t^2 - 1)h_t$ is a martingale difference sequence with $\sigma_\eta^2 = E(\eta_t^2) = \frac{\delta^2(\kappa^{(\varepsilon)}-1)}{1-(\kappa^{(\varepsilon)}-1)a}$; $\Phi(B), \beta(B)$ are AR and MA polynomials of degree r such that $\Phi_i = \alpha_i + \beta_i,\ i = 1, 2, \cdots, r$.

(iv) $0 < (\kappa^{(\varepsilon)} - 1)a < 1$.

Since the process $\{X_t^2\}$ in (5.6) is stationary and possesses the moving average ([11]) representation given in (5.2), the theorem 5.2 below can be established using the theorem 5.1. Further applications can be found in [9].

Theorem 5.2 *The kurtosis for the GARCH process in (5.4) and (5.5) is given by*

$$\kappa^{(X)} = \frac{\kappa^{(\varepsilon)}}{1 - \left(\kappa^{(\varepsilon)} - 1\right)a}. \tag{5.7}$$

Proof: From (5.3) of theorem 5.1, we have

$$\kappa^{(X)} = \frac{\delta^2 + \sigma_\eta^2(1+a)}{\delta^2}.$$

Substituting σ_η^2 and rearranging terms, theorem 5.2 follows.

5.2 Examples

- When $l = 1$ and $m = 0$, we have an ARCH(1) process with $\Phi_1 = \alpha$ and $a = \sum_{j=1}^{\infty} \Psi_j^2 = \frac{\alpha^2}{1-\alpha^2}$. Substituting in (5.7) gives the kurtosis (2.3) for an ARCH(1).

- When $l = 1,\ m = 1$, we have a GARCH(1,1) process with $\Phi_1 = \alpha + \beta, \beta_1 = \beta$ and $a = \sum_{j=1}^{\infty} \Psi_j^2 = \frac{\alpha^2}{1-(\alpha+\beta)^2}$. Substituting in (5.7) gives the kurtosis (2.4) for a GARCH(1,1).

When the marginal distribution of ε_t is normal, the corresponding GARCH process is called a Gaussian GARCH (r, r) process.

5.3 Kurtosis of Normal GARCH

Any GARCH process $\{X_t\}$ satisfy $\kappa^{(X)} > 3$ and hence the excess kurtosis $\kappa^{(G)} = \kappa^{(X)} - 3$ is known as the GARCH kurtosis. Therefore, it is easy to show that the corresponding $\kappa^{(G)}$ can be expressed as

$$\kappa^{(G)} = \frac{\kappa^{(\varepsilon)} - 3 + 3a(\kappa^{(\varepsilon)} - 1)}{1 - \big(\kappa^{(\varepsilon)} - 1\big)a}$$

and when $\{\varepsilon_t\}$ follows a Gaussian distribution, the above result reduces to

$$\kappa^{(G)_N} = \frac{6a}{1 - \big(\kappa^{(\varepsilon)} - 1\big)a}\,.$$

Further results related to kurtosis and applications can be found in [6], [10], [8], [4], [12], [3] and the references there in.

6 Conclusions

Financial returns are often modelled as stationary (autoregressive) time series with innovations having conditional heteroscedastic errors, especially with GARCH innovations. This paper, extends the results to any 4^{th} order stationary processes which have ARMA type representation. The kurtosis is useful for identifying the marginal distribution of volatility processes and is expressed in a simple form in terms of the model parameters and autocorrelation of the squared observed processes.

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References

[1] Bai, X., Russell, J.R. and Tiao, G.C. (2003). Kurtosis of GARCH and Stochastic Volatility Models with Non-normal Innovations, *Journal of Econometrics*, **114**, 349-360.

[2] Bollerslev, T. (1986). Generalised Autoregressive Conditional heteroscedasticity, *Journal of Econometrics*, **31**, 307-327.

[3] Cain, M.K., Zhang, Z. & Yuan, KH. (2017), Univariate and multivariate skewness and kurtosis for measuring nonnormality: Prevalence, influence and estimation, *Behav Res.* **49,** 1716-1735. https://doi.org/10.3758/s13428-016-0814-1

[4] Cisar, P. (2010). Skewness and Kurtosis in Function of Selection of Network Traffic Distribution, *Acta Polytechnica Hungarica,* **7,** 95-106.

[5] Engle. R. F, (1982).Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation, *Econometrica,* **50,** 987-1008.

[6] He, C. and Terašvirta, T. (1999). Fourth moment structure of the GARCH (p, q) process. *Econometric Theory*, **15**, 824-846.

[7] Pearson, E.S., (1930). A further development of tests for normality, *Biometrika,* **22 (1-2),** 239-249.

[8] Peiris,S. and Thavaneswaran, A.(2007). An Introduction to Volatility models with Indices, *Applied Mathematics Letters*, **20**, 177-182.

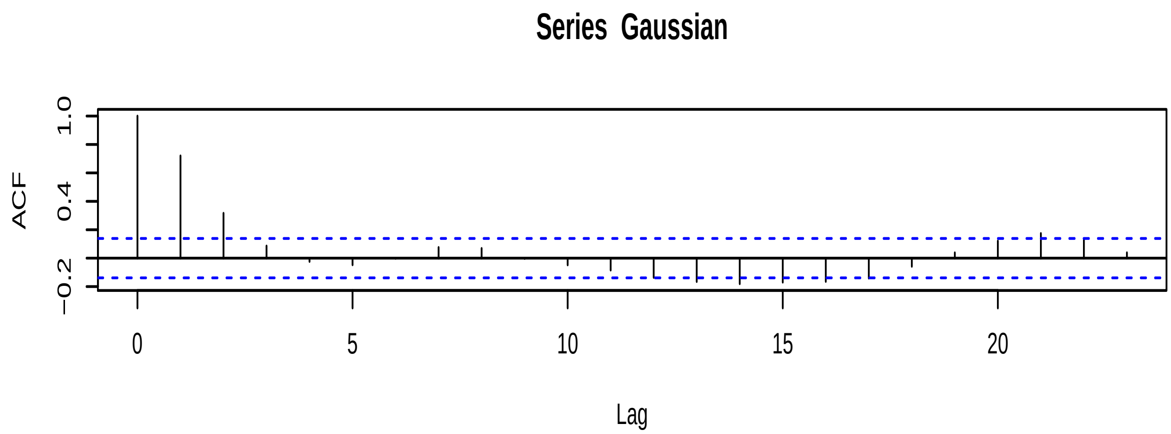
[9] Peiris,S., Allen,D. and Peiris,U. (2005). Generalized AR Models with Conditional Heteroscedasticity: An Application to Financial Time Series, *Proceedings of the Statistics in Finance Conference*, 75-83 (University of Wollongong, NSW, Australia).

[10] Thavaneswaran, A., Appadoo, S. and Peiris, S. (2005). Forecasting volatility, *Statistics and Probability Letters*, **75**, 1-10.

[11] Wold, H. (1954). *A study in the Analysis of Stationary Time Series* (2nd edition) Stockholm: Almquist and Wiksell.

[12] Yuan, L., He, Y., Huang, J., Sun, Y. (2010). A New Neural-Network-Based Fault Diagnosis Approach for Analog Circuits by Using Kurtosis and Entropy as a Preprocessor, *IEEE Transactions On Instrumentation and Measurement,* **59(3),** 586-595.

Appendix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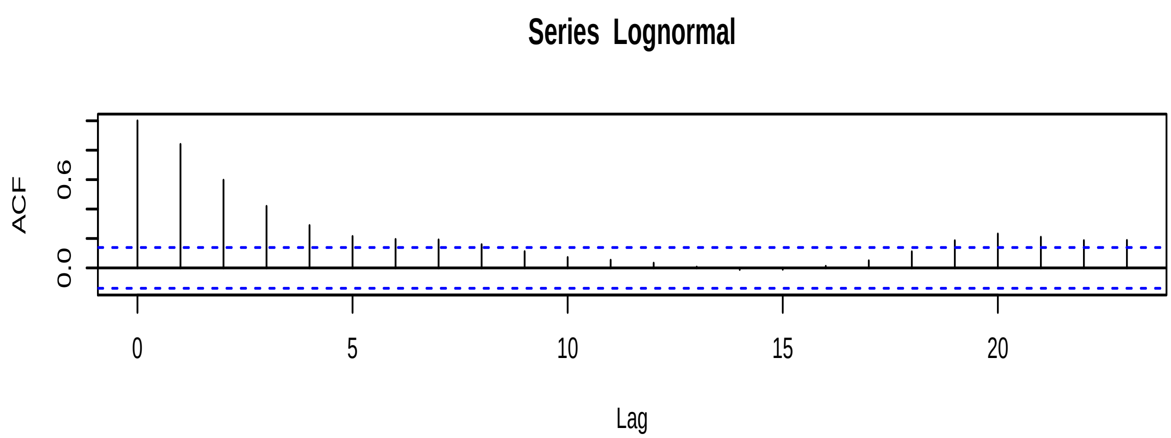
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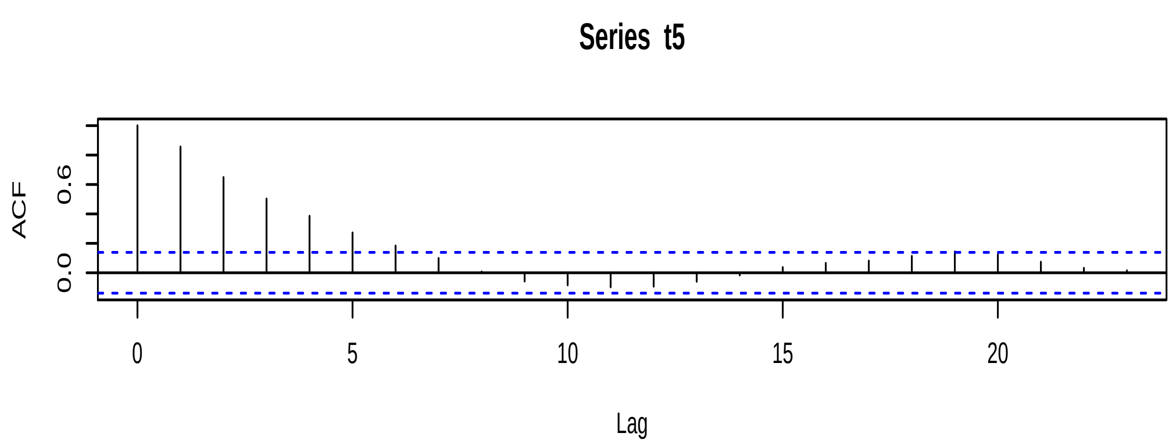
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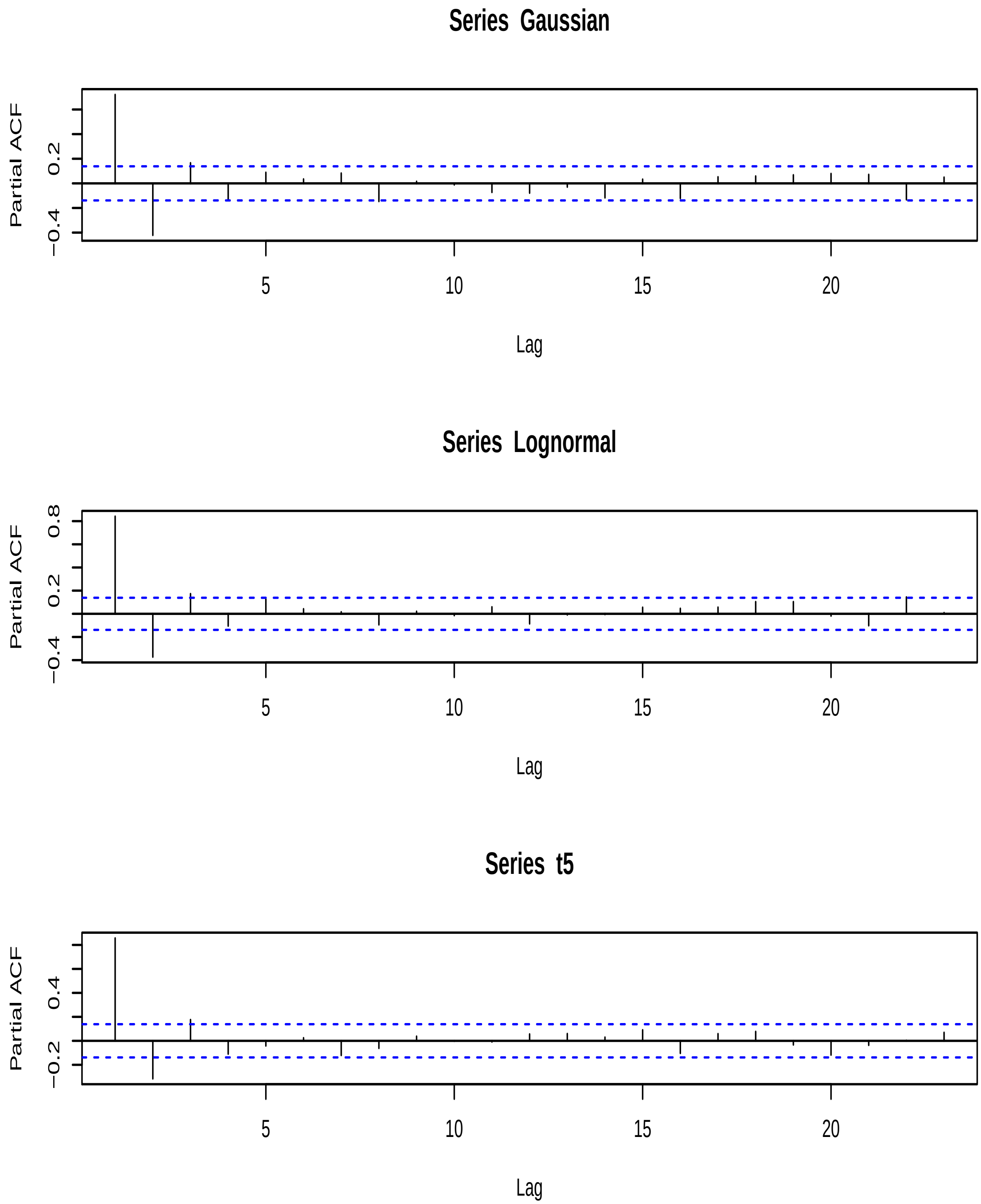
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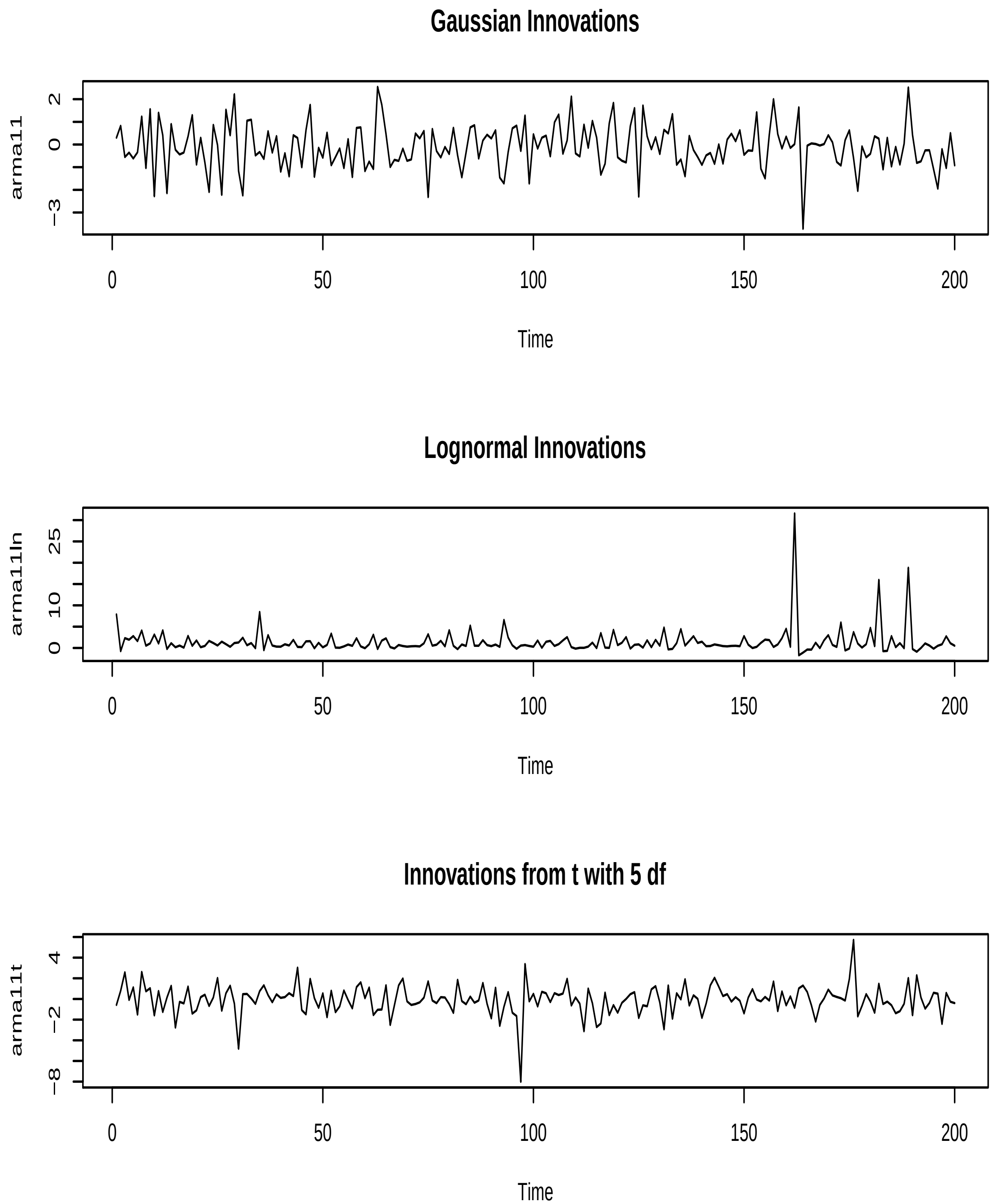
Gaussian Innovations
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Time
Lognormal Innovations
arma11ln
Time
Innovations from t with 5 df
arma11t
Time