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A Study of the Optimal Asset Allocation as Raising Taxes on the Rich

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Abstract

We study the effects of introducing taxation in classical continuous-time optimization problems with utility from consumption and optimal asset allocation as taxation on the rich. This paper employs the framework of original Merton's model to a new optimal portfolio selection that consists of a riskless asset as well as a risk asset. The aim of this article is to analyze the portfolio strategies that are adopted a dynamic model of consumption, as the impact on optimal portfolio rules concerns the contribution-hedge strategy. We thus emphasize that the current practice of taxing the rich only is appropriate when trying to reduce the distortions of the taxation system on the portfolio behavior of the investor, and that taxation applied on contributions would be more adapted.

JEL classification numbers: D14, G11, G23.
Keywords: dynamic asset allocation, Merton portfolio problem, European put option.

1 Introduction

Wealth inequality is the most obvious measure of the gap between the rich and the poor in economics activity. The top 1 percent of US households held about 35 percent of the economy's wealth, thereby owning more assets than the bottom 90 percent together in 2007 (Wolff, 2010).

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There's just no doubt about it, Oxfam's claim[4] that by 2016 the richest 1% could own as much or the same wealth as the bottom 99% is not so wildly implausible. These inequalities have provoked a vivid debate on whether wealth should be redistributed, and the possibility of reaching a more fair wealth distribution has historically been a common rationale on taxing high-income people.

This article responds the Piketty's argument at *Capital in the Twenty First Century*. His main proposal that is defined to contain every kind of asset is a comprehensive international agreement to construct a progressive tax on individual wealth. There has been a renewed interest in the normative point of view on capital and wealth taxation in recent years. (Diamond and Saez, 2011; Cagetti et al., 2009). Usual topics in these discussions are long-term expenditure cuts of social security payments and subsidies, or increments in wealth taxation. However, tax policy increasingly anticipates taxing the rich man. Recently, Obama administration official will look for raising taxes on rich to finance cuts for middle class. The plan would also increase the top capital-gains tax rate, to 28 percent from 23.8 percent, for couples with incomes above $500,000 annually. People with higher income and wealth are suggested to bear a greater share of the tax burden. The pioneering work by Goolsbee (2000) estimated that the short-term elasticity of taxable income (ETI) with respect to the net of tax share exceeded one, however achieved that the elasticity after one year was approximately 0.4 and probably closer to zero. He also concluded that the impact of executive salaries was almost completely a short-term shift in the timing of compensation rather than a continuous alteration and came entirely from a substantial increase in the exercise of stock options by the highest-income executives' taxpayers of the tax rate increases.

Gruber and Saez (2002) expanded the previous studies in some important features applying a panel of tax returns that spanned several major shifts in tax rate regimes. The different variation in tax rates from the long time interval covered by their panel allowed them to more accurately investigate and consider heterogeneity in contract to income and other taxpayer characteristics.

In related research, on the other hand with Gruber and Saez, Previous researcher Carroll (1998) studied on a time interval when tax rates increased employing a sample that included many high-income taxpayers. Hereafter, he surprisingly found an elasticity of taxable income with respect to the after-tax share of 0.4, about the similar estimated result of Gruber and Saez (2002) and lower than their estimation for high-income taxpayers. In this debate, the behavioral impacts of the wealthy to raise taxes on the rich are of particular interest. This issue is motivated by the notion that high income taxpayers may be more reactive to taxes both because they face higher marginal tax rates and may have more chances to response to alter in tax policy. As a result, raising in tax rates at the top of the income distribution can have great implications for tax revenues and economic meaning or activity. Moreover, the recent debates over future tax policy in the U.S. have focused central issue on the taxation of the high end of the income distribution, these behavioral responses of the rich have received increased attention.

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[4]According to a research by anti-poverty charity Oxfam, The wealthiest 1% will soon hold more than the rest of the world's population. The charity's work displays that the share of the world's wealth owned by the richest 1% increased from 44% in 2009 to 48% last year. On current trends, Oxfam explains it expects the wealthiest 1% to own more than 50% of the world's wealth by 2016.Source from BBC news 18 January, 2015. http://www.bbc.co.uk/news/business-30875633

This paper offers the first analysis of the implications for dynamic asset allocation of taxation on the rich. Wealthy person also responds by increasingly favoring the higher-return risky asset. This stochastic process is expected to increase in real terms over time and might be correlated with the investment performance of the 'risky asset'.

Who Are the Rich?
Who is rich and who is not? The answer to that question depends on the measure of affluence chosen, and what dividing line one chooses. Different measures may be more or less appropriate, income and wealth affluence lines are used to define and identify the rich. Although longitudinal data sets that follow people over ten years or more are now available. Even if conceptually attractive, a lack of data that tracks people over a lifetime precludes empirical examination of the latter measures. Not surprisingly, some candidates for a measure of affluence are annual consumption, taxable income, wealth, lifetime income and consumption; depending on the issue at hand .We assume the wealth process satisfies the following geometric Brownian motion (GBM), but with specification:

$$dW(t) = \begin{Bmatrix} \alpha_\omega W(t)dt + \sigma_\omega W(t)d{Z_t}^W & W(t) \geq W(r) \\ 0 & W(t), < W(r) \end{Bmatrix} \tag{1}$$

Where $W(r)$ is the wealth threshold to be rich people, α_ω return to assets, σ volatility of risky assets, and $d{Z_t}^W$ Wiener increment, $W(t)$ wealth taken into the wealthy taxpayers.

2 Economy Model

The rich investors take into consideration the risk and financial management decision for wealth utility maximization. Therefore adopt the option hedging strategies to reduce risk on the asset allocation of wealth, $W(t)$ indicates the wealth taken into rich man, $T-t$ is time to maturity, α instantaneous expected return to risky assets, r return to safe assets, $\mathcal{K}$ strike price. Solving the agent's utility maximization problem, we can define protected wealth

$$\mathbb{w}(t) \equiv \frac{1}{r}\left[1-e^{-r(T-t)}\right] - \mathcal{K}e^{-r(T-t)} \tag{2}$$

The surplus wealth is

$$\overline{\mathbb{w}}(t) \equiv W(t) - \mathbb{w}(t) \tag{3}$$

It duplicates and values a put option on a synthetic security with optimally invested wealth as $\mathbb{w}_\lambda(t)$, the subscript and the condition follow Cox and Huang (1989). Remaining initial surplus wealth $\mathbb{w}(0) - \mathbb{w}_\lambda(0)$ is invested in a European put option on optimally-invested wealth. The initial surplus optimally-invested wealth, $\mathbb{w}_\lambda(0)$ ensure that sufficient wealth remains to guarantee a positive bequest. The put's value subsequently becomes

$$\mathcal{P}(\mathbb{w}_\lambda(t), t) \equiv E_t^Q \max[0, \mathcal{K} - \mathbb{w}_\lambda(T)], \tag{4}$$

where the superscript $\mathbb{Q}$ on the right-hand side is expressed as the value of an expectation taken under the risk-neutral measure. Therefore, the surplus wealth is given by

$$\overline{\mathbb{w}}(t) = \mathbb{w}_\lambda(t) + \mathcal{P}(\mathbb{w}_\lambda(t), t) \tag{5}$$

Up to this point, the risk-neutral specialization of the process defined by Eq. (1) has instantaneous return r and (constant) instantaneous volatility $\bar{x}^*\sigma$. According to the standard option theory, it says that replicating the option by Eq. (4) with this asset and the safe asset requires going short by an amount

$$N(-d_1)\mathbb{w}_\lambda(t)e^{-\int_t^T q(\mu)d\mu} \tag{6}$$

in the safe asset, and long an amount

$$N(-d_2)\mathcal{K}e^{-r(T-t)} \tag{7}$$

where $\int_t^T q(\mu)d\mu$ is the same as investment(stock) dividend q in the synthetic risky asset and

$$d_1 = \frac{ln\left(\frac{\mathbb{w}_\lambda(t)}{\mathcal{K}}\right) + \left(r + \frac{(x^*\sigma)^2}{2} - q\right)(T-t)}{\bar{x}\sigma(T-t)} \tag{8}$$

$$d_2 = d_1 - \bar{x}\sigma(T-t)$$

where $N(\,)$ denotes the Normal distribution. The put with the underlying risky asset at time t therefore requires going short an amount in the underlying risky asset as follows:

$$\bar{x}^*(t)N(-d_1)\mathbb{w}_\lambda(t)\mathcal{K}e^{-\int_t^T q(\mu)d\mu} \tag{9}$$

2.1 The Investor's Asset Allocation

Followed by application of Eqs (2) and (3) to substitute into $\overline{\mathbb{w}}(t)$, the optimal dollar investment $O^*(\mathrm{t})$ in risky assets is given by

$$O^*(\mathrm{t}) = \bar{x}^*(t)\mathbb{w}_\lambda(t) - \bar{x}^*(t)N(-d_1)\mathbb{w}_\lambda(t)e^{-\int_t^T q(\mu)d\mu} \tag{10}$$

$$= \bar{x}^*(t)\left[\overline{\mathbb{w}}(t) - \mathcal{P}(\mathbb{w}_\lambda(t), t) - N(-d_1)\mathbb{w}_\lambda(t)e^{-\int_t^T q(\mu)d\mu}\right] \tag{11}$$

Combined with $x^*(t)$ of equation (26) on behind section get

$$\left(\frac{\alpha-r}{\gamma\sigma^2}\right)\left[\overline{\mathbb{w}}(t) - N(-d_2)\mathcal{K}e^{-r(T-t)}\right] \tag{12}$$

or ,equivalently,

$$\left(\frac{\alpha-r}{\gamma\sigma^2}\right)\left[W(t) - \frac{1}{r}(1 - e^{-r(T-t)}) + \mathcal{K}e^{-r(T-t)}(1 - N(-d_2))\quad\right] \tag{13}$$

Divide Eq. (13) through by $W(t)$,we obtain to the following main result:

Proposition 1
Consider the model's state variable and parameters in terms of above context, the optimal proportionate investment $x^*(t)$ in risky assets of is given by

$$x^*(t) = \left(\frac{\alpha - r}{\gamma\sigma^2}\right)\left[1 - \frac{1}{rW(t)}(1 - e^{-r(T-t)}) + \frac{\mathcal{K}}{W(t)}e^{-r(T-t)}(1 - N(-d_2))\right] \tag{14}$$

Moreover, from equation (15) one can obtain optimal proportionate investment $x^*(t)$ in risky assets after imposition taxes τ as

$$\left(\frac{\alpha - r}{\gamma\sigma^2}\right)(1-\tau)\left[W(t) - \frac{1}{(1-\tau)r}(1 - e^{-(1-\tau)r(T-t)}) + \mathcal{K}e^{-(1-\tau)r(T-t)}(1 - N(-d_2))\right] \tag{15}$$

See e.g. Bruhn, K. (2013) for more detailed discussions of the equation (15) when there are the chargeable wealthy people under tax system.
The right-hand side of Eq. (14) consists of twofold. The first, i.e., $\frac{\alpha - r}{\gamma\sigma^2}$ is familiar from Merton (1969). The second was introduced by Merton (1971). Its implications for dynamic asset allocation are discussed by Karatzas and Shreve (1998) who give several worked-out examples containing option-related components in their solutions. Without taking into account dynamic asset allocation, Carroll (2002) gives both evidence and theory in support of the proposition that luxury bequests raise the average position of risky assets in portfolios.

2.2 Two Assets Allocation Model

We shall assume that the rich wealth funds can trade continuously two assets which consist of money market account (the Bond) and the second asset is a risky security (the stock) in an economy. Following the optimal portfolio model studied by Merton (1971), Miao (2010), we start from deriving Bellman equation usually refers to the dynamic programming equation associated with continuous-time optimization problems. The portfolio selection participants derive utility from intertemporal consumption $\boldsymbol{C}$ of this good and the terminal wealth at time T . We should assume a probability space $(\Omega, \mathcal{F}, \mathcal{P})$and a filtration $\{\mathcal{F}_t\}$.Uncertainty is generated by standard Brownian motion Z_t in the dynamics model. The following equations governing the dynamics of the money market account (bond) and stock are expressed as;

$$B_t = B_0 exp(rt) \tag{16}$$

and

$$dS_t = \alpha S_t dt + \sigma S_t dZ_t \tag{17}$$

or ,equivalently,

$$S_t = S_0 exp\left\{\sigma Z_t + (\alpha - \frac{\sigma^2}{2})t_,\right\} \tag{18}$$

The σ indicates the volatility of risky assets. Parameter B_0 denotes the initial investment on the money market account. The two processes satisfy $D(0)=$ $\boldsymbol{I}(0)=\mathbf{0}$ and both are right continuous, non-decreasing adapted. The notation of trading strategies are (***D, I***), and the processes *D* and *I* are cumulative amount of purchases and sales of stock. The evolution of the amount invested in the money market account and stock process can be given as:

$$\begin{cases} dB_t = rB_t dt - dI_t + dD_t \\ dS_t = \alpha S_t dt + \sigma S_t dZ_t + dI_t - dD_t \end{cases} \tag{19}$$

For tractability, quantitative derivation to optimal investment portfolio strategy of the rich, we use CRRA utility function of the final wealth, that is, $U(W) = \frac{W^{1-\gamma}}{1-\gamma}, for\ \ 0 < \gamma < 1$, γ is the constant relative risk aversion parameter. The portfolio selection chooses optimal investment strategies and so as to maximize the final wealth at a deterministic time on behalf of the plan participants. The value function at time T is given as;

$$J(B,S,t;T) = Max_{(D,I)} E\left[\frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}\right], \tag{20}$$

where $W = B_T + S_T$ is the portfolio from both the riskless and the risky assets. It guarantees that ***B*** and ***S*** would be chosen to be strictly positive.Moreover; the participant makes intermediate consumption decision on the admissible consumption space $\mathbb{C}$, which satisfies

$$\int_0^t |C_s|\, ds < \infty, \ , \forall t \in [0,T] \tag{21}$$

The parameter values satisfy:

$$0 < \frac{\alpha - r}{\gamma\sigma^2} < 1. \tag{22}$$

Consideration above assumptions, consumption is yield available to the consumers through the money market account. The portfolio problem is given by:

$$J(C,B,S,t;T) = \underset{C_t, B_t, S_t : t>0}{Max} E\left[\int_0^T e^{-\theta t} \frac{{C_t}^{1-\gamma}}{1-\gamma} dt + e^{-\theta t} \frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}\right] \tag{23}$$

Subject to;

$$\begin{aligned} dB_t &= rB_t dt - C_t dt - dI_t + dD_t \\ dS_t &= \alpha S_t dt + \sigma S_t dZ_t + dI_t - dD_t \end{aligned} \tag{24}$$

The constraints equation is equivalent to:

$$dW_t = (rB_t + \alpha S_t - C_t)dt + \sigma S_t dZ_t. \tag{25}$$

where the notation is: ***E*** expectations operator, *T* age at death (assumed known), θ rate of time preference and the time discount rate, γ is the coefficient of relative risk aversion and is assumed to greater than or equal to 1, W wealth and $d\boldsymbol{Z_t}$ is Wiener increment. Under this setting, we may establish that the result have located an optimum, the solution can be summarized as follows:

Proposition 2
The optimal investment portfolio is a constant proportion of constrained optimally-invested surplus wealth, optimal amount invested in stock:

$$S^* = \frac{\alpha-r}{\gamma\sigma^2}W \text{ and } x^*(t) = \frac{\alpha-r}{\gamma\sigma^2} \tag{26}$$

Where

$x^*(t) = S^*/\,W$

Appendix A provides the proof of equation (26).
Moreover, the agent's policy functions satisfy the following equation

$$C(t) = a(t)^{-\frac{1}{\gamma}}[W(t) + b(t)] \tag{27}$$
$$S(t) = \frac{\alpha-r}{\gamma\sigma^2}[W(t) + b(t)]$$

The proof of equation (27) see Appendix B. Optimal investment policy involves investing a constant fraction of wealth in the stock, independent of the investor's horizon. As long as $\alpha > r$, the fund always holds the stock in its portfolio. The ratio of the amount invested in stock and money market account is:

$$\pi^* = \frac{S^*}{B} = \frac{\frac{\alpha-r}{\gamma\sigma^2}W}{\left(1-\frac{\alpha-r}{\gamma\sigma^2}\right)W} = \frac{\alpha-r}{\gamma\sigma^2-\alpha+r}. \tag{28}$$

Now replacing $\boldsymbol{S}$ with the optimal value $S^* = \frac{\alpha-r}{\gamma\sigma^2}W$, in the HJB equation and rearrange, we find the ordinary differential equation of a in time *t* as:

$$a(t)^{-\frac{1-\gamma}{\gamma}} \cdot \frac{\gamma}{1-\gamma} + \frac{a(t)'}{1-\gamma} + a(t)r + \frac{(\alpha-r)^2}{2\gamma\sigma^2}a(t) - \frac{\theta}{1-\gamma}a(t) = 0. \tag{29}$$

Formalizing it to:

$$\frac{da}{dt} = -\gamma a(t)^{-\frac{1-\gamma}{\gamma}} - \left[(1-\gamma)r + \frac{(1-\gamma)(\alpha-r)^2}{2\gamma\sigma^2} - \theta\right]a(t) \tag{30}$$

As a result, we will obtain[5]

$$a(t)^{\frac{1}{\gamma}} = \left(a(0)^{\frac{1}{\gamma}} - \frac{\gamma}{\eta}\right)e^{\frac{-\eta(T-t)}{\gamma}} + \frac{\gamma}{\eta} \quad \text{i.e. } a(t) = \left[\left(a(0)^{\frac{1}{\gamma}} - \frac{\gamma}{\eta}\right)e^{\frac{-\eta(T-t)}{\gamma}} + \frac{\gamma}{\eta}\right]^{\gamma} \tag{31}$$

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[5]The details see also Appendix C.

Where $\eta = (1-\gamma)r + \frac{(1-\gamma)(\alpha-r)^2}{2\gamma\sigma^2} - \theta$

Thus, from extended eq. (24) we can have

$$dW(t) = [rW(t) + (\alpha - r)S(t) - C(t) + Y(t)]dt + \sigma S_t dZ_t \quad (32)$$

Substituting eq. (26)-(27) into wealth process obtained in above (32) gives us

$$dW(t) = \left[rW(t) + \frac{(\alpha-r)^2}{\gamma\sigma^2}(W(t) + b(t)) - a(t)^{-\frac{1}{\gamma}}[\![W(t + b(t)]\!] + Y(t)\right]dt + \frac{(\alpha-r)}{\gamma\sigma}(W(t) + b(t))dZ_t \quad (32.1)$$

Suppose that

$$b(t) = \int_t^T Y(\mu)e^{-r(\mu-t)}d\mu$$

Where $Y(t)$ denote the labor income depending on age. We know that

$$db(t) = [-Y(t) + rb(t)]dt \quad (33)$$

Thus

$$d(W(t) + b(t)) = \left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)\left((W(t) + b(t)\right)dt + \frac{\alpha-r}{\gamma\sigma}\left((W(t) + b(t)\right)dZ(t) \quad (34)$$

And wealth accumulation within lifetime, Let $X(t)$ be the total wealth, i.e. the sum of physical wealth and human wealth.

$$X(t) = W(t) + b(t)$$

Thus

$$dX(t) = \left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)X(t)dt + \frac{\alpha-r}{\gamma\sigma}X(t)dZ(t) \quad (35)$$

We can apply Euler equation and yield

$$X(t) = \left(\frac{a(t)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)t + \frac{\alpha-r}{\gamma\sigma}Z(t)X(0) \quad (36)$$

Using the boundary condition

$$a(T) = \chi(1-\zeta)^{1-\gamma} \quad (37)$$

Note that pre-tax the end-of-life wealth is

$$W(T) = X(T)$$
$$= \left(\frac{a(T)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T + \frac{\alpha-r}{\gamma\sigma}Z(T)\right]X(0)$$
$$= (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}(a(0))^{-\frac{1}{\gamma}} exp\left[\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T + \frac{\alpha-r}{\gamma\sigma}Z(T)\right]X(0) \tag{38}$$

And get the following after-tax result

$$W(T) = X(T) =$$
$$(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} a(0)^{-\frac{1}{\gamma}} exp\left(\frac{(1-\tau)r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T + \frac{\alpha-r}{\gamma\sigma}Z(T)X(0) \tag{38.1}$$

The details see Appendix D.

3 Research Method

3.1 Intergenerational Connection with Certain Life

Following Zhu (2010), now suppose $T, 2T, 3T \ \dots\ nT, \ \dots$ be the rich at time of generation 1,2, …n, …. Let $X_1 = X(T), X_2 = X(2T), X_3 = X(3T) \dots,\ X_n = X(nT), \dots$ Therefore

$$X(n+1) = X(n+1)T \tag{39}$$
$$= (1-\xi)W\big((n+1)T\big) + z(0)$$

Combining with eq.38, we may yield

$$= \left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T \ \ + \frac{\alpha-r}{\gamma\sigma}Z(T)\right]X(nT) + z(0)$$

Which is equivalent to

$$\left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T \ \ + \frac{\alpha-r}{\gamma\sigma}Z(T)\right]X_n + z(0) \tag{40}$$

Comply with $\rho_{n+1} = \left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T \ \ + \frac{(\alpha-r)}{\gamma\sigma}Z(T)\right]$

Accordingly, the result of Sornette (2006) could be employed that ρ_{n+1} is lognormally distribution.

Thereby $X_{n+1} = \rho_{n+1}X_n + z(0)$ (41)

3.2 Bequest Distribution with Pareto tail

The next step invokes that the bequest distribution has a Pareto upper tail. Then Sornette (2006) pointed out the bequest follows a distribution with a Pareto upper tail, if there exists a ν such that $E\rho_{n+1}^{\nu} = 1$. Remark that ρ_{n+1} is log-normally distributed.

Therefore

$$E\rho_{n+1}^{\nu} = \left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{\nu}{\gamma}} exp\left[\nu\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T \;\; + \frac{1}{2}\nu^2\frac{(\alpha-r)^2}{\gamma\sigma^2}T\right] = 1 \tag{42}$$

Rearrange them and yield the pretax result

$$\nu\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) + \frac{1}{2}\nu^2\frac{(\alpha-r)^2}{\gamma\sigma^2} = \frac{1}{T}\frac{\nu}{\gamma}\log\left(\frac{a(0)}{\chi(1-\xi)}\right)$$

$$\nu = \gamma\left(\frac{\frac{1}{T}\log\left(\frac{a(0)}{\chi(1-\xi)}\right) - r + \theta}{\frac{(\alpha-r)^2}{2\sigma^2}} - 1\right) \tag{43}$$

By Reed (2006), the starting wealth displays an asymptotic Pareto upper tail under tax system i.e.

$$P(x(0) > x) \sim x^{-\nu} \tag{44}$$

where

$$\nu = \gamma\left(\frac{\frac{1}{T}\log\left(\frac{a(0)}{\chi(1-\xi)}\right) + \theta - (1-\tau)r}{\frac{(\alpha-r)^2}{2\sigma^2}} - 1\right)$$

and

$$G = \frac{1}{(2\nu-1)}$$ [6]

Above equation (44) states that a sufficient condition for the convergence of the wealth distribution to the Pareto distribution is that the wealth differentiation is driven only by luck. It can be shown that this condition is not only sufficient, but also necessary, to ensure the Pareto distribution; see Levy (2003).

4 Numeric Illustration

Firstly, we consider the parameterization of the model. Consideration is given to the resulting the wealthy tax policy implications and how these suggest the optimal rule of thumb is an appropriate rule for investor's asset purchases. Numeric results are then discussed, and we consider with Hence, we set $\alpha - r/\gamma\sigma^2$ to 0.25.Its value is$\frac{\alpha-r}{\gamma\sigma^2} = 0.25$, Merton (1973) describe exactly satisfies the requirement two assets sufficient liquidity market conditions. In Fig.1 parameters setting: initial wealth=\$100,000(thousand), strike price = \$92,000 (thousand), initial age = 45, final age = 89, tax rate $\tau = 0.1,\ 0.2,\ 0.25,$

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[6]The Gini coefficient measures the deviation of the Lorenz curve from the equivalent distribution line connecting [0, 0] and [1, 1], which is illustrated by the straight line with slope 1. Specifically, the Gini coefficient is twice the area between the Lorenz curve and the equidistribution line. The Gini coefficient for the Pareto distribution is then calculated to be $G = 1 - 2\left(\int_0^1 L(F)dF\right) = \frac{1}{(2\nu-1)}$, where $\alpha \geq 1$ (see Gastwirth 1972).

$0.3,\ 0.35, 0.4$,rate of time preference=2% p.a., investment dividend rate=3%, giving q as 0.03 , expected return to risky assets=2% p.a., volatility of risky assets=20% p.a., The latter two values are from Lockwood (2012, Table 3).

Figure 2 illustrates the simulated results show that my model replicates the Gini coefficient of the wealth distribution for a particular initial value of wealth and a particular set of model parameters.Gini and Lorenz curve parameters setting: $\theta = 0.04, r = 0.02, \gamma$=2.5, $\alpha = 0.08, \sigma = 0.2, \zeta = 0.19$, $\chi = 15,\ t \in [45{,}89]$.

The rich may have inherited more, either in terms of financial resources or in terms of human capital, broadly defined. If inherited endowment is the major source of inequality, from a one-generation perspective there is a slight latent economic cost from a tax levy system that redistributes the advantages of this endowment. The rich may have different skills than everyone else, rather than more of the same kind of skills. This characterization certainly rings true, as the higher the bequest motive χ , or the lower the estate tax ζ,the smaller is v . Thus the impacts of χ , and ζon v are in line with our intuition about the role of bequest on wealth inequality: the more persistent the bequest process[7], the higher is the inequality in wealth distribution.

A smaller v implies a fatter tail of wealth distribution. Castaneda et al. (2003) study the steady-state economic implications of abolishing estate taxation, and find that abolishing estate taxation leads to very little change in wealth inequality. Cagetti and De Nardi (2009) study the effect of abolishing estate taxation on the stationary wealth distribution with different policy vary simulation in calibrated models. They also find that abolishing the estate taxation has little influence on the wealth inequality in each experiment result.

5 Implications for Policy Advice

In figure 1 we plot the red-dotted line strips out the effect of our synthetic put on optimally invested wealth before tax, thereby shedding light on the empirical importance of looking beyond the solution resulting from unconstrained dynamic programming. Herein figure 1 is similar to the share of risky assets of the portfolio line with Ding, J. et al. (2014).At the initial age of 45, and in the case of the solution that rules out negative bequests (i.e., the solution that incorporates a synthetic put option), the estimated share of risky assets is 29.27%, so our example suggests that at the outset of retirement it is not important in practice to account for luxury bequests when allocating assets. This difference is consistent with the fact that the required synthetic put has considerable time value at the outset of retirement. At the final age of 89, and in the case of the solution that rules out negative bequests, the expected share of risky assets is 35.11%. Bodie et al. (1992) show that labor income can make a great difference to asset allocation early in

[7]Atkinson (1970) and many following researchers prefer to assume that income is a continuous variable. It indicates that the population is latent infinite, but the sample can be finite. The finite population and discrete variables are at first simple concepts to understand whereas infinite population and continuous variables are too hard to accept. However as far as derivations and computations are concerned, continuous variables bring about integral calculus which is a simple issue once we know some primary theorems. Taking into account a continuous random variable opens the method for considering particular parametric densities such as the lognormal or the Pareto which have played an important role in income distribution research.

working life. On the other hand, the synthetic European put option makes hardly any difference to asset allocation late in retirement period, similar to decay over time in its value. The main behavior assumption invoked by Merton's (1969) model is that investors only use securities that they know about in constructing their optimal portfolios. In sensitivity analysis on taxes rich, along with the tax rate increasing, the proportion of risky assets is also associated with the decline.

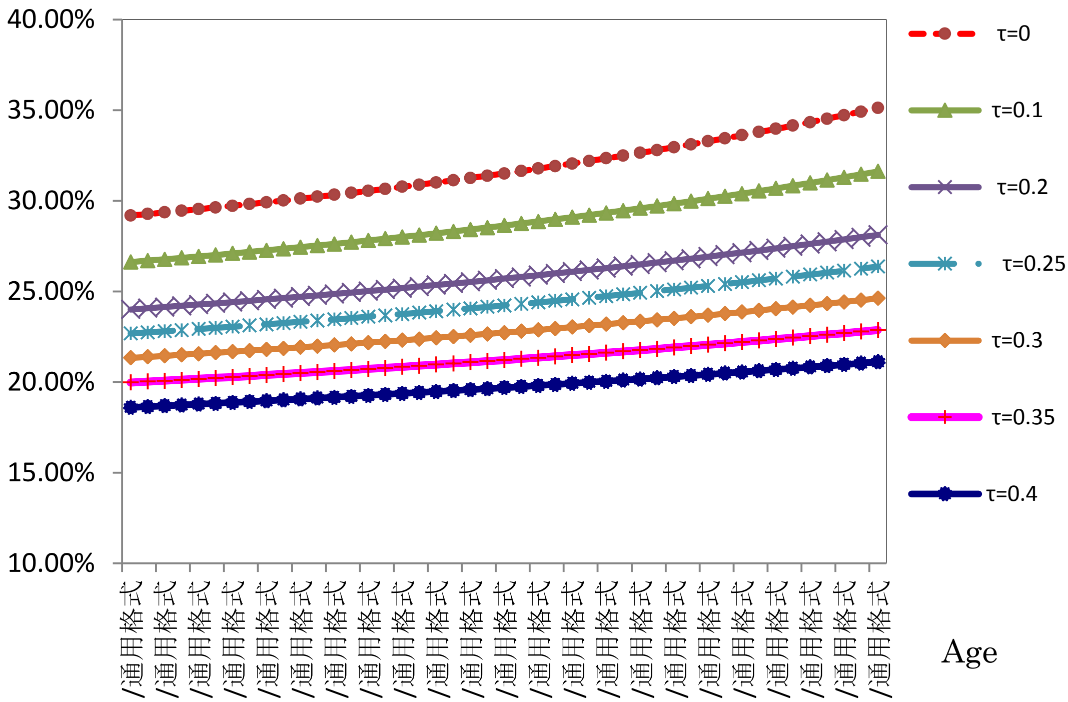

Figure 1: Asset Allocation under different tax regimes distribution

Thus the straight line represents perfect equality, and any departure from this 45-degree line represents inequality; see e.g., dotted line in figure 2. The simulation graph shows that the Lorenz curve on the red dashed line after tax the rich get close to 45-degree line, which displays the tax levied on the wealthy to reduce inequities and the distribution of wealth allocated more evenly. There is one important feature of the solution that should be pointed out: The wealth tax on the rich is fully demonstrated phenomenon tackling wealth inequality (can be reduced inequality).It play a corrective function on externalities of the gap between rich and poor.

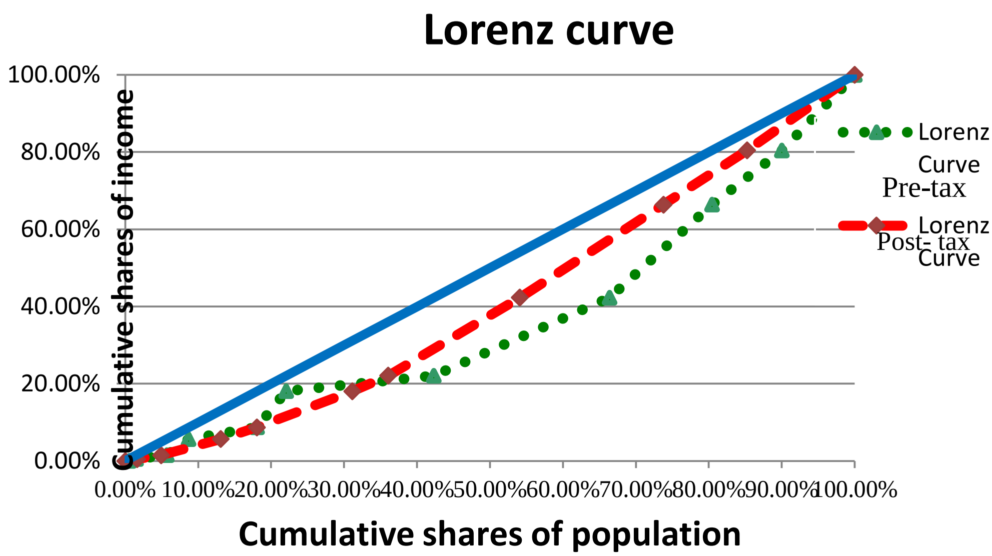

Figure 2: Compared with pre-tax and post- tax Lorenz curve

6 Conclusion

We study the dynamics of the wealth distribution in an economy with infinitely lived agents, intergenerational transmission of wealth, and redistributive taxing rich policy. We also demonstrate that wealth accumulation with idiosyncratic investment risk and lifetimes a Pareto wealth distribution can be produced. From a policy perspective, by levying a wealth transfer tax and redistributing revenue among the young generation, the government can further reduce the concentration of wealth. The higher the tax τ on the rich, the lower is the variance of wealth, while average wealth holdings are not affected. As a consequence, the coefficient of variation is reduced by the tax. Hence, the government can follow a wealthy taxation policy in order to reduce wealth inequality. While these results hold for the coefficient of variation as a measure of inequality, simulation suggests that they also hold for other, the Gini coefficient seems more popular measure. Taxing wealthy reduces not only the Gini coefficient but also the coefficient of variation. Future work could check whether the taxation result also survives under these more general specifications.

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Appendix

Appendix A.
To solve the optimal consumption and investment problem, the approach of stochastic dynamic optimization is considered. We have

$$J(C,B,S,t;T) = Max_{C,S}\left\{\frac{C^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho}E[J(C',B',S',t+\Delta t;T)]\right\} . \tag{A1}$$

The discounting term over such time interval is given by $\frac{1}{1+\rho\Delta t}$, and the utility over the time interval of length Δt is $\frac{C^{1-\gamma}}{1-\gamma}\Delta t$. Therefore the (A1) equation becomes:

$$J(C,B,S,t;T) = Max_{C,S}\left\{\frac{C^{1-\gamma}}{1-\gamma}\Delta t + \frac{1}{1+\rho\Delta t}E[J(C',B',S',t+\Delta t;T)]\right\} . \tag{A2}$$

Multiplying both RHS and LHS by a factor of $1+\rho\Delta t$ and rearrangement the terms, dividing by Δt and let it go to 0, the Bellman equation yields:

$$\rho J = Max_S\left\{\frac{C^{1-\gamma}}{1-\gamma} + \frac{1}{dt}E[dJ]\right\} \tag{A3}$$

Thus, according to Ito's lemma :

$$dJ = \left[\frac{dJ}{dt} + (rB + \alpha S - C)\frac{dJ}{dW} + \frac{1}{2}\sigma^2 S^2\frac{d^2J}{dW^2}\right]dt + \rho S\frac{dJ}{dW}dZ \tag{A4}$$

Substituting it to the Bellman equation, then get the Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{C^{1-\gamma}}{1-\gamma} + J_t + J_W(rB + \alpha S - C) + \frac{1}{2}J_{WW}\sigma^2 S^2 - \rho J = 0 \tag{A5}$$

First order condition with respect to consumption on the HJB equation yields:

$$J_W = \frac{\partial}{\partial C}\frac{C^{1-\gamma}}{1-\gamma} = C^{-\gamma} \tag{A6}$$

The optimal consumption is the given as:

$$C^* = (J_W)^{-\frac{1}{\gamma}}. \tag{A7}$$

Substituting the optimal consumption into the HJB equation becomes:

$$\frac{C^{1-\gamma}}{1-\gamma} + J_t + J_W(rB + \alpha S - C^*) + \frac{1}{2}J_{WW}\sigma^2 S^2 - \rho J = 0. \tag{A8}$$

Applying the condition $W = B + S$ to eliminate B from the equation, and conjecture that the value function J must be linear to $\frac{W^{1-\gamma}}{1-\gamma}$, and takes the form;

$$J(C,B,S,t;T) = a(t;T)\frac{W^{1-\gamma}}{1-\gamma} \tag{A9}$$

Where $\mathbf{a}(t;T) > 0, \forall t \in [0,T]$ is a horizon dependent function.

Substituting C^* into $(J_W)^{-\frac{1}{\gamma}} = a^{-\frac{1}{\gamma}}W, J\ into\ a'\frac{W^{1-\gamma}}{1-\gamma}$,and $J_t\ into\ a'\frac{W^{1-\gamma}}{1-\gamma}$ in the HJB equation yields that:

$$\frac{a^{-\frac{1-\gamma}{\gamma}}}{1-\gamma}W^{1-\gamma} + a'\frac{W^{1-\gamma}}{1-\gamma} + aW^{-\gamma}\left(rW + (\alpha - r)S - a^{\frac{-1}{\gamma}}W\right) - \frac{\gamma}{2}aW^{-1-\gamma}.\sigma^2 S^2 - \rho a\frac{W^{1-\gamma}}{1-\gamma} = 0 \tag{A10}$$

First order condition on $\boldsymbol{s}$ obtains the optimal portfolio invested in stock:

$$S^* = \frac{\alpha - r}{\gamma\sigma^2}W \tag{A11}$$

This completes the proof of equation (26).

Appendix B: Proof of Proposition 2.

The agent lives from 0 to T. For the agent who the wealth pass through threshold to be the rich at time u,the value of his idiosyncratic risky asset at time t .The rich have portfolio selection problem between a risky asset and a riskless asset. The rich agent's problem becomes

$$J(W,t) = \underset{C_t,B_t,S_t:t>0}{Max} E\left[\int_t^T e^{-\theta(\mu-t)}\frac{C(\mu)^{1-\gamma}}{1-\gamma}d\mu + e^{-\theta(\mu-t)}\frac{(1-\zeta)(B_T+S_T)^{1-\gamma}}{1-\gamma}\right] \tag{B1}$$

$$dW(\mu) = \left[rW(\mu) + (\alpha - r)S_\mu - C(\mu) + Y(\mu)\right]d\mu + \sigma S_\mu dZ_\mu \tag{B2}$$

Suppose that

$b(t) = \int_t^T Y(\mu)e^{-r(\mu-t)}d\,\mu$ i.e.

$db(t) = [-Y(t) + rb(t)]dt$

Thus

$$\begin{aligned} d(W(t)+b(t)) \\ &= \left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)\big((W(t)+b(t)\big)dt \\ &+ \frac{\alpha-r}{\gamma\sigma}\big((W(t)+b(t)\big)dZ(t) \end{aligned}$$

By Hamilton-Jacobi-Bellman approach obtain

$$\theta J(W,t) = \underset{C_t,B_t,S_t:t>0}{Max}\left\{\frac{C(t)^{1-\gamma}}{1-\gamma} + J_w(W,t)[rW(t) + (\alpha-r)S(t) - C(t) + Y(t)] + \frac{1}{2}J_{ww}(W,t)\sigma^2 S(t)^2 + J_t(W,t)\right\} \tag{B3}$$

We have the F.O.C.

$$C(t)^{-\gamma} = J_w(W,t) \tag{B4}$$

$$J_w(W,t)(\alpha-r) = -J_{ww}(W,t)\sigma^2 S(t) \tag{B5}$$

Guess

$$J(W,t) = \frac{a(t)}{1-\gamma}[W(t)+b(t)]^{1-\gamma} \tag{B6}$$

Where

$b(t) = \int_t^T Y(\mu)e^{-r(\mu-t)}d\mu$

$$J_w(W,t) = a(t)[W(t)+b(t)]^{-\gamma} \tag{B7}$$

$$J_{ww}(W,t) = -\gamma a(t)[W(t)+b(t)]^{-\gamma-1} \tag{B8}$$

After arrangement, We have

$$C(t) = a(t)^{-\frac{1}{\gamma}}[W(t)+b(t)] \tag{B9}$$

$$S(t) = \frac{\alpha-r}{\gamma\sigma^2}[W(t)+b(t)] \tag{B10}$$

Appendix C

One can also derive the formula for constant relative risk aversion utility (CRRA), the analytic form of $J(W,t;T)$ itself can be obtained as follows

$J(W,t;T) = a(t;T)\,\frac{W(t)^{1-\gamma}}{1-\gamma}$

Where $a(t;T)$ satisfies the following ordinary differential equation

$$\frac{1}{1-\gamma}\frac{a'}{a} + \frac{\gamma}{1-\gamma}\,e^{-\frac{\theta}{\gamma}t}a^{-\frac{1}{\gamma}} + r + \frac{(\alpha-r)^2}{2\gamma\sigma^2} = 0 \tag{C1}$$

Hereafter the prime symbol is used to denote the derivative with respect to time and solve the **Bernoulli's equation** form, analytical solution of equation (C1) with zero bequest at time T can be obtained as:

$$a(t;T) = e^{-\theta t}\left(\frac{e^{\eta(T-t)}-1}{\eta}\right)^{\gamma} \quad \text{(C2)}$$

Where

$$\eta = \frac{1-\gamma}{\gamma}\left[r + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right] - \frac{\theta}{\gamma} \quad \text{(C3)}$$

with the terminal condition:

$$a(T,T) = 1 \quad \text{(C4)}$$

We can derive a at each time t numerically by discretization $a_t = a_{t-1} + \Delta a_t$ and work backward from the terminal time. Optimal consumption contains a horizon dependent fraction of wealth, which is independent of wealth at hand:

$$c_t^* = a(t;T)^{\frac{-1}{\gamma}}\, W_t \quad \text{(C5)}$$

It can be easily shown that in the infinite horizon case, optimal consumption is a constant proportion of wealth:

$$c_t^* = \frac{1}{\gamma}\left[\theta - (1-\gamma)r - \frac{(1-\gamma)(\alpha-r)^2}{2\gamma\sigma^2}\right] W_t \quad \text{(C6)}$$

as the same given by Merton (1969).

Appendix D.

Furthermore, take into account taxes on rich condition, the investor makes contingent plans for a bequest χ , The consumption C is made through the money market account. The participant has a CRRA utility function over consumption and terminal wealth, and that maximize expected utility, Agent's problem

$$\max_{c(t),S(t)}\left\{E_t\int_{\mathrm{t}}^{T}\frac{C(\mu)^{1-\gamma}}{1-\gamma}e^{-\Theta(\mu-t)}d\mu + \chi\frac{[(1-\zeta)W(T)]^{1-\gamma}}{1-\gamma}e^{-\Theta(T-t)}\right\} \quad \text{(D1)}$$

subject to a budget constraint

$$dW(\mu) = [(1-\tau)rw(\mu) + (\alpha-(1-\tau)r)S(\mu) - C(\mu) + Y(\mu)]d\mu + (1-\tau)\sigma W(\mu)dZ(t)$$

where τ is capital income tax rate. ζ is estate tax rate. The agent's human wealth

$$b(t) = \int_{\mathrm{t}}^{\mathrm{T}} Y(\mu)e^{-(1-\tau)r(\mu-t)}d\mu$$

Plugging these expressions into the HJB, we have the agent's policy functions after tax are

$$c(t) = a(t)^{-\frac{1}{\gamma}}[W(t)+b(t)] \quad \text{(D2)}$$

$$S(t) = \frac{(1-\tau)\alpha-(1-\tau)r}{\gamma\sigma^2(1-\tau)^2}[W(t)+b(t)] \quad \text{(D3)}$$

And

$$d[W(t)+b(t)] = \left[(1-\tau)r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right][W(t)+b(t)]dt + \frac{\alpha-r}{\gamma\sigma}[W(t)+b(t)]dZ(t) \quad \text{(D4)}$$

Combining with equation (35), we know

$$dX(t) = \left[(1-\tau)r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right]X(t)dt + \frac{\alpha-r}{\gamma\sigma}X(t)dZ(t) \quad \text{(D5)}$$

The end-of-life wealth post tax is

$$W(T) = X(T) = (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} a(0)^{-\frac{1}{\gamma}} exp\left(\frac{(1-\tau)r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T + \frac{\alpha-r}{\gamma\sigma} Z(T)X(0) \qquad \text{(D6)}$$