Communications in Mathematical Finance, vol.4, no.2, 2015, 1-47
ISSN: 2241-1968 (print), 2241-195X (online)
Scienpress Ltd, 2015

Valuing the Probability of Generating Negative Interest Rates under the Vasicek One-Factor Model

Stéphane Dang-Nguyen[1] and Yves Rakotondratsimba[2]

Abstract

The generation of scenarios for interest rates is needed in many contexts, as in the valuation capital requirements (under Solvency II or Basel 3 regulation frameworks), in other risk management tasks (the application of a risk measure to a portfolio) as well as in the pricing of the financial contracts. For such purposes, a model for the term structure, as the famous Vasicek one-factor model, is needed. Though it is very often considered as a benchmark, mainly due to its tractability, unfortunately it generates negative interest rates with a non-null probability.

Our purpose in this paper is to analyse to what extend this model can be used to the generation of yield curve scenarios, at one or more future time horizons and under both historical and risk-neutral measures given its inconsistency. In the first case, the spot rate is defined in terms of a realization of a Gaussian variable and the bounds avoiding negative

[1] Alef-Servizi Spa, Viale Regina Margherita 140, 00198 Roma, Italy.
E-mail: stephane.dang-nguyen@alef.it
[2] ECE Paris Graduate School of Engineering, 37 quai de Grenelle CS71520, 75 725 Paris 15, France. E-mail: w_yrakoto@yahoo.com

Article Info: *Received* : July 7, 2015. *Revised* : August 12, 2015
Published online : November 10, 2015

yields are analysed. In the second case, the problem is described involving of a hitting time in order to value the probability to obtain negative yields during the simulation. Moreover, some numerical examples are provided in order to illustrate the computation of the probability.

Mathematics Subject Classification: 60G15; 60G40
Keywords: Interest rate term structure; Monte-Carlo pricing and risk management; hitting time; Vasicek Model

1 Introduction

1.1 The context

Either for the determination of capital requirements (as in the Solvency II and Basel 3 regulation frameworks), for the risk management tasks (as in the application of a risk measure to a portfolio) or for the pricing of simple and complex financial instruments, it is useful to perform a generation of future interest rates (IR) that are economically acceptable. For such a purpose, there is a need to introduce a model for the IR term structure, as for example the famous *Vasicek* one-factor model (1-VM) of [1] which is encompassed into the class of Gaussian Affine Term Structure Model (GATSM). For the sake of simplicity, our analyses are limited to the case of 1-VM, but the situation studied here would shed light the difficulties appearing with any general GATSM.

As shown in [2], the 1-VM is an attractive benchmark model for IR, essentially due to its explicit properties and tractability[3]. However, some limitations arise as: generation of negative IR, poor fitting of the initial IR term structure, perfect correlation between different rates of the yield curve. This last inconvenience lead people, both from markets and academics, to switch to an extended model with two or more uncertainty factors that can belong to the

[3] under the 1-VM, the discount factor as well as options on discount bonds are known in closed form expression allowing one to value a wide range of financial contracts like bonds or basic interest-rate sensitive derivatives.

GATSM class. Of course alternative models avoiding negative IR do exist (as for example [3], [4], [5], [6] and [7]) but, the attempt to be consistent with the economic reality make them too complex[4] and less tractable[5], such that very often practitioners tend to give preference to the economically inconsistent GATSM models and the 1-VM plays a benchmark role.

Theoretically, as is explained below, the 1-VM induces negative IR with a non-zero probability, which are in general economically meaningless although they are observed on market for currencies like Euro, Japanese Yen and Swiss Franc since January 2015 for some maturities. In fact, no investor would accept an investment agreement to certainly loose money[6]. Very often it is claimed elsewhere in the literature, in textbooks like [8], or various papers, e.g. [9], that the probability of such a pathology with the 1-VM to arise is very small, even if it seems that no explicit consideration of the situation is really available. In fact, some reference mention that the spot rate may become negative and value the corresponding probability. However, we show that negative value of this rate do not necessarily imply negative interest rates. Moreover, [9] shows that negative spot rate of the 1-VM can cause troubles in the pricing of some derivatives and of the bonds with a long maturity. Consequently, the inconvenience linked to the model and its extensions can be neglected facing to the benefit it could bring. No close look to these difficulties is really available in the literature to the best of our knowledge, despite the importance of the 1-VM and GATSM in the generation of IR.

Our purpose in this paper is to analyse to what extend the 1-VM can be used to generate scenarios for the IR at one or more future time horizon, given its inconsistency. The generation under the 1-VM can be used for the valuation

[4] on the calibration and/or analytical point of view. For example, the conditional distribution of the spot rate in the [3] model is described by a non-central chi-squared distribution which is more difficult to handle than the Gaussian distribution corresponding to the same rate under the 1-VM.

[5] as Monte-Carlo or other numerical approaches are needed for the pricing even for some basic IR contracts if non closed form expressions are available.

[6] as with a zero-coupon whose the price exceeds one unit. However, in the sequel of the financial crisis such negative IR were observed for different financial contracts like the LIBOR rates for the Swiss Franc and German bonds were issued at the primary auction of July 18, 2018. Moreover, the deposit rate of European Central Bank became negative at June 11, 2014.

of an asset and/or liabilities portfolio a one future time, provided that its value can be expressed in terms of discount factor, e.g. a bond portfolio. However, complex financial contracts or insurance policies imply cash-flow exchanges at various future dates, hence the previous case, focusing on one simulation horizon has to be extended. We attempt here to remedy to these observations by providing both theoretical analysis and numerical examples. Moreover, we analyse two approaches that are mathematically similar but different on a financial point of view: The spot rate can be generated under the historical or risk-neutral measure. The first case corresponds to risk management tasks while the second one is performed for pricing purposes.

1.2 Our contribution

Under the 1-VM, a simulated value of the IR for a given time to maturity at one future time horizon is just the result of a realization of a Gaussian random variable (referred here as a shock). This observation is the basis of the analysis of negative IR given a future time horizon. Firstly, we make explicit that any shock below some level leads to a negative IR. This bound depends on the considered time-horizon, the IR maturity, the model parameters (hence on the generation under the historical or risk-neutral measure) and some initial state variable (which should be viewed as an instantaneous short rate[7]). Consequently, *the generation of negative IR remains unavoidable, even the user has made efforts to obtain good parameters calibration of the model, because of the Gaussian property of the 1-VM.* However, the user expects that this level is low or negative enough such that the probability to simulate a shock below this level should be very small. This is the rationale behind the claim, seen in the literature, about the good reason to maintain in use the 1-VM (or more generally the GATSM) despite this limitation. Secondly, in order to prevent to the harmful consequence of a brute application of the 1-VM[8], we suggest a manner to tweak the model in order to discard negative IR. However

[7]This variable is not observed on the markets as corresponding to an IR with an infinitesimal maturity. However, usually an IR with a short term maturity as one or three month maturity can be used as a proxy.

[8]for example providing the simulated negative IR to a valuation or risk-management system that is not designed to handle them correctly.

by so doing, we recognize that this model and its extensions encompassing by GATSM are definitely not always good models for IR simulation. Thirdly, we focus on the generation of IR with different time to maturities. We derive a restriction on the parameters that make the level non-increasing, thus it is sufficient to focus on the shortest time to maturity. Furthermore, this non-increasing property is not necessary when dealing with finite number of maturities because the maximal level of a finite set can be easily computed. Since there is no well-documented studies supporting these claims, we hope, with this paper, to provide to the reader an explicit reference on the question. These theoretical results are presented in section 2.

Moreover in the context of *Monte-Carlo* simulations, the IR have to be simulated for various future time horizons and for different time to maturities, extending the previous analysis. Assuming constant time to maturities, the problem of the generation of negative IR can be formulated in terms of a hitting time. In fact, according to the previous observations, the model inconsistency appears when the spot rate (generated under the historical or risk-neutral measure) is lower than the aforementioned level but this process can be observed continuously or at discrete times. It will be shown that the first case is easier to handle on a numerical point of view. However, the second one is a natural formulation in the context of *Monte-Carlo* simulations. Actually, the short rate is generated at discrete future times and the IR have to be non-negative at these dates so as handle economically sound scenarios. Again, the previously mentioned rationale holds and 1-VM is valued defining an acceptable probability for negative IR. As a consequence, the cumulative probabilities of the hitting times allow one to obtain a maximal simulation horizon given the parameters of the model. To the best of our knowledge, this formulation in terms of hitting time and the application of the representation of hitting times for 1-VM are a second contribution of this paper. These results are presented in section 3.

Though technical reasons as those mentioned above are presented, we also bring here various numerical examples in section 4 aiming at scrutinizing the validity and limit of the 1-VM with respect to the generation of negative IR. These market conditions will show that negative IR are always a concern to consider before and after the financial crisis, even if this problem is more pronounced since 2007. This fact reinforces the idea to switch to alternative

IR models. However, given the complexity of these advanced models, it makes sense to perform an accurate examination of the validity of the former classical tractable models facing various market conditions. Furthermore, the GATSM can serve as a shadow rate model underlying another suitable model for the IR (the 1-VM is directly used as shadow rate in [4]). This is partly the reason of our present work here. Moreover, we also perform here some empirical study of the sensitivity of the 1-VM to produce negative IR, with respect to the model parameters. In fact, there are various ways to calibrate the model[9]. The sensitivity study may help to understand to which parameter(s) must receive a special care during the calibration in order to limit the harmful effect resulting from the model inconsistency. This numerical analysis is applied for the simulation at one future date or involving the hitting times.

2 The generation of negative IR at one future date

In this section, we focus on the simulation at one future horizon, denoted t, of one or several zero-coupon bonds (ZCB) prices under the historical or risk-neutral measure. In a first time, the 1-VM is presented (2.1), next a condition avoiding negative prices is obtained and a restriction on the parameters simplifying this problem in the bond portfolio context is derived (2.2). Since the 1-VM is Gaussian, this problem is equivalent to a restriction of the normal variables (or shocks) driving the spot rate (2.3). Then, the ZCB prices are reformulated integrating these shocks so as to provide a financial interpretation (2.4). Lastly, the simulation of a ZCB portfolio without the previous constraint is discussed (2.5).

2.1 An introduction to the 1-VM

As described in [1] or [2], the instantaneous short rate $(r_t(\cdot))_{t\geq 0}$ for the

[9] for example, the cross-sectional analysis, i.e. minimizing the distance between the model and market prices, can provide various estimated that depend on the initial point used by the optimization procedure.

1-VM, under a risk-neutral probability measure $\mathbb{Q}$, is driven by the stochastic differential equation (SDE):

$$dr_t(\cdot) = \kappa \left[\theta - r_t(\cdot)\right] dt + \sigma dW_t(\cdot) \tag{1}$$

where the non-negative constants κ, σ and θ represent the mean reversion speed, the long-term mean and the volatility of the instantaneous spot rate and $(W_t(\cdot))_{t\geq 0}$ denotes a standard $\mathbb{Q}$-Brownian motion. The dot notation for each expression in the following is used in the sequel to differentiate between random and deterministic/constant quantities. The SDE driving the spot rate dynamics, under the historical probability measure $\mathbb{P}$, is obtained with a change of measure. We adopt the affine form of market risk premium given in [10], hence the dynamics of the process, denoted $(r_{t,\mathbb{P}}(\cdot))_{t\geq 0}$, become:

$$dr_{t,\mathbb{P}}(\cdot) = \left[\kappa\theta + \lambda_1 - (\kappa - \lambda_2) r_{t,\mathbb{P}}(\cdot)\right] dt + \sigma d\tilde{W}_t(\cdot) \tag{1$'$}$$

where λ_1 and λ_2 are two constants and $\left(\tilde{W}_t(\cdot)\right)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$. Note that SDE (1$'$) can be written in form of (1) using the parameters $\kappa_{\mathbb{P}}$, $\theta_{\mathbb{P}}$ and $\sigma_{\mathbb{P}}$ defined by $\kappa_{\mathbb{P}} = \kappa - \lambda_2$, $\theta_{\mathbb{P}} = (\kappa\theta + \lambda_1)/\kappa_{\mathbb{P}}$ and $\sigma_{\mathbb{P}} = \sigma$. Since the volatility coefficients are the same under SDE (1) and (1$'$), they are not differentiated in the following. Moreover, if λ_1 and λ_2 are null, then SDE (1) and (1$'$) coincide.

It is always assumed in the following that $0 < t$, hence the instant 0 can be seen as the present-time and t is a future-time horizon. Using the Itô's lemma, the spot rate $r_t(\cdot)$ driven by SDE (1) satisfies:

$$r_t(\cdot) = \exp\left(-\kappa t\right) r_0 + \kappa\theta \mathrm{b}\left(t;\kappa\right) + \sigma \mathrm{b}^{\frac{1}{2}}\left(t;2\kappa\right) \varepsilon_{t|0}(\cdot) \tag{2}$$

where the function $\mathrm{b}\left(u;\alpha\right)$ is defined by:

$$\mathrm{b}\left(u;\alpha\right) = \frac{1}{\alpha}\left[1 - \exp\left(-\alpha u\right)\right] \tag{3}$$

and the term $\varepsilon_{t|0}(\cdot)$ represents a standard normal Gaussian random variable:

$$\varepsilon_{t|0}(\cdot) = \mathrm{b}^{-\frac{1}{2}}\left(t;2\kappa\right) \exp\left(-\kappa t\right) \int_0^t \exp\left(\kappa u\right) dW_u(\cdot) \tag{4}$$

According to eq. (2), the value of $r_t(\cdot)$ is actually dictated by the standard Gaussian variable $\varepsilon_{t|0}(\cdot)$. If the spot rate is driven by SDE (1$'$), then eqs. (2) and (4) are rewritten using the real-world parameters $\kappa_{\mathbb{P}}$, $\theta_{\mathbb{P}}$ and σ:

$$r_{t,\mathbb{P}}(\cdot) = \exp\left(-\kappa_{\mathbb{P}} t\right) r_0 + \kappa_{\mathbb{P}}\theta_{\mathbb{P}} \mathrm{b}\left(t;\kappa_{\mathbb{P}}\right) + \sigma \mathrm{b}^{\frac{1}{2}}\left(t;2\kappa_{\mathbb{P}}\right) \varepsilon_{t|0;\mathbb{P}}(\cdot) \tag{2$'$}$$

and:

$$\varepsilon_{t|0;\mathbb{P}}(\cdot) = \mathrm{b}^{-\frac{1}{2}}\left(t; 2\kappa_{\mathbb{P}}\right) \exp\left(-\kappa_{\mathbb{P}} t\right) \int_0^t \exp\left(\kappa_{\mathbb{P}} u\right) d\tilde{W}_u(\cdot) \tag{4'}$$

Under the 1-VM of eq. (1), it is well established (see [2]), that the time-t (random) price $P(t, t+\tau)(\cdot)$, of a ZCB with the time to maturity τ, for $0 < \tau$, is given by the formula:

$$P(t, t+\tau)(\cdot) \equiv P\left(\tau; r_t(\cdot); \kappa, \theta, \sigma\right) = \exp\Big[-\mathrm{b}\left(\tau; \kappa\right) r_t(\cdot) + \mathrm{a}\left(\tau; \kappa, \theta, \sigma\right)\Big] \tag{5}$$

with:

$$\mathrm{a}\left(\tau; \kappa, \theta, \sigma\right) = -\mathrm{c}_1 \mathrm{b}^2\left(\tau; \kappa\right) - \mathrm{c}_2\left[\tau - \mathrm{b}\left(\tau; \kappa\right)\right] \tag{6}$$

and using the notations $\mathrm{c}_1 \equiv \mathrm{c}_1(\kappa, \sigma) = \frac{\sigma^2}{4\kappa}$ and $\mathrm{c}_2 \equiv \mathrm{c}_2(\kappa, \theta, \sigma) = \theta - \frac{\sigma^2}{2\kappa^2}$. A striking point with eq. (5) is that any zero-coupon price $P(t, t+\tau)(\cdot)$ can be seen as a function of the time-t random state variable $r_t(\cdot)$, the time to maturity τ and the model parameters κ, θ and σ.

According to eq. (5), a generated series of ZCB prices $P(t, t+\tau_1)(\cdot)$, $\cdots$, $P(t, t+\tau_M)(\cdot)$ for non-negative and increasing time to maturities $\tau_1, \cdots, \tau_M$, as required for example in valuation of a portfolio of IR contracts as Coupon-Bearing-Bonds (CBB), Interest-Rate-Swaps (IRS), depends on the time-t value of the spot rate. The generation of this time series depends on the purpose of the analysis. In fact, in the pricing of financial instruments and insurance policies, the spot rate $r_t(\cdot)$ is generated under $\mathbb{Q}$ from the present time to t, i.e. using SDE (1), then the future ZCB prices are computed using eq. (5) since the simulated discount factor is the conditional expectation of the integral of the spot rate from t to $t+\tau$ given the value of $r_t(\cdot)$. For risk-management tasks, the approach is slightly different. In fact, the path of the spot rate is described under the historical measure from the present time to the future time t, hence using SDE (1′). Then, the discount factor is valued according to eq. (5) using the simulated value of the spot rate $r_{t,\mathbb{P}}(\cdot)$ instead of $r_t(\cdot)$. In order to have a formal distinction among these two generated discount factors, $P_{\mathbb{P}}(t, t+\tau)(\cdot)$ denotes the ZCB generated for risk-management purposes. Consequently, a series denoted $P_{\mathbb{P}}(t, t+\tau_1)(\cdot)$, $\cdots$, $P(t, t+\tau_M)(\cdot)$ of ZCB prices are used for risk management purposes.

2.2 Model price and realistic situation

The future model price $P(t,t+\tau)(\cdot)$ or $P_{\mathbb{P}}(t,t+\tau)(\cdot)$ in which form is given by eq. (5), appears to be an acceptable market price[10] whenever the term inside the exponential expression is a negative real number. This is the case whenever the time-t value of the state variables $r_t(\cdot)$ and $r_{t;\mathbb{P}}(\cdot)$ are greater than a bound, denoted $\mathcal{B}(\tau;\kappa,\theta,\sigma)$:

$$\begin{aligned}\mathcal{B}(\tau;\kappa,\theta,\sigma) &= \frac{1}{\mathrm{b}(\tau;\kappa)}\mathrm{a}(\tau;\kappa,\theta,\sigma) \\ &= -\frac{1}{\mathrm{b}(\tau;\kappa)}\left(c_1\mathrm{b}^2(\tau;\kappa)+c_2\left[\tau-\mathrm{b}(\tau;\kappa)\right]\right)\le r_t(\cdot) \qquad (7)\end{aligned}$$

and for risk management condition (7) is formulated as:

$$\mathcal{B}(\tau;\kappa,\theta,\sigma)\le r_{t;\mathbb{P}}(\cdot) \qquad (7')$$

It comes from eqs. (7) and (7′) that the bounds avoiding negative interest rates are not necessary positive or negative and this fact is illustrated in section 4. However, in various papers and textbooks like [8], only the problem of a negative value of the spot rate is considered. However, this condition does not necessary imply negative interest rates as shown by the two above equations. Moreover, there is no reason that these inequalities hold for any time to maturity τ, with $0<\tau$. This is a problem which can be encountered when dealing with one or a series of ZCB prices. It can be observed that $0<\mathrm{c}_1$ since the parameters are positive and $0\le\mathrm{c}_2$ if and only if:

$$\sigma^2\le 2\kappa^2\theta \qquad (8)$$

This inequality means that the instantaneous spot rate volatility coefficient has to be bounded by the constant $2\kappa^2\theta$ and one can state that:

Lemma 2.1. *Under condition (8), the mappings:*

$$\tau\in(0,\infty)\longmapsto \mathrm{b}(\tau;\kappa)\in\left(0,\frac{1}{\kappa}\right)$$

and:

$$\tau\in(0,\infty)\longmapsto(\tau-\mathrm{b}(\tau;\kappa))\in(0,\infty)$$

are non-decreasing.

[10] in the sense that $0<P(t,t+\tau)\le 1$ or $0<P_{\mathbb{P}}(t,t+\tau)(\cdot)\le 1$.

Consequently, these mappings defined with the historical parameters are also non-decreasing under condition (8). From this lemma and inequalities (7) and (7′), we can state that:

Lemma 2.2. *Under condition (8), if the time-t state variable $r_t(\cdot)$ is positive, then the model price $P(t, t+\tau)(\cdot)$ is acceptable to represent a possible market price, otherwise an issue may arise. In a similar vein, under condition (8), if the time-t state variable $r_{t;\mathbb{P}}(\cdot)$ is positive, then the model price $P_{\mathbb{P}}(t, t+\tau)(\cdot)$ is acceptable to represent a possible market price, otherwise an issue may arise.*

Actually the question of acceptability has to be considered essentially when t is a future-time horizon, since very often at the present time some market prices of ZCB may be already available[11]. The analysis of such a situation is the purpose of the next subsection.

2.3 Future price under the 1-VM

Given a future horizon t, as seen in eq. (5), the prices $P(t, t+\tau)(\cdot)$ and $P_{\mathbb{P}}(t, t+\tau)(\cdot)$ are functions of $r_t(\cdot)$ and $r_{t;\mathbb{P}}(\cdot)$, which are conditionally Gaussian random variables defined by some shocks $\varepsilon_{t|0}(\cdot)$ and $\varepsilon_{t|0;\mathbb{P}}(\cdot)$. Actually the following can be stated from eqs. (2) and (7) as well as (2′) and (7′).

Proposition 2.3. *Under the Vasicek one-factor model under SDE (1), the expression $P(t, t+\tau)(\cdot)$ can be considered as an acceptable market price if and only if the shock $\varepsilon_{t|0}(\cdot)$ is not too negative in the sense that:*

$$\mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma\right) \leq \varepsilon_{t|0}(\cdot) \tag{9}$$

where the bound $\mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma\right)$ is given by:

$$\begin{aligned}&\mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma\right)\\ &\qquad \equiv \frac{1}{\sigma \mathrm{b}^{\frac{1}{2}}\left(t; 2\kappa\right)} \left(\frac{\mathrm{a}\left(\tau; \kappa, \theta, \sigma\right)}{\mathrm{b}\left(\tau; \kappa\right)} - \left[\exp\left(-\kappa t\right) r_0 + \kappa\theta \mathrm{b}\left(t; \kappa\right)\right] \right)\end{aligned} \tag{10}$$

[11] however it often arises as with the case of 1-VM, the prices given by the model do not fit exactly those available on the market. One can overcome this unpleasant by switching to extended 1-VM as with the [11] one-factor model for example or, equivalently, applying to the 1-VM a deterministic shift extension as in [2].

Under the historical dynamics of SDE (1′), i.e. for the price $P_{\mathbb{P}}(t,t+\tau)(\cdot)$*, the shock has to be not too negative in sense of:*

$$\mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}}\right) \leq \varepsilon_{t|0;\mathbb{P}}(\cdot) \tag{9'}$$

where the bound $\mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}}\right)$ *is defined by:*

$$\begin{aligned}&\mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}}\right)\\ &\quad\equiv \frac{1}{\sigma \mathrm{b}^{\frac{1}{2}}\left(t;2\kappa_{\mathbb{P}}\right)}\left(\frac{\mathrm{a}\left(\tau;\kappa,\theta,\sigma\right)}{\mathrm{b}\left(\tau;\kappa\right)} - \left[\exp\left(-\kappa_{\mathbb{P}}t\right)r_0 + \kappa_{\mathbb{P}}\theta_{\mathbb{P}}\mathrm{b}\left(t;\kappa_{\mathbb{P}}\right)\right]\right)\end{aligned} \tag{10'}$$

From eq. (9), the (future) yield of the ZCB price $P(t,t+\tau)(\cdot)$ satisfies:

$$\begin{aligned}&\tau \in (0,\infty) \longmapsto R(t,t+\tau)(\cdot)\\ &\qquad \equiv \frac{1}{\tau}\sigma \mathrm{b}^{\frac{1}{2}}\left(t;2\kappa\right)\mathrm{b}\left(\tau;\kappa\right)\left[\varepsilon_{t|0}(\cdot) - \mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma\right)\right]\end{aligned} \tag{11}$$

In a similar vein using eq. (9′), the yield of the price $P_{\mathbb{P}}(t,t+\tau)(\cdot)$ is reformulated as:

$$\begin{aligned}&\tau \in (0,\infty) \longmapsto R_{\mathbb{P}}(t,t+\tau)(\cdot)\\ &\qquad \equiv \frac{1}{\tau}\sigma \mathrm{b}^{\frac{1}{2}}\left(t;2\kappa_{\mathbb{P}}\right)\mathrm{b}\left(\tau;\kappa\right)\left[\varepsilon_{t|0;\mathbb{P}}(\cdot) - \mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}}\right)\right]\end{aligned} \tag{11'}$$

Eq. (11) (and (11′)) shows that if the inequality (9) (or (9′)) is satisfied by $\varepsilon_{t|0}(\cdot)$ (or $\varepsilon_{t|0;\mathbb{P}}(\cdot)$), then the yield $R(t,t+\tau)(\cdot)$ (or $R_{\mathbb{P}}(t,t+\tau)(\cdot)$) will be positive and conversely. From proposition 2.3, one can state that, when used as a generator of IR scenarios for the future time-horizon t, the 1-VM always generates negative IR with time to maturity τ for any shock $\varepsilon_{t|0}(\cdot)$ (or $\varepsilon_{t|0;\mathbb{P}}(\cdot)$) satisfying $\varepsilon_{t|0}(\cdot) < \mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma\right)$ (or $\varepsilon_{t|0;\mathbb{P}}(\cdot) < \mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}}\right)$ for risk-management purposes). Therefore these quantities allows one to judge if the 1-VM calibrated on the market yield curve will preserve the non-negativity property of the ZCB prices $P(t,t+\tau)(\cdot)$ or $P_{\mathbb{P}}(t,t+\tau)(\cdot)$.

It is reasonable to assume that the initial instantaneous short rate r_0 is positive. Consequently, in this case, the quantity $\exp\left(-\kappa t\right)r_0 + \kappa\theta \mathrm{b}\left(t;\kappa\right)$ is also positive. On the other hand, one has $\mathrm{a}\left(\tau;\kappa,\theta,\sigma\right) < 0$ whenever the instantaneous short rate volatility σ is sufficiently small in the sense of condition (8). Therefore under this condition one has $\mathcal{E}(t,\tau;r_0;\kappa,\theta,\sigma) < 0$. If condition (8) does not hold, it may arise that $0 \leq \mathcal{E}(t,\tau;r_0;\kappa,\theta,\sigma)$, hence the range of acceptable shocks is reduced. This fact means that the 1-VM model is rather suitable to generate acceptable future yield of the ZCB $R(t, t+\tau)(\cdot)$ under condition (8). Similar conclusions can be drawn about a spot rate generated by the historical measure and its use for the computation of the yield $R_{\mathbb{P}}(t, t+\tau)(\cdot)$.

Using the Gaussian property of the shocks $\varepsilon_{t|0}(\cdot)$ and $\varepsilon_{t|0;\mathbb{P}}(\cdot)$, the probabilities of negative yields are computed:

Proposition 2.4. *If the spot rate is generated under the risk-neutral measure, then the probability that the one factor Vasicek model generates unrealistic future zero-coupon prices $P(t, t+\tau)(\cdot)$ is given by:*

$$\pi(t, \tau; r_0; \kappa, \theta, \sigma) = \Phi\left[\mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma\right)\right] \tag{12}$$

with $\Phi\left(\right)$ denotes the cumulative distribution function of the standard Gaussian normal random variable. If the historical measure is used for the generation of the path, then the corresponding probability is:

$$\pi(t, \tau; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}) = \Phi\left[\mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right)\right] \tag{12'}$$

According to propositions 2.3 and 2.4, when the intention is to generate IR at a future time-horizon t, the first action to do is to compute the level $\mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma\right)$ or $\mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right)$ depending on the purpose of the simulation. This allows one to appreciate the suitability (or not) of using the 1-VM for pricing and/or risk management purposes. Of course if the corresponding probability $\pi(t, \tau; r_0; \kappa, \theta, \sigma)$ or $\pi(t, \tau; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}})$ are very small or acceptable, then the 1-VM model appears to be suitable to generate the yield of the ZCB $R(t, t+\tau)(\cdot)$ or $R_{\mathbb{P}}(t, t+\tau)(\cdot)$. Numerical examples are provided in subsection 4.1 for illustrations.

An other natural question, linked to the treatment of a portfolio or price series is that, if the probability $\pi(t, \tau; r_0; \kappa, \theta, \sigma)$ is acceptable for a given time to maturity $\tau = \tau_1$, then what can be said about all the other probabilities associated to the time to maturities τ_m's. This leads us to ask about the monotonicity of the mappings defining the thresholds. Using lemma 2.1, the following can be stated.

Proposition 2.5. *Under condition (8), the mappings:*

$$\tau \in (0, \infty) \longmapsto \mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma\right)$$

and:

$$\tau \in (0, \infty) \longmapsto \mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right)$$

define a decreasing function.

As a consequence if the price series $P(t, t+\tau_0)(\cdot), \cdots, P(t, t+\tau_M)(\cdot)$ for $\tau_1 \leq \cdots \leq \tau_M$ are considered, then under condition (8) one has $\mathcal{E}\left(t, \tau_M; r_0; \kappa, \theta, \sigma\right) \leq \cdots \leq \mathcal{E}\left(t, \tau_1; r_0; \kappa, \theta, \sigma\right)$. This means that just a good definition of $P(t, t+\tau_1)(\cdot)$ implies the same situation for all the remaining ZCB prices $P(t, t+\tau_i)(\cdot)$, with $2 \leq i \leq M$. Similar conclusions can be drawn about a series of prices $P_{\mathbb{P}}(t, t+\tau_1)(\cdot), \cdots, P_{\mathbb{P}}(t, t+\tau_M)(\cdot)$ generated by the spot rate under the historical measure.

2.4 ZCB prices and shocks

Conceptually, under the 1-VM, the shocks defining the future ZCB prices (or the associated yields) are any real numbers since their distribution is conditional Gaussian. However given that a real market ZCB price should be positive and less than one, then from the practical point of view only shocks inside some convenient real intervals deserve to be considered. In this subsection, we try to perform a close look to the situation.

Using eqs. (11) and (11′), the prices $P(t, t+\tau)(\cdot)$ and $P_{\mathbb{P}}(t, t+\tau)(\cdot)$ are rewritten in terms of a functions of the shocks $\varepsilon_{t|0}(\cdot)$ and $\varepsilon_{t|0;\mathbb{P}}(\cdot)$:

$$\begin{aligned} &\mathrm{P}\left(\varepsilon_{t|0}(\cdot); \tau, t; r_0; \kappa, \theta, \sigma\right) \\ &\qquad = \exp\left(-\sigma \mathrm{b}^{\frac{1}{2}}\left(t; 2\kappa\right) \mathrm{b}\left(\tau; \kappa\right)\left[\varepsilon_{t|0}(\cdot) - \mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma\right)\right]\right) \end{aligned} \tag{13}$$

and:

$$\begin{aligned} &\mathrm{P}\left(\varepsilon_{t|0;\mathbb{P}}(\cdot); \tau, t; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right) \\ &\qquad = \exp\left(-\sigma \mathrm{b}^{\frac{1}{2}}\left(t; 2\kappa_{\mathbb{P}}\right) \mathrm{b}\left(\tau; \kappa\right)\left[\varepsilon_{t|0;\mathbb{P}}(\cdot) - \mathcal{E}\left(t, \tau; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right)\right]\right) \end{aligned} \tag{13′}$$

In this subsection, denote by $\mathrm{P}\left(\varepsilon_{t|0}(\cdot)\right)$ and $\mathrm{P}_{\mathbb{P}}\left(\varepsilon_{t|0;\mathbb{P}}(\cdot)\right)$ the two mappings of eqs. (13) and (13′). These equations mean that the ZCB price at a future-time horizon t follows from the effect of a risk-driver realization $\varepsilon_{t|0}(\cdot)$ or $\varepsilon_{t|0;\mathbb{P}}(\cdot)$. To grasp the value or risk related to a given position, it is common to make some scenarios related to the risk-driver. Actually this is done because the market value or risk pending on a financial instrument is actually, in general, an involved function of the risk driver and no monotonicity property is satisfied. However, in the framework of 1-VM, some monotonicity property is available and deserves to be analysed. The reason is that it allows to get a global view of the position situation in an economical manner as no simulation is really needed.

From eqs. (13) and (13′), it is clear that for given t, τ, r_0, κ, θ, $\kappa_{\mathbb{P}}$, $\theta_{\mathbb{P}}$ and σ the mappings:

$$\varepsilon \in (-\infty, \infty) \longmapsto \mathrm{P}\left(\varepsilon\right) \in (0, \infty)$$

and:

$$\varepsilon \in (-\infty, \infty) \longmapsto \mathrm{P}_{\mathbb{P}}\left(\varepsilon\right) \in (0, \infty)$$

define decreasing functions. As a consequence one can state the following.

Proposition 2.6. *If for the future time-horizon t one has a view on shock $\varepsilon(\cdot)$ such that*

$$\varepsilon_* \leq \varepsilon(\cdot) \leq \varepsilon_{**} \tag{14}$$

for some fixed real numbers ε_* and ε_{**}, then the Vasicek model generated price is bounded below and above as:

$$\mathrm{P}(\varepsilon_{**}) \leq P(t, t+\tau)(\cdot) \leq \mathrm{P}(\varepsilon_*) \tag{15}$$

and:

$$\mathrm{P}_{\mathbb{P}}(\varepsilon_{**}) \leq P_{\mathbb{P}}(t, t+\tau)(\cdot) \leq \mathrm{P}_{\mathbb{P}}(\varepsilon_*) \tag{15$'$}$$

As under the 1-VM, the shocks are actually realizations of a standard normal Gaussian random variable then the double-inequality (14) is satisfied for $\varepsilon_* = -5$ and $\varepsilon_{**} = 5$ with a probability more than 99.9999%. It means that the price $P(t, t+\tau)(\cdot)$ generated by the 1-VM should be roughly bounded below and above by $\mathrm{P}(5)$ and $\mathrm{P}(-5)$ and similar conclusions can by drawn about the price $P_{\mathbb{P}}(t, t+\tau)(\cdot)$.

Moreover, according to eqs. (13) and (13′), it may be observed that:

$$\varepsilon \in (-\infty, \infty) \longmapsto \mathrm{P}\left(\varepsilon\right) \in (0, \infty)$$

and:

$$\varepsilon \in (-\infty, \infty) \longmapsto \mathrm{P}_{\mathbb{P}}(\varepsilon) \in (0, \infty)$$

define a one-to-one mapping, as for example to each ZCB price $P(t, t+\tau)(\cdot) \in (0, \infty)$ corresponds to a shock $\varepsilon \in (-\infty, \infty)$ as:

$$\varepsilon = \mathcal{E}(t, \tau; r_0; \kappa, \theta, \sigma) - \frac{1}{\sigma \mathrm{b}^{\frac{1}{2}}(t; 2\kappa)\, \mathrm{b}(\tau; \kappa)} \ln P(t, t+\tau)(.) \tag{16}$$

In a similar vein, each ZCB price $P_{\mathbb{P}}(t, t+\tau)(\cdot) \in (0, \infty)$ correspond to a shock $\varepsilon_{\mathbb{P}} \in (-\infty, \infty)$ as:

$$\varepsilon_{\mathbb{P}} = \mathcal{E}(t, \tau; r_0; \kappa, \theta, \sigma) - \frac{1}{\sigma \mathrm{b}^{\frac{1}{2}}(t; 2\kappa_{\mathbb{P}})\, \mathrm{b}(\tau; \kappa)} \ln P_{\mathbb{P}}(t, t+\tau)(.) \tag{16$'$}$$

To get a view the shocks of eqs. (16) and (16′) is not so natural since they have no direct meaning on financial point of view. But it is rather common that the market practitioners have some ideas about low and high returns of the bond prices for a given horizon. For these expectations, we state the following:

Proposition 2.7. *Assume that at the future time-horizon t the return of $P(t, t+\tau)(\cdot)$ is seen to be bounded below and above as:*

$$\rho_* \leq \frac{P(t, t+\tau)(\cdot) - P(0, t+\tau)}{P(0, t+\tau)(\cdot)} \leq \rho_{**} \tag{17}$$

for some real numbers ρ_* and ρ_{**}, with $-1 < \rho_* \leq \rho_{**}$. Then, the shock $\varepsilon_{t|0}(\cdot)$ realizing the price $P(t, t+\tau)(\cdot)$ should satisfy the double-inequality:

$$e(\rho_{**}) \leq \varepsilon_{t|0}(\cdot) \leq e(\rho_*) \tag{18}$$

where $e(\rho)$ is given by:

$$e(\rho) = \mathcal{E}(t, \tau; r_0; \kappa, \theta, \sigma) - \frac{1}{\sigma \mathrm{b}^{\frac{1}{2}}(t; 2\kappa)\, \mathrm{b}(\tau; \kappa)} \ln\left[(1+\rho) P(0, t+\tau)\right] \tag{19}$$

Similarly, if the return of $P_{\mathbb{P}}(t, t+\tau)(\cdot)$ is seen to be bounded as:

$$\rho_{*,\mathbb{P}} \leq \frac{P_{\mathbb{P}}(t, t+\tau)(\cdot) - P(0, t+\tau)}{P(0, t+\tau)(\cdot)} \leq \rho_{**,\mathbb{P}} \tag{17$'$}$$

then, the shock $\varepsilon_{t|0;\mathbb{P}}(\cdot)$ realizing the price $P_{\mathbb{P}}(t, t+\tau)(\cdot)$ should satisfy the double-inequality:

$$e_{\mathbb{P}}(\rho_{**,\mathbb{P}}) \leq \varepsilon_{t|0;\mathbb{P}}(\cdot) \leq e_{\mathbb{P}}(\rho_{*,\mathbb{P}}) \tag{18$'$}$$

where $e_{\mathbb{P}}$ is defined by:

$$e_{\mathbb{P}}(\rho) = \mathcal{E}(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}}) - \frac{1}{\sigma \mathrm{b}^{\frac{1}{2}}(t;2\kappa_{\mathbb{P}})\,\mathrm{b}(\tau;\kappa)} \ln\left[(1+\rho)P(0,t+\tau)\right] \quad (19')$$

With such a result it appears that if the main focus is about returns less than ρ_{**} (as when considering some loss level), then only shocks greater than $e(\rho_{**})$ have to be considered.

2.5 Simulation of an IR portfolio

In the regulation framework of Solvency II or Basel 3, as well as for pricing purposes, very often one has to generate scenarios for the IR at one future time-horizon t and for various maturities. This leads to define scenarios for the yield-curve $R(t,t+\tau_1)(\cdot),\cdots,R(t,t+\tau_M)(\cdot)$ or $R_{\mathbb{P}}(t,t+\tau_1)(\cdot),\cdots,R_{\mathbb{P}}(t,t+\tau_M)(\cdot)$ with non-negative and increasing time to maturities $\tau_1,\cdots,\tau_M$. It is important that each of these yields has an economical meaning (in the sense to be at least positive).

The simulation is done by considering some realizations $\varepsilon_{t|0}$ or $\varepsilon_{t|0;\mathbb{P}}$ of the standard Gaussian random variable, then to apply formula (11) or (11′) for each time to maturity $\tau = \tau_i$, with $1 \le i \le M$. The consistency, on the economics point of view, for the entire yield curve means that all the bounds $\mathcal{E}(t,\tau_i;r_0;\kappa,\theta,\sigma)$ or $\mathcal{E}(t,\tau_i;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}})$ should be less than the shock $\varepsilon_{t|0}$ or $\varepsilon_{t|0;\mathbb{P}}$. According to proposition 2.5, and under condition (8), the mappings $\tau \in (0,\infty) \longmapsto \mathcal{E}(t,\tau;r_0;\kappa,\theta,\sigma)$ and $\tau \in (0,\infty) \longmapsto \mathcal{E}(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}})$ define decreasing functions. When condition (8) is not satisfied, it is possible to make a close-look at these two mappings. In fact, if a curve made by a finite number of time to maturities, denoted M, is considered, then it would be easy to use the following:

Proposition 2.8. *Assume that the initial instantaneous rate r_0 is positive and the model parameters κ, θ, σ and $\kappa_{\mathbb{P}}$, $\theta_{\mathbb{P}}$ are given. Let us consider two integers $m^\star$ and $m^\star_{\mathbb{P}}$ such that:*

$$\mathcal{E}(t,\tau_{m^\star};r_0;\kappa,\theta,\sigma) = \max\left\{\mathcal{E}(t,\tau_i;r_0;\kappa,\theta,\sigma)\,|i \in \{1,\ldots,M\}\right\} \quad (20)$$

and:

$$\mathcal{E}\left(t, \tau_{m^\star_{\mathbb{P}}}; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right) \\ = \max\left\{\mathcal{E}\left(t, \tau_i; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right) | i \in \{1, \ldots, M\}\right\} \qquad (20')$$

Then, all acceptable yields $R(t, t+\tau_i)(\cdot)$ and $R_{\mathbb{P}}(t, t+\tau_i)(\cdot)$ for $1 \leq i \leq M$ as well as zero-coupon bond prices $P(t, t+\tau_i)(\cdot)$ and $P_{\mathbb{P}}(t, t+\tau_i)(\cdot)$ can be simulated by using all shocks $\varepsilon_{t|0}(\cdot)$ and $\varepsilon_{t|0;\mathbb{P}}(\cdot)$ satisfying:

$$\mathcal{E}\left(t, \tau_{m^\star}; r_0; \kappa, \theta, \sigma\right) \leq \varepsilon_{t|0}(\cdot) \qquad (21)$$

and for risk management purposes:

$$\mathcal{E}\left(t, \tau_{m^\star}; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right) \leq \varepsilon_{t|0;\mathbb{P}}(\cdot) \qquad (21')$$

The corresponding probabilities are valued using eqs. (12) and (12′) with the above thresholds of eqs. (20) and (20′).

For all considered maturities τ_m, there is no reason that all the probabilities $\pi(t, \tau_m; r_0; \kappa, \theta, \sigma)$ or $\pi(t, \tau_m; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}})$ are negligible from the perspective of the user. Making the restriction on shocks to be greater than $\mathcal{E}\left(t, \tau_{m^\star}; r_0; \kappa, \theta, \sigma\right)$ or $\mathcal{E}\left(t, \tau_{m^\star_{\mathbb{P}}}; r_0; \kappa, \theta, \sigma, \kappa_{\mathbb{P}}, \theta_{\mathbb{P}}\right)$ allows one to limit the possible harmful effect causing by the model inadequacy. As illustrated below, this last may generate too many negative IR if directly applied to the simulation. However, tweaking the model as we suggest here in proposition 2.8 does not solve the definitive inconsistency of the model to represent market prices. Though in this section we have limited our analysis to this simple benchmark 1-VM, it appears that a similar, but more difficult, problem arises in the IR simulation under GATSM.

3 The generation of negative IR at various future times

The results of the previous section are extended for the generation of a time series of ZCB prices that are needed for the pricing of financial contracts in the context of *Monte-Carlo* simulations. In a first time, only one ZCB with constant time to maturity is considered and the negative yield problem is formulated in terms of hitting time in a continuous and discrete framework (3.1). The first case corresponds to a continuously observed spot rate while the second one stands for a short rate observed at discrete times, as performed in pricing and risk management using *Monte-Carlo* simulations. In a second time, these random variables are extended for the simulation of time series of ZCB prices with different but constant time to maturities (3.2). Using the results of subsections 2.3 and 2.5, only one maturity plays a key role and the extended hitting time is identified to a hitting described in subsection 3.1.

3.1 The generation of a time series of a constant-maturity bond price

In section 2, the ZCB price at a future time $0 < t$, $P(t, t+\tau)(\cdot)$ or $P_{\mathbb{P}}(t, t+\tau)(\cdot)$, have been considered and expressed in terms of shocks $\varepsilon_{t|0}$ or $\varepsilon_{t|0;\mathbb{P}}$. However, in the pricing or risk valuation of complex financial contracts, a series of prices $P(t_1, t_1+\tau)(\cdot), \cdots, P(t_N, t_N+\tau)(\cdot)$ or $P_{\mathbb{P}}(t_1, t_1+\tau)(\cdot), \cdots, P_{\mathbb{P}}(t_N, t_N+\tau)(\cdot)$, for different future strictly positive and increasing dates $t_1, \cdots, t_N$ and a constant time to maturity τ, have to be generated, for example in the valuation of a variable-rate loan indexed to the yield of constant maturity bond. Consider that a path of the spot rate, driven by SDE (1) or (1′), is acceptable, provided that all the ZCB prices of the series are always lower than one. In the previous section, a bond has been obtained in eqs. (7) and (7′). In a context of multiple simulations, we propose to analyse this problem focusing on the random variable modelling the first hitting time of the bound, which is a hitting time. In fact, the cumulative probability of this random variable allows one to value the probability to have generated negative yields from the present time to a future simulated date.

This condition can be formulated using hitting times using a continuous or discrete time. In fact, if the path of the processes $r_t(\cdot)$ or $r_{t,\mathbb{P}}(\cdot)$ is continuously available from time 0 to t_N, then the path will be considered acceptable for pricing if:

$$t_N < \mathrm{T}\left[\mathcal{B}(\tau)\right](\cdot) \equiv \inf\left\{t > 0 \,|r_t(\cdot) \leq \mathcal{B}(\tau; \kappa, \theta, \sigma)\right\} \tag{22}$$

and for risk management if:

$$t_N < \mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \equiv \inf\left\{t > 0 \left| r_{t,\mathbb{P}}(\cdot) \leq \mathcal{B}\left(\tau; \kappa, \theta, \sigma\right)\right.\right\} \tag{22$'$}$$

The hitting times $\mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ and $\mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ are defined on a continuous path of the processes $r_t(\cdot)$ and $r_{t,\mathbb{P}}(\cdot)$, therefore can be approximated using proposition A.1 given in appendix A.1 with the exact or asymptotic coefficients.

Actually in the *Monte-Carlo* simulations for pricing and risk-management, the processes $r_t(\cdot)$ and $r_{t,\mathbb{P}}(\cdot)$ are generated with a discrete time step, denoted δ. Consequently, the path is not available on the time interval $[0, t_N]$ rather at times in form of $t_i = i\delta$, $i \in \mathbb{N}^*$. The two hitting times of eqs. (22) and (22$'$) are therefore redefined as:

$$t_N < \mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \equiv \inf\left\{t_i = i\delta \in \delta\mathbb{N}^* \left| r_{t_i}(\cdot) \leq \mathcal{B}\left(\tau; \kappa, \theta, \sigma\right)\right.\right\} \tag{23}$$

and:

$$t_N < \mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \equiv \inf\left\{t_i = i\delta \in \delta\mathbb{N}^* \left| r_{t_i,\mathbb{P}}(\cdot) \leq \mathcal{B}\left(\tau; \kappa, \theta, \sigma\right)\right.\right\} \tag{23$'$}$$

Consider a given $t^* > 0$ such that $\mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) = t^*$. Then $t^* = \mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ if and only if there exists $i^* \in \mathbb{N}^*$ such that $t^* = i^*\delta$. Otherwise, we have $\mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) = t^* \leq \mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ since that the spot rate can be lower than the bound $\mathcal{B}\left(\tau; \kappa, \theta, \sigma\right)$ at time t^* and not at other observation dates in form of $i\delta$. Therefore, condition (23) is weaker than condition (22). Moreover, the previous observation implies that for any $t > 0$, we have if $\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \leq t$, then $\mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \leq t$, consequently $\left\{\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \leq t\right\} \subseteq \left\{\mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \leq t\right\}$, hence $\mathbb{Q}\left\{\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \leq t\right\} \leq \mathbb{Q}\left\{\mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \leq t\right\}$. This latter inequality means that the use of the continuous hitting time produces an overestimation of the probability to obtain negative yields, hence this approach is prudential. Similar conclusions can be drawn for the hitting times $\mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ and $\mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$. The spot rate process can be formulated in terms of auto-regressive process or order one as shown by eqs. (A.16) and (A.16$'$) of appendix A.2. Consequently, proposition A.2 can be used for the computation of the densities of the discrete hitting times $\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ and $\mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$. However, this result implies high order integration that can be difficult to value numerically, especially for long time series. In order to circumvent this problem, one can use the density of the hitting time $\mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ or $\mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ instead of their discrete counterparts $\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ and $\mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$ when δ is small enough according to a convergence result of [12]:

Proposition 3.1. *If $\delta \to 0^{+}$, then:*

$$\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \to \mathrm{T}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$$

and:

$$\mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right) \to \mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau\right)\right]\left(\cdot\right)$$

in distribution of probability.

Proof. See Appendix A.3. □

However, the numerical examples of subsection 4.2 show that the overestimation error can be not considered as negligible for long simulation lengths for some parameters of the 1-VM, especially with long simulation steps δ.

The introduction of hitting times adds mathematical complexity since the probability of the spot rate process to hit the barrier is delicate to value in both continuous and discrete cases. However, a final user like a risk-manager, can just focus on the estimated probability since the final result may be more important than the mathematical theory on its point of view. As a consequence, the hitting times are tools to make an analysis of the adoption of 1-VM before running complex systems that perform the simulations like in the *Solvency II* framework. Such systems may require the simulation of various risk drivers, e.g. the risk-free IR, currency, credit, technical risks, as well as access to the company databases (for the simulation of the assets and liabilities). Thus, a fast analysis before the simulation can be needed.

3.2 The generation of a time series of several bond prices

The generation of a ZCB price times series at strictly positive and increasing future dates $t_1, \cdots, t_N$ for a constant time to maturity τ may not be sufficient in the valuation of financial contracts or insurance policies, hence we consider the problem of the simulation of ZCB prices at future strictly positive and increasing dates $t_1, \cdots, t_N$, each of them with constant, non-negative, and increasing time to maturities $\tau_1, \cdots, \tau_M$. The conditions of eqs. (22), (22′), (23) and (23′) are extended in the continuous time for pricing purposes:

$$t_N < \mathrm{T}^{\star}(\cdot) \equiv \inf\left\{\mathrm{T}\left[\mathcal{B}\left(\tau_i\right)\right](\cdot)\,|i \in \{1, \ldots, M\}\right\} \tag{24}$$

and for risk management tasks:

$$t_N < \mathrm{T}^{\star}_{\mathbb{P}}(\cdot) \equiv \inf\left\{\mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_i\right)\right](\cdot)\,|i \in \{1, \ldots, M\}\right\} \tag{24′}$$

In a similar vein, the discrete hitting time $\mathrm{T}^{\delta}(\tau)(\cdot)$ is extended for pricing valuations:

$$t_N < \mathrm{T}^{\delta,\star}(\cdot) \equiv \inf\left\{\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau_i\right)\right](\cdot)\,|i \in \{1, \ldots, M\}\right\} \tag{25}$$

and $\mathrm{T}^{\delta}_{\mathbb{P}}(\tau)(\cdot)$ is extended for risk-management computations:

$$t_N < \mathrm{T}^{\delta,\star}_{\mathbb{P}}(\cdot) \equiv \inf\left\{\mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_i\right)\right](\cdot)\,|i \in \{1, \ldots, M\}\right\} \tag{25′}$$

As in section 2, condition (8) allows to simplify the problem. In fact, as in proposition 2.5, the bound $\mathcal{B}(\tau;\kappa,\theta,\sigma)$ has a decreasing property with respect to the time to maturity τ, hence conditions (24), (24′), (25) and (25′) can be simplified:

Proposition 3.2. *Under condition (8), the mapping:*

$$\tau \in (0,\infty) \longmapsto \mathcal{B}(\tau;\kappa,\theta,\sigma)$$

defines a decreasing function. As a consequence, the hitting times of conditions (24), (24′), (25) and (25′) become $\mathrm{T}^{\star}(\cdot) \equiv \mathrm{T}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$, $\mathrm{T}^{\star}_{\mathbb{P}}(\cdot) \equiv \mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$, $\mathrm{T}^{\delta,\star}(\cdot) \equiv \mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$ *and* $\mathrm{T}^{\delta,\star}_{\mathbb{P}}(\cdot) \equiv \mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$.

This proposition means that the lowest time to maturity τ_1 plays a key role in the control of the acceptability in the simulation of the path of $r_t(\cdot)$ or $r_{t;\mathbb{P}}(\cdot)$ as in section 2. Moreover, combining propositions 3.1 and 3.2, the hitting times $\mathrm{T}^{\delta,\star}(\cdot)$ and $\mathrm{T}^{\delta,\star}_{\mathbb{P}}(\cdot)$ converge to $\mathrm{T}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$ and $\mathrm{T}^{\star}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$:

Corollary 3.3. *Under condition (8), if $\delta \to 0^{+}$, then:*

$$\mathrm{T}^{\delta}(\tau)(\cdot) \to \mathrm{T}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$$

and:

$$\mathrm{T}^{\delta}_{\mathbb{P}}(\tau)(\cdot) \to \mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_1\right)\right](\cdot)$$

in distribution of probability.

If condition (8) is not fulfilled, then a similar reasoning than in proposition 2.8 is applied. At first, the comparison of the bounds $\mathcal{B}(\tau;\kappa,\theta,\sigma)$ and $\mathcal{E}(t,\tau_i;r_0;\kappa,\theta,\sigma)$ of eqs. (7) and (10) as well as the $\mathcal{B}(\tau;\kappa,\theta,\sigma)$ and $\mathcal{E}(t,\tau;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}})$ of eqs. (7) and (10′) shows that they have the same variations with respect to the time to maturity τ, hence the two integers $m^\star$ and $m^\star_{\mathbb{P}}$ of proposition 2.8 could have been defined using the quantity $\mathcal{B}(\tau;\kappa,\theta,\sigma)$ instead of $\mathcal{E}(t,\tau_i;r_0;\kappa,\theta,\sigma)$ and $\mathcal{E}(t,\tau_i;r_0;\kappa,\theta,\sigma,\kappa_{\mathbb{P}},\theta_{\mathbb{P}})$, $i=1,\cdots,M$ in eqs. (20) and (20′).

Proposition 3.4. *Consider the two integers $m^\star$ and $m^\star_{\mathbb{P}}$ of proposition 2.8 defined by eqs. (20) and (20′). Then, the hitting times of conditions (24), (24′), (25) and (25′) become* $\mathrm{T}^\star(\cdot)\equiv\mathrm{T}\left[\mathcal{B}\left(\tau_{m^\star}\right)\right](\cdot)$, $\mathrm{T}^\star_{\mathbb{P}}(\cdot)\equiv\mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_{m^\star_{\mathbb{P}}}\right)\right](\cdot)$, $\mathrm{T}^{\delta,\star}(\cdot)\equiv\mathrm{T}^{\delta}\left[\mathcal{B}\left(\tau_{m^\star}\right)\right](\cdot)$ *and* $\mathrm{T}^{\delta,\star}_{\mathbb{P}}(\cdot)\equiv\mathrm{T}^{\delta}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_{m^\star_{\mathbb{P}}}\right)\right](\cdot)$.

As in corollary 3.3, the hitting times $\mathrm{T}^{\delta,\star}(\cdot)$ and $\mathrm{T}^{\delta,\star}_{\mathbb{P}}(\cdot)$ converge when δ is small.

Corollary 3.5. *Consider the two integers $m^\star$ and $m^\star_{\mathbb{P}}$ of proposition 2.8 defined by eqs. (20) and (20′). If $\delta\to 0^+$, then:*

$$\mathrm{T}^{\delta,\star}(\cdot)\to\mathrm{T}\left[\mathcal{B}\left(\tau_{m^\star}\right)\right](\cdot)$$

and:

$$\mathrm{T}^{\delta,\star}_{\mathbb{P}}(\cdot)\to\mathrm{T}_{\mathbb{P}}\left[\mathcal{B}\left(\tau_{m^\star_{\mathbb{P}}}\right)\right](\cdot)$$

in distribution of probability.

4 Numerical examples

As mentioned in the introduction, we aim to provide in this paper some illustrations related to the generation of negative IR when using the 1-VM. In contrast with the general results of sections 2 and 3, pertaining to the model, the findings obtained in this section part are rather linked to the particularity of the model parameters under consideration. The conclusions drawn reflect some past realities and are provided for illustrations and understanding of the results. But they should not be appropriated for any general use as the market conditions may be very different. In a first subsection 4.1, the probabilities to obtain a negative yield are illustrated when one bond price at one future time is generated following section 2. In a second subsection 4.2, the hitting times of section 3 related to the generation of a time series of one constant maturity ZCB are analysed.

4.1 The generation of a bond at one future time

In order to analyse the maximal shocks and the probability to obtain negative yields, we use different parameters sets estimated[12] before and after the financial crisis of 2007 and all of them are summarized in table 1.

Table 1: The parameters for the numerical illustrations.

Parameter	P_1	P_2	P_3	P_4	P_5
r_0	0.0250	0.0250	0.0250	0.0001	0.0010
α	0.0650	0.0650	0.0650	0.1405	0.1100
γ	0.1292	0.1292	0.1292	0.0652	0.0100
σ	0.0175	0.0175	0.0175	0.0230	0.0175
λ_1	-	-0.005	0.005	-	-
λ_2	-	-	-0.14	-	-
Eq (8)	0.000786	0.000136	0.005188	0.002044	-0.000064

The first three parameters sets, P_1, P_2 and P_3 were estimated in [10] using the *US Treasury* yields from 1970 to 2001 with the 3 months, 1 and 10 years time to maturity, thus they represent a estimated before the financial crisis. The authors mention that the mean interest rate is equal to 5.2 %. In order to

[12]Various methods may be used to calibrate the 1-VM model depending on the nature of data at hand. If a times series of the spot rate is available, then one can use the Maximum-Likelihood (see [2]) and the estimated parameters are under the historical measure. If the spot rate is not observed, then one can assume that some yields are observed without errors unlike others then use the maximum of likelihood, as explained in [10]. Other methods are the Efficient Method of Moments (see [10]) and the *Kalman* filter, e.g [13] and [10]. The estimated parameter set contains the historical ones and the risk premium. Lastly, the cross-sectional analysis, minimizing the distance between the market and model prices, allows one to obtain the risk-neutral parameters.

have some significant probabilities, we chose a current value of the spot rate equal to 2.5 % for all three parameters sets. The first one, P_1, corresponds to the risk-neutral parameters that can be used for pricing purposes, hence the shock and the corresponding probabilities are valued with eqs. (10) and (12). The two other sets P_2 and P_3, containing two different historical dynamics, can be used for risk-management purposes, therefore the maximal shock and corresponding probabilities come from eqs. (10′) and (12′). The parameter set P_4 was estimated using the *US Treasury* yield curve at December 31, 2013. This parameters set is estimated making a cross-sectional analysis, i.e. minimizing the square market-model yield error under the constraint that condition (8) is fulfilled, hence the parameters are under the risk-neutral measure $\mathbb{Q}$. The observed yields have 1, 3 and 6 mouths, 1, 2, 3, 5, 7, 10, 20 and 30 years time to maturity and are available on the website of the *U.S. Department of the Treasury*. The last parameter set P_5 is constructed from P_1 so as to not fulfil condition (8) but to keep positive yields for time to maturity until 30 years. The implied term structure is increasing up to a time to maturity of seven years then is decreasing. The simulation horizons under consideration here are 1 and 10 days, 1, 3, 6 months and 1 and 5 years and the maturities of the bonds are 1 day, one week, 3, 6, 9 months and one to 5, 10, 15 or 50 years with a yearly step[13].

The probabilities to obtain a negative yield using the “pre-crisis” sets P_1, P_2 and P_3 are represented in figure 1. At first, the three parameters sets satisfy condition (8), therefore according to proposition 2.5 the shock is decreasing with the time to maturity of the bond τ, hence the probability to obtain a negative yield is also decreasing with respect to the time to maturity. Secondly, the two parameters set P_1 and P_3 exhibit a negligible probability after a time to maturity of 10 and 5 years, therefore the 1-VM is acceptable to simulate bonds with long term maturities. For short term bonds, the highest probability for the set P_1 around 5 percent for short bonds and for the set P_3 is around two percent. As a consequence, all these probabilities can be judged as negligible, especially for the parameter set P_3. Note that the probabilities are increasing with the simulation horizon in the parameter set P_1 but this observation does not hold for the parameter set P_3, since the negative yield probabilities are

[13] As shown in the figure above, the probabilities are negligible after a time to maturity, these probabilities can be ignored.

higher for the intermediate simulation horizons of 6 months and one year. As a consequence, the parameter set P_1 will produce negative yields with a low probability if the simulation horizon is small while the parameter set P_3 is suitable for short term and long term simulation horizon. The probabilities related to the parameter set P_2 share conclusion with those related to the set P_1 since they are decreasing with the time to maturity of the bond, are negligible for a time to maturity longer than 10 years and increasing with the simulation horizon. However, they are not negligible for short term maturities bonds, especially for the simulation horizon of 5 years. This parameter set is therefore suitable for short term simulation but is subject to more troubles than P_1 and P_3 for long term simulation horizon.

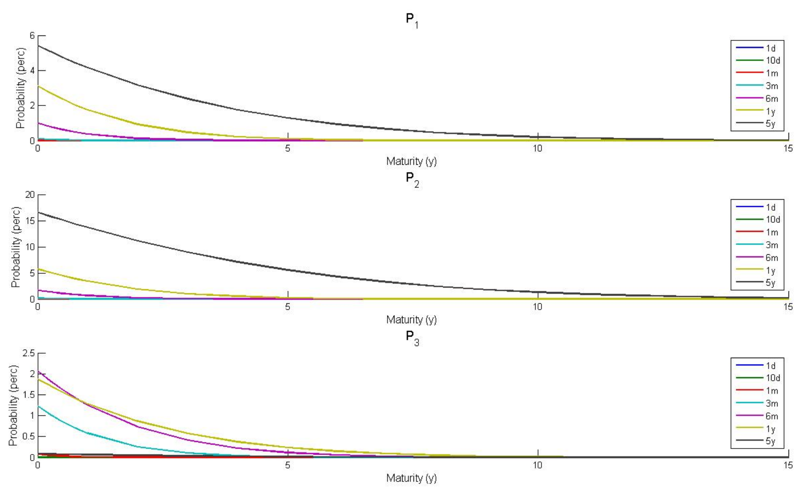

Figure 1: Probability to generate a negative yield at the 1 and 10 days, 1, 3, 6 months and 1 and 5 years simulation horizon using the parameters set P_1, P_2 and P_3.

In the sequel of the financial crisis of 2007, low interest rates are observed and some of them are negative. The current value of the spot rate r_0 and the long term equilibrium value θ are expected to be lower, therefore according to eqs. (10) and (10′) the levels that avoid negative yields are lower, thus their probability is higher. Using the simulation horizons of figure 1, the probabilities of negative yields are given in figure 2. At first it can be noted that the probabilities to obtain a negative yield for low time to maturity bonds are far from being negligible. For the simulation horizon of 10 days, the probability to have negative yields for the 1 day time to maturity bond reaches about 45 %. For short maturities bonds, the simulation horizon with the lowest probabilities is the 5 horizon but it provides the highest probabilities for the medium bonds. However, it should be noted that all these probabilities are negligible after the time to maturity of 15 years, as for the parameters sets P_1 and P_2. Secondly, for short term bonds, the probabilities to obtain negative yields are the probabilities are increasing with the simulation horizon then decreasing as for the parameter set P_3. For medium and long term, these probabilities are non-decreasing with the simulation horizon as for the parameters sets P_1 and P_2.

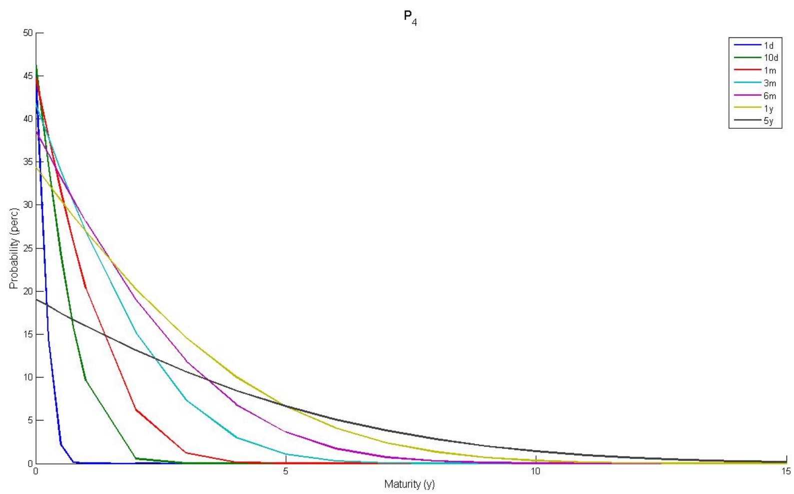

Figure 2: Probability to generate a negative yield at the 1 and 10 days, 1, 3, 6 months and 1 and 5 years simulation horizon using the parameters set P_4.

In order to illustrate the sensitivity of the parameters and the corresponding negative yield probabilities, we use the risk-neutral parameter set P_1 with the simulation horizon of one year and we apply shocks on all the parameters that correspond to ± 20% of the original parameters' values. At first, even after these perturbations, condition (8) is fulfilled. Secondly, according to figure 1, the probability of the shock $\varepsilon_{t|0}(\cdot)$ to be lower than the bound $\mathcal{E}\left(t,\tau;r_0;\kappa,\theta,\sigma\right)$ is negligible after 10 years, hence we consider the bonds with time to maturities equal to 1 day, one week, 3, 6, 9 months and one to 10 years and the obtained probabilities are represented in figure 3. The curves with a "x" marking stand for an upwards shock and the curves with the "o" represent the opposite variation. This figure shows that an upwards shock on the volatility coefficient σ, a downwards shock on the current value r_0, the mean reversion speed κ and equilibrium value γ tend to produce higher negative yields probabilities, while the opposite shocks reduce the corresponding probabilities. The parameter with the highest effect is the volatility coefficient. Moreover, the initial value has a stronger effect than the mean reversion speed on short term bonds (up to 3 years circa) and for longer maturities bonds, the initial value has a lower weaker effect than the mean reversion speed. Lastly, the equilibrium value has a low influence on the negative yield probabilities, especially for short term bonds.

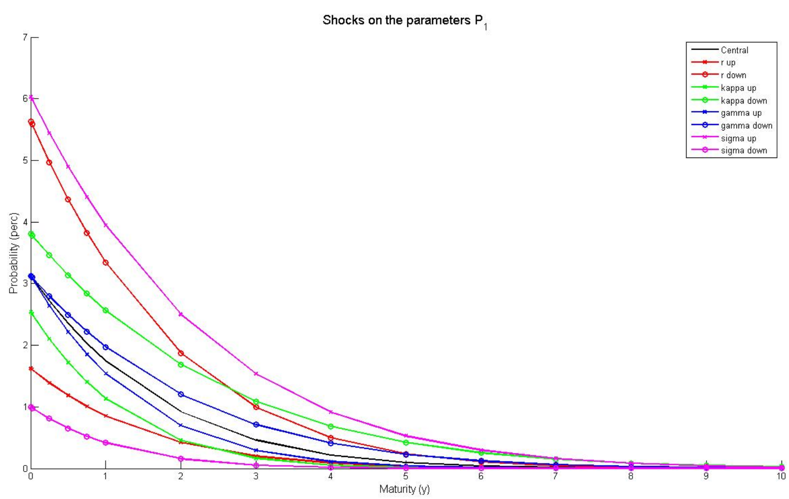

Figure 3: Probability to generate a negative yield at the 1 year simulation horizon applying shocks of $\pm$ 20% on the parameters set P_1.

Lastly, the parameter set P_5 allows one to analyse the problems that can be faced when condition (8) is not fulfilled. Using the simulation horizon of figures 1 and 2 but with higher maximal time to maturity of the bond equal to 50 years, the corresponding probabilities of negative yields are represented in figure 4. At first, it can be noted that the probabilities are decreasing for short term maturities (until 10 years) then are non-decreasing for bonds with long time to maturity for all considered simulation horizons. Consequently, the probability of negative yields for these bonds tends to one as shown by the curve standing for the one day simulation horizon. In fact, it can be observed that the level is positive, generating more troubles in the simulation of the yield curve. Moreover, the maturity determining the acceptable bonds has to be computed using proposition 2.8 and not the decreasing property of proposition 2.5. Because of the observed "U-shaped" probabilities exhibited for the various simulation horizons, one could exclude short term and long term bonds and consider for simulation purposes only bonds with medium-term time to maturity.

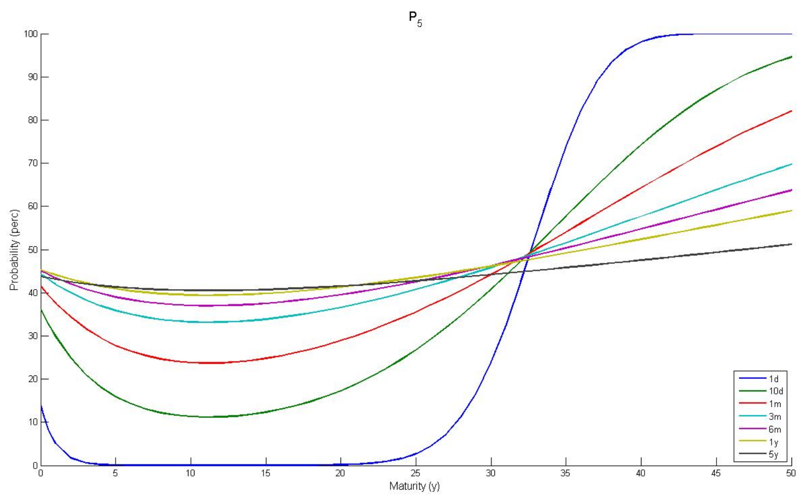

Figure 4: Probability to generate a negative yield at the 1 and 10 days, 1, 3, 6 months and 1 and 5 years simulation horizon using the parameters set P_5.

4.2 The generation of a time series of a constant-maturity bond price

The price of one ZCB with one year maturity using the parameters set P_1 is analysed using the hitting times introduced in section 3. The cumulative probability function of the hitting times is obtained up to a 30 years simulation lenghts under various simulation time steps. The continuous hitting time, i.e. a null time step, is valued using proposition A.1 using the first 100 exact coefficients. This function is represented using a time step equal to 1/10000. Using the same time step, this function is obtained by *Monte-Carlo* simulations so as to control the approximation error using 500 000 simulated paths. Then, using the simulation time steps of 1 and 10 days, 1, 3, 6 months and one year, the cumulative probability of the discrete hitting time are obtained by *Monte-Carlo* simulations using 500 000 simulated paths since their valuation will imply high order integration. All these functions are represented in figure 5. Firstly, the approximated and *Monte-Carlo* cumulative probabilities of the continuous hitting time are close at the exception of the small simulation horizons, meaning that the error implied by the approximation method is small, but the approximation method is not efficient for small simulation lengths. Secondly, the cumulative probabilities are decreasing functions with respect to the simulation step δ, confirming that using the continuous hitting time instead of its discrete counterpart is prudential. For short simulation steps like a day, the discrete and continuous cumulative probabilities are close but, when the simulation horizon increases, the difference is significant. For example, at the 30 years simulation length, the cumulative probability for the continuous hitting times is twice the cumulative probability of the discrete hitting time with a yearly step using the first exact 100 terms of the approximation. Moreover, it should be noted that the order of magnitude of the cumulative probabilities is higher than the probabilities of the previous section using the same parameters sets of figure 1, therefore the negative IR are a concern in the simulation of the process for medium or long term simulations lengths.

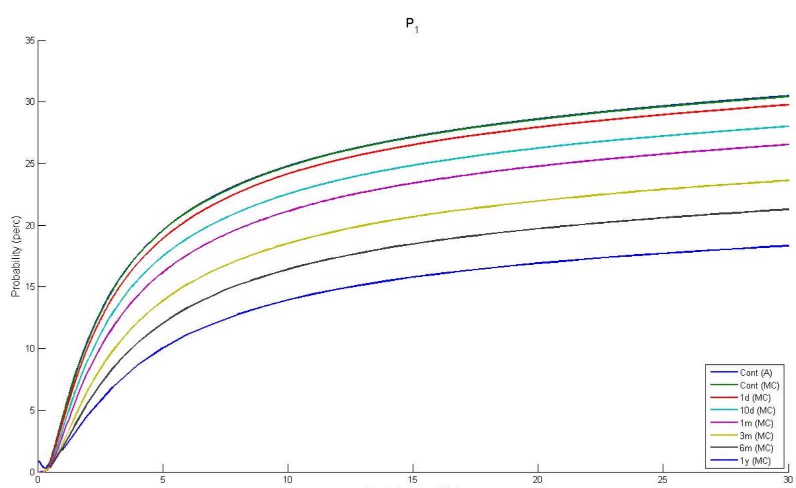

Figure 5: Cumulative probability to generate a negative yield of the one year bond using the 1 and 10 days, 1, 3, 6 months and 1 years time step in the simulation under the parameters set P_1.

Since the difference among the continuous and discrete densities for long simulation steps and long simulation lengths is not negligible, we propose to modify the level defining the continuous hitting time so as to match the discrete cumulative function. For pricing purposes, this approach consists firstly in valuing the vector which components are $\mathbb{Q}\left\{T^{\delta}\left[\mathcal{B}(\tau)\right](\cdot)\leq i\delta\right\}$ for $i = 1,\cdots,N^{*}$, then finding a modified bound, denoted $\tilde{\mathcal{B}}$, minimizing the distance between the previous vector and the vector which i-th term is $\mathbb{Q}\left\{T\left[\tilde{\mathcal{B}}\right](\cdot)\leq i\delta\right\}$ for $i=1,\cdots,N^{*}$, i.e. the bond $\tilde{\mathcal{B}}$ is the solution to the problem:

$$\tilde{\mathcal{B}} = \arg\min_{B}\sum_{i=1}^{N^{*}}\left[\mathbb{Q}\left\{T^{\delta}\left[\mathcal{B}(\tau)\right](\cdot)\leq i\delta\right\}-\mathbb{Q}\left\{T\left[B\right](\cdot)\leq i\delta\right\}\right]^{2} \tag{26}$$

In a similar vein, for risk management purposes, the modified bond $\tilde{\mathcal{B}}_{\mathbb{P}}$ is the solution to:

$$\tilde{\mathcal{B}}_{\mathbb{P}} = \arg\min_{B}\sum_{i=1}^{N^{*}}\left[\mathbb{P}\left\{T^{\delta}_{\mathbb{P}}\left[\mathcal{B}(\tau)\right](\cdot)\leq i\delta\right\}-\mathbb{P}\left\{T_{\mathbb{P}}\left[B\right](\cdot)\leq i\delta\right\}\right]^{2} \tag{26'}$$

This extrapolation approach is inspired by the continuity adjustment of [14] but to the best of our knowledge, no closed-form expression for the modified bond is available in the context of the hitting time associated to the 1-VM, thus we propose to use the numerical solution of eqs. (26) and (26′). Moreover, the cumulative probabilities $\mathbb{Q}\left\{T\left[\tilde{\mathcal{B}}\right](\cdot)\leq i\delta\right\}$ and $\mathbb{P}\left\{T_{\mathbb{P}}\left[\tilde{\mathcal{B}}\right](\cdot)\leq i\delta\right\}$ are valued in the numerical examples using the approximation of proposition A.1, hence have a little error. We set $N^{*}=15$, as a trade-off between accuracy of the approximated bond and computational valuations of the multi-dimensional distribution. Then, we compute the adjustment of eq. (26) for the parameter set P_1 for the simulation steps of 3 and 6 months as well as one year[14]. These densities are represented in figure 6 among with the densities obtained with *Monte-Carlo* simulations using 500 000 simulated paths. For the three considered cumulative probabilities, the continuity adjustment provides a good approximation of the simulated densities, especially for a yearly simulation step $\delta=1$. However, unreported experiments show that the approximation error is not satisfying for lower simulations steps, even for a monthly simulation

[14] Unfortunately, this numerical solution is not performing for lower simulations steps. In our opinion, this fact is due to the approximation error for small simulations horizons as shown by figure 5.

step $\delta = 1/12$. The adjustment of eqs. (26) and (26′) aims at giving a pragmatic solution for long simulations steps and has limitations for medium term simulation steps. Note that for small simulation steps, the difference between the continuous and discrete cumulative distribution can be small according to figure 5. However, a formal derivation of a continuity adjustment like in [14] can be a research investigation.

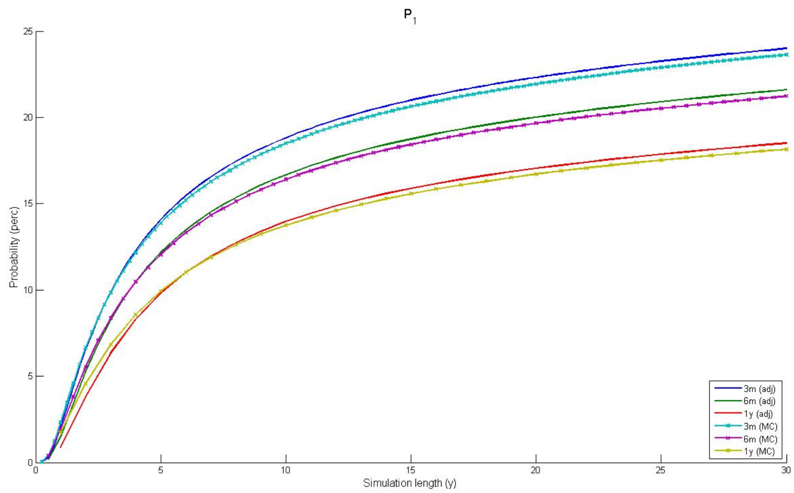

Figure 6: Cumulative probability to generate a negative yield of the one year bond according to the adjusted continuous and empirical discrete hitting time using the 1 and 10 days, 1, 3, 6 months and 1 years time step in the simulation under the parameters set P_1.

Using the parameters set P_4, we consider the continuous and simulated densities of figure 5 as well as the adjusted densities of figure 6. As in the previous figures 5 and 6, the empirical cumulative probabilities are obtained with 500 000 simulations and the order for the continuity adjustment is $N^* = 15$. At first, it can be noted, from the comparison of figures 5 and 7, that the cumulative probabilities for short simulation horizons are higher for low simulation steps and low lengths of simulations. However, at the 30 years simulations lengths, using a yearly step, the cumulative probabilities are similar. Secondly, the cumulative probabilities are strongly increasing up to the 5 years simulation length then are slightly increasing unlike those implied by the parameter set P_1 for all simulation steps. Thirdly, the continuity adjustment provides a good approximation of the cumulative density for simulation steps starting from 10 days (and not for the daily simulation step). Again, the performance of the proposed adjustment for medium simulation steps is a weakness.

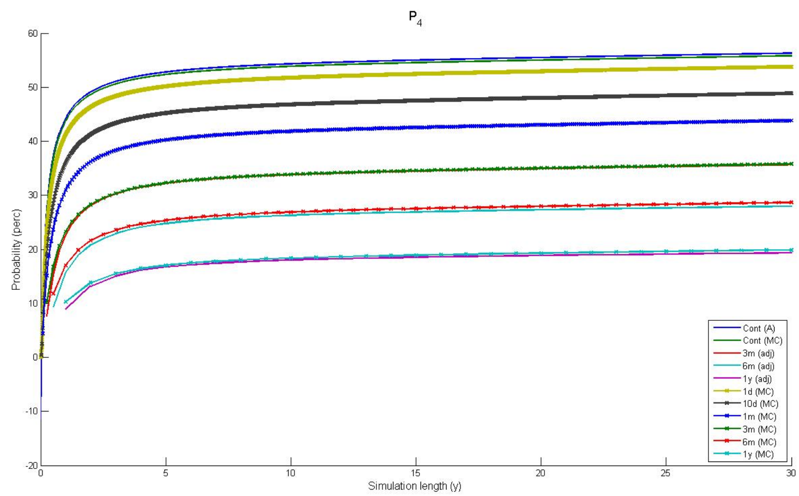

Figure 7: Cumulative probability to generate a negative yield of the one year bond using the 1 and 10 days, 1, 3, 6 months and 1 years time step according to the adjusted continuous and empirical discrete hitting time under the parameters set P_4.

The same shocks of figure 3 are applied to the the parameters set P_1 and for the sake of simplicity, only the continuous hitting time densities associated to the negative yield of the one year maturity bond are analysed and the obtained densities are represented in figure 8 with a yearly step. As previously, the curves with a “x” marking stand for an upwards shock of 20 percent of the parameters value while the curves with a “o” marking represent a downwards variation of the parameters value. As in figure 3, the volatility coefficient has the majors effects on the probability density, increasing it value increase the cumulative probability and vice versa. A lower value of the equilibrium value, mean-reversion speed and current value of the spot rate also increase the cumulative probability and vice versa but the effect of the latter parameter is more significant on the short simulation horizons unlike the two first ones.

Three conclusions can be drawn from these numerical examples. Firstly, these market conditions show that negative IR are always a concern to consider when building economical scenarios, especially in the context of *Monte-Carlo* simulations. Moreover, for a single generation, it appears that for a given

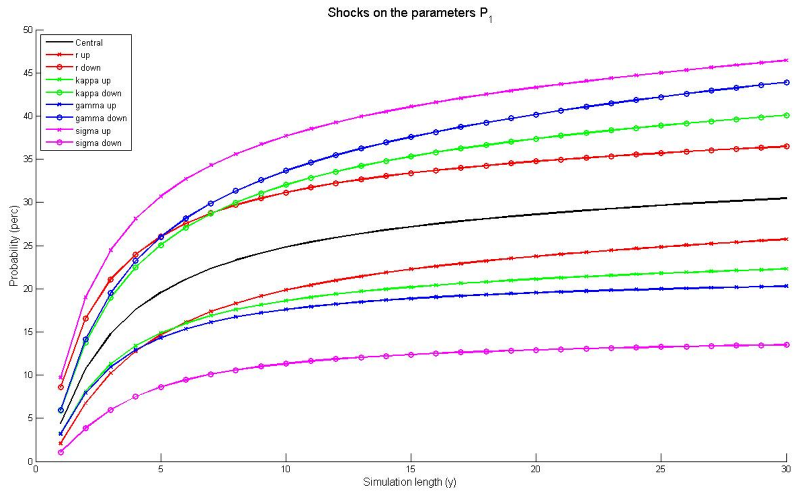

Figure 8: Cumulative probability to generate a negative yield of the one year bond applying shocks of ± 20% on the parameters set P_1.

time-horizon very often the 1-VM can be used to generate IR term structure when the corresponding time to maturities are sufficiently large under condition (8). Consequently, the acceptance or rejection use of the model has to be appreciated depending on the situation at hand, but not globally as it is generally believed that the 1-VM seems useless after the 2007 financial crisis. Secondly, it can be seen that even in the pre-financial crisis period, the generation of negative IR under the 1-VM can be a concern for short maturity as 1/10 day(s) or 1 month. This situation should warn against some practitioners action. In fact, calibrating the model on a regular basis (within a daily or weekly frequency) and using it for a short/medium time horizon is believed to reduce the model weaknesses. Thirdly, the considered market conditions confirm that the problem of generation of negative rates is more pronounced during and after the 2007 financial crisis rather than before. This reinforces the idea to switch to alternative IR models as those introduced in [4], [5], [6] and [7].

5 Conclusion

The 1-VM is a famous benchmark model for the IR simulation. Despite its severe limitations, its explicit properties and tractability make the 1-VM to be always used for both theoretical and practical purposes. In fact, the 1-VM is encompassed on the general GATSM class and can describe the shadow rate process for realistic IR models. Also, the determination of regulatory capital requirements, as in the Basel 3 and Solvency II frameworks, calls for a model to generate scenarios for IR at some future time horizons. This may be performed with the 1-VM, at least in order to check some outputs, e.g. market values or capital requirements, comparing them to the same outputs obtained from another complex model.

With the theoretical considerations and IR market observations we provide in this work, it is a hard fact that the 1-VM generate negative IR, which are economically inconsistent. The generation of negative IR arises before and after the 2007 financial crisis[15] even the 1-VM is correctly calibrated. We have derived here a very explicit condition for any yield-rate, prevailing at one future-time horizon, to be negative when generated with the 1-VM (proposition 2.3) and, given the Gaussian framework, the corresponding probability can be valued (proposition 2.4). We have obtained a condition on the parameters implying a non-increasing of the level determining negative IR with respect to the time to maturity of the ZCB. Consequently, when the generation of IR of various maturities at one future date is considered, the shortest maturity plays a key role in the parameters restriction is fulfilled (proposition 2.5). Moreover, if a finite number of maturities is considered, then one can find a maturity that has a central role whether the parameters restriction holds or not (proposition 2.8). We also have suggested a manner to tweak the model in order to discard negative IR at one future time horizon (propositions 2.6 and 2.7).

We have extended the analysis on the simulation of the IR for various future time horizons, motivated by the pricing and risk management of complex financial contracts and insurance policies. Firstly, in case of a time series of one ZCB price with a constant time to maturity, this problem has been formulated in terms of a continuous hitting time, which density can be easily approximated. However, the simulation is performed using a discrete time step, hence the hitting time has to be reformulated with a discrete one. Adapting the proof of [12], the discrete hitting time converges to the continuous one when the time

[15]though the inconsistency is more pronounced, for the post-crisis period

steps tends to zero (proposition 3.1). Secondly, when time series of ZCB prices with various time to maturities are considered, the parameters restriction allows to consider only the shortest time to maturity (proposition 3.2). If this condition is not fulfilled, the maturity with a key role can be identified for a finite number of generated time series (proposition 3.4). According to these observation, the convergence results can be adapted (corollaries 3.3 and 3.5).

With the market observations considered here, it appears that for short-time horizons, the IR generated by 1-VM can have a high probability to be negative when its time to maturity is very short as 1 or 10 day(s). The situation become more viable when the time to maturity increases under the parameter restriction. Such a finding may be useful as a warning about some belief among practitioners. In fact, a way to overcome the limitation of a benchmark model (as the Black-Scholes-Merton) is to make calibrations on a short and regular basis. The model forecasting power is believed to be good when short horizon and maturities are used. Our above mentioned empirical observations finding seems contradicts this tenet. Our analyses of the market observations, before and after the 2007 financial crisis, lead to conclude that, the 1-VM is definitely an inconsistent model. However there are some situations (depending on the time-horizon, the model parameters and the IR time to maturity under consideration) where the 1-VM becomes an acceptable model to generate future IR scenarios. Therefore we have proposed a simple way to tweak the model in order to prevent against the harmful consequences resulting from a brute application of the 1-VM.

Our study here is just restricted to the simple benchmark 1-VM. But the analyses and observations done here should help the reader to be aware with the hidden problems underlying any extended multi-factors *Vasicek* model or GATSM or HJM models. These latter models are largely presented and used in the literature, but actually there is no clear analyses about their economical inconsistencies as we have done here. This issue deserves to be analysed at the present time where various attempts do exist to find HJM type models in the context of multiple curve modelling, as it seems useful to consider after the 2007 financial crisis. Actually to the best of our knowledge, no explicit investigation does emerge related to the economical sense of the IR produced by these models.

A Appendices

In this appendix, some results related to the hitting time of the 1-VM and autoregressive processes are introduced so as to obtain the densities used in the numerical examples as well as the proof of proposition 3.1.

A.1 The hitting time of a Vasicek Process

Consider a process $(x_t(\cdot))_{t\geq 0}$ driven by SDE in form of (1) or (1′) under a measure $\mathbb{M}$ with initial value x_0, let $-\infty < y < x_0$ be a constant and define the hitting time $\mathrm{T}(y)(\cdot) = \inf\{t > 0 \,|x_t \leq y\}$. As mentioned in [15], this hitting time has various applications in financial modelling like shadow models of IR (see [4]), path dependent options pricing or credit risk modelling.

Several methods have been proposed in order to approximate the density of $\mathrm{T}(y)(\cdot)$ and we follow [15] and [16], expanding the density in terms of eigenfunctions. Other methods are presented in [17], in terms of series, integral and *Bessel* bridge. Denote by ${}_1F_1\left(a, b, z\right)$ for $a, z \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ the *Kummer* confluent hyperbolic function:

$$
{}_1F_1\left(a, b, x\right) = \sum_{i=0}^{\infty} \frac{(a)_i z^i}{(b)_i i!}
$$

where $(p)_m$ stands for the rising factorial[16]. The *Hermite* function, $\mathrm{H}_\nu(z)$, is given by:

$$
\mathrm{H}_\nu(z) = 2^\nu \sqrt{\pi} \left[\frac{{}_1F_1\left(-\frac{\nu}{2}, \frac{1}{2}, z^2\right)}{\Gamma\left(\frac{1-\nu}{2}\right)} - 2z \frac{{}_1F_1\left(\frac{1-\nu}{2}, \frac{3}{2}, z^2\right)}{\Gamma\left(-\frac{\nu}{2}\right)} \right]
$$

According to [15] and [16], the density of $\mathrm{T}(y)(\cdot)$ can be expressed using this function:

Proposition A.1. *For $-\infty < y < x_0$, the density of $\mathrm{T}(y)(\cdot)$ has the following form for $0 < t$:*

$$
f_{\mathrm{T}(y)(\cdot)} = \sum_{i=1}^{\infty} c_i(\kappa, \theta, \sigma) \lambda_i(\kappa, \theta, \sigma) \exp\left(-\lambda_i(\kappa, \theta, \sigma) t\right) \qquad \text{(A.3)}
$$

[16] $(p)_0 = 1$ and $(p)_m = p(p + 1) \cdots (p + m - 1)$.

where the first coefficient $\lambda_i(\kappa,\theta,\sigma)$ *is defined by:*

$$\nu_i(\kappa,\theta,\sigma) = \lambda_i(\kappa,\theta,\sigma)/\kappa \tag{A.4}$$

and $\nu_i(\kappa,\theta,\sigma)$, *for* $0 < \nu_1(\kappa,\theta,\sigma) < \cdots < \nu_i(\kappa,\theta,\sigma) < \cdots$, *are the positive roots of the equation:*

$$\mathrm{H}_\nu\left(\frac{\bar{y}(\kappa,\theta,\sigma)}{\sqrt{2}}\right) = 0 \tag{A.5}$$

The coefficients $c_i(\kappa,\theta,\sigma)$ *are defined by:*

$$c_i(\kappa,\theta,\sigma) = -\frac{\mathrm{H}_{\nu_i(\kappa,\theta,\sigma)}\left(\frac{\bar{x_0}(\kappa,\theta,\sigma)}{\sqrt{2}}\right)}{\nu_i(\kappa,\theta,\sigma)\frac{\partial}{\partial\nu}\left[\mathrm{H}_\nu\left(\frac{\bar{y}(\kappa,\theta,\sigma)}{\sqrt{2}}\right)\right]_{\nu=\nu_i(\kappa,\theta,\sigma)}} \tag{A.6}$$

where $\bar{x_0}(\kappa,\theta,\sigma) = \sqrt{2\kappa}\,(x_0-\theta)/\sigma$ *and* $\bar{y}(\kappa,\theta,\sigma) = \sqrt{2\kappa}\,(y-\theta)/\sigma$. *Moreover,* $\lambda_i(\kappa,\theta,\sigma)$ *and* $c_i(\kappa,\theta,\sigma)$ *have the large-i asymptotics:*

$$\lambda_i^\star(\kappa,\theta,\sigma) = \kappa\nu_i^\star(\kappa,\theta,\sigma) \tag{A.7}$$

and:

$$c_i^\star(\kappa,\theta,\sigma) = \frac{(-1)^{i+1}2\sqrt{k_i}}{(2k_i-1/2)\left(\pi\sqrt{k_i}-2^{-1/2}\right)} \tag{A.8}$$

where $\nu_i^\star(\kappa,\theta,\sigma) = 2k_i - 1/2$ *and* $k_i = i - \frac{1}{4} + \frac{\bar{y}^2(\kappa,\theta,\sigma)}{\pi^2} + \frac{\bar{y}(\kappa,\theta,\sigma)\sqrt{2}}{\pi}\sqrt{i-\frac{1}{4}+\frac{\bar{y}^2(\kappa,\theta,\sigma)}{2\pi^2}}$.

It follows form eq. (A.3) that the probability to have not hit the bound y at time $0 < t$ is given by:

$$\mathbb{M}\left\{t < \mathrm{T}(y)(\cdot)\right\} = \sum_{i=1}^{\infty} c_i(\kappa,\theta,\sigma)\exp\left(-\lambda_i(\kappa,\theta,\sigma)t\right) \tag{A.9}$$

Moreover, the representation in terms of series of eq. (A.3) allows one to approximate the density with a truncated number of terms controlling the error as discussed in [15]. In one example, the author proposes to truncate the series and to use the exact coefficients of eqs. (A.4) and (A.6) for the initial terms and their approximated values of eqs. (A.7) and (A.8) for the remaining terms and this solution shows a little bias.

A.2 The hitting time of a AR(1) process

Consider a AR(1) process $(\tilde{x}_n(.))_{n\in\mathbb{N}}$ driven by the transition equation:

$$\tilde{x}_n(.) = \alpha\tilde{x}_{n-1}(.) + \beta + \eta_n \tag{A.10}$$

where α and β are constants such that $0 < \alpha < 1$, $\beta = 1-\alpha$ and η_n is a family of i.i.d. centred normal variables of variance $\frac{\sigma^2}{2\kappa}(1-\alpha^2)$[17]. As a consequence, the conditional distribution of $\tilde{x}_n$ given $\tilde{x}_0 = x_0$ is Gaussian variable of mean:

$$\mathbb{E}\left[\tilde{x}_n|\tilde{x}_0 = x_0\right] = \alpha^n x_0 + \beta\left(\frac{1-\alpha^n}{1-\alpha}\right) = \alpha^n x_0 + \theta\left(1-\alpha^n\right) \tag{A.11}$$

and variance:

$$\mathbb{V}\left[\tilde{x}_n|\tilde{x}_0 = x_0\right] = \frac{\sigma^2}{2\kappa}\left(1-\alpha^{2n}\right) \tag{A.12}$$

Moreover, the covariance of $\tilde{x}_p$ and $\tilde{x}_n$ for $p \geq n$ is:

$$\Sigma\left[\tilde{x}_p, \tilde{x}_n|\tilde{x}_0 = x_0\right] = \alpha^{p-n}\mathbb{V}\left[\tilde{x}_n|\tilde{x}_0 = x_0\right] \tag{A.13}$$

since the variables η_i, for $i \geq n$ are independent from $\tilde{x}_i$. Let $y < x_0$ be a constant and define the discrete hitting time:

$$\mathrm{T}^\delta(y)(\cdot) = \min\left\{i \in \mathbb{N}^* : \tilde{x}_i \leq y\right\} = \min\left\{i \in \mathbb{N}^* : X_i \leq y\right\} \tag{A.14}$$

The density of this random variable can be computed recursively using the orthant probabilities as mentioned by [18]:

Proposition A.2. *Let $\mathcal{D}_n(\tilde{x}_1, \cdots, \tilde{x}_n)$ be the set defined by*

$$\mathcal{D}_n(\tilde{x}_1, \cdots, \tilde{x}_n) = \bigcup_{i=1}^{n}\left\{\tilde{x}_i > y\right\}.$$

The probability that $\mathrm{T}^\delta(y)(\cdot)$ equals n, for $1 \leq n$ is given by:

$$\mathbb{M}\left\{\mathrm{T}^\delta(y)(\cdot) = n|\tilde{x}_0 = x_0\right\} = \tag{A.15}$$
$$\begin{cases} 1 - \mathbb{M}\left\{\mathcal{D}_1(\tilde{x}_1)|\tilde{x}_0 = x_0\right\} & \textit{if } n = 1 \\ \mathbb{M}\left\{\mathcal{D}_{n-1}(\tilde{x}_1, \cdots, \tilde{x}_{n-1})|\tilde{x}_0 = x_0\right\} - \mathbb{M}\left\{\mathcal{D}_n(\tilde{x}_1, \cdots, \tilde{x}_n)|\tilde{x}_0 = x_0\right\} & \textit{if } n > 1 \end{cases}$$

[17] This specification of the AR(1) is adopted so as to prepare the formulation for the 1-VM.

Proof. 1. If $n = 1$, then:

$$\begin{aligned}
&\mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot) = 1|\tilde{x}_0 = x_0\right\} \\
&\quad = \mathbb{M}\left\{\tilde{x}_1 \leq y|\tilde{x}_0 = x_0\right\} = 1 - \mathbb{M}\left\{\tilde{x}_1 > y|\tilde{x}_0 = x_0\right\} \\
&\quad = 1 - \mathbb{M}\left\{\mathcal{D}_1\left(\tilde{x}_1\right)|\tilde{x}_0 = x_0\right\}
\end{aligned}$$

2. If $n > 1$, then:

$$\begin{aligned}
&\mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot) = n|\tilde{x}_0 = x_0\right\} \\
&\quad = \mathbb{M}\left\{\tilde{x}_n \leq y, \tilde{x}_{n-1} > y, \cdots, \tilde{x}_1 > y|\tilde{x}_0 = x_0\right\} \\
&\quad = \mathbb{M}\left\{\tilde{x}_{n-1} > y, \cdots, \tilde{x}_1 > y|\tilde{x}_0 = x_0\right\} \\
&\qquad - \mathbb{M}\left\{\tilde{x}_n > y, \tilde{x}_{n-1} > y, \cdots, \tilde{x}_1 > y|\tilde{x}_0 = x_0\right\} \\
&\quad = \mathbb{M}\left\{\mathcal{D}_{n-1}\left(\tilde{x}_1, \cdots, \tilde{x}_{n-1}\right)|\tilde{x}_0 = x_0\right\} - \mathbb{M}\left\{\mathcal{D}_n\left(\tilde{x}_1, \cdots, \tilde{x}_n\right)|\tilde{x}_0 = x_0\right\}
\end{aligned}$$

□

Since the process $\tilde{x}_n$ is Gaussian, then for all $1 \leq n$, one has:

$$\mathbb{M}\left\{\mathcal{D}_n\left(\tilde{x}_1, \cdots, \tilde{x}_n\right)|\tilde{x}_0 = x_0\right\} = \int\limits_{\mathcal{O}_n} \frac{1}{(2\pi)^{n/2}|\mathbf{\Sigma}_n|^{1/2}} e^{-\frac{1}{2}(x-m_n)^T \mathbf{\Sigma}_n^{-1}(x-m_n)} dx$$

where $\mathcal{O}_n = (y, \infty)^n$, m_n is the mean vector which elements are given by eq. (A.11) and $\mathbf{\Sigma}_n$ is the covariance matrix which diagonal elements are given by eq. (A.12) and remaining elements by eq. (A.13). Eq. (A.15) shows that the probability $\mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot) = n|\tilde{x}_0 = x_0\right\}$ involves the valuation of a n-dimensional integral which can be difficult to value on a numerical point of view.

Assume that the process spot rate process $r_t(\cdot)$ under the probability measure $\mathbb{M}$ has the form of eqs. (2) and (2′). This allows one to write the discretization of the processes $r_t(\cdot)$ as autoregressive processes of order one in form of eq. (A.10):

$$\begin{aligned}
r_{t_{i+1}}(\cdot) &= \exp\left[-\kappa\delta\right] r_{t_i}(\cdot) + \kappa\theta \mathrm{b}\left(\delta; \kappa\right) + \sigma \mathrm{b}^{\frac{1}{2}}\left(\delta; 2\kappa\right) \varepsilon_{i+1|i}(\cdot) \\
&= \alpha r_{t_i}(\cdot) + \beta + \eta_{t_i}(\cdot)
\end{aligned} \tag{A.16}$$

and:

$$\begin{aligned}
r_{t_{i+1},\mathbb{P}}(\cdot) &= \exp\left[-\kappa_{\mathbb{P}}\delta\right] r_{t_i}(\cdot) + \kappa\theta \mathrm{b}\left(\delta; \kappa_{\mathbb{P}}\right) + \sigma \mathrm{b}^{\frac{1}{2}}\left(\delta; 2\kappa_{\mathbb{P}}\right) \varepsilon_{i+1|i,\mathbb{P}}(\cdot) \\
&= \alpha_{\mathbb{P}} r_{t_i,\mathbb{P}}(\cdot) + \beta_{\mathbb{P}} + \eta_{t_i,\mathbb{P}}(\cdot)
\end{aligned} \tag{A.16′}$$

where $\alpha = \exp[-\kappa\delta]$, $\alpha_{\mathbb{P}} = \exp[-\kappa_{\mathbb{P}}\delta]$, $\beta = (1-\alpha)$, $\beta_P = (1-\alpha_{\mathbb{P}})$, $\eta_{t_i}(\cdot)$ is a Gaussian variable of null mean and variance $\sigma^2 \mathrm{b}(\delta; 2\kappa)$ and $\eta_{t_i,\mathbb{P}}(\cdot)$ is a Gaussian variable of null mean and variance $\sigma^2 \mathrm{b}(\delta; 2\kappa_{\mathbb{P}})$. As a consequence, the previous results related to the discrete hitting time can be applied for the spot rate generated under both historical and risk-neutral measures.

A.3 Proof of proposition 3.1

A.3.1 Preliminary results

Consider the process $(x_t(\cdot))_{t\geq 0}$ instead of its discrete observations $(\tilde{x}_n(.))_{n\in\mathbb{N}}$. An alternative representation of x_t is:

$$x_t = x_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \frac{\sigma}{\sqrt{2\kappa}} W^{\mathbb{M}}\left(e^{2\kappa t} - 1\right) e^{-\kappa t} \tag{A.17}$$

Define the hitting time $\mathrm{T}(y)(\cdot)$, corresponding to the continuous version of $\mathrm{T}^{\delta}(y)(\cdot)$, by:

$$\mathrm{T}(y)(\cdot) = \inf\{t > 0 : x(t) \leq y\} \tag{A.18}$$

At first, it follows from eqs. (A.14) and (A.18) that $\mathrm{T}(y)(\cdot) \leq \mathrm{T}^{\delta}(y)(\cdot)$ for all $0 < \delta$. The proof relies on the following lemma proven by *Lochowski* [12]:

Lemma A.3. *Let $W^{\mathbb{M}}(t)$ be a standard Brownian motion under $\mathbb{M}$, $\iota > 0$ and $0 \leq t_1 \leq \cdots \leq t_n$ be a sequence satisfying $t_{k+1} \geq (n^6+1)t_k$ and $t_{k+1} \geq t_k + \iota$ for all $1 \leq k \leq n-1$. Then for every $\epsilon > 0$:*

$$\mathbb{M}\left\{\max\left\{W^{\mathbb{M}}(t_1), \cdots, W^{\mathbb{M}}(t_n)\right\} \leq \epsilon\right\} \leq \xi_n\left(\frac{\epsilon}{\sqrt{\iota}}\right) \tag{A.19}$$

where the bound is defined by:

$$\xi_n(x) = \frac{2}{n} + 2ne^{-\frac{n^2}{8}} + \Phi(x)^{n-1} \tag{A.20}$$

Eq. (A.19) implies that for each $\epsilon > 0$

$$\mathbb{M}\left\{\min\left\{W^{\mathbb{M}}(t_1), \cdots, W^{\mathbb{M}}(t_n)\right\} \geq -\epsilon\right\} \leq \xi_n\left(\frac{\epsilon}{\sqrt{\iota}}\right) \tag{A.21}$$

In fact:

$$\begin{aligned}
&\mathbb{M}\left\{\min\left\{W^{\mathbb{M}}(t_1),\cdots,W^{\mathbb{M}}(t_n)\right\}\geq-\epsilon\right\}\\
&\quad=\mathbb{M}\left\{-\min\left\{W^{\mathbb{M}}(t_1),\cdots,W^{\mathbb{M}}(t_n)\right\}\leq\epsilon\right\}\\
&\quad=\mathbb{M}\left\{\max\left\{-W^{\mathbb{M}}(t_1),\cdots,-W^{\mathbb{M}}(t_n)\right\}\leq\epsilon\right\}
\end{aligned}$$

Since $W^{\mathbb{M}}(t)$ is a $\mathbb{M}$-Brownian motion, $-W^{\mathbb{M}}(t)$ is also a $\mathbb{M}$-Brownian motion, hence satisfies eq. (A.19) and eq. (A.21) follows.

A.3.2 Proof of proposition 3.1

This proof follows the original proof of *Lochowski* [12]:

Proof. At first assume that $y \leq \theta$ and focus on the probability

$$p(t,\sqrt{\delta},y)=\mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot)\geq t+\sqrt{\delta}|\mathrm{T}(y)(\cdot)=t\right\} mboxforall\ \ t>0 \ \text{ and } \ \delta$$

such that

$$0<\delta\leq\min\left\{\frac{1}{16},\frac{\ln(2)^2}{4\kappa^2}\right\}.$$

Using the definitions of the hitting times $\mathrm{T}^{\delta}(y)(\cdot)$ and $\mathrm{T}(y)(\cdot)$ of eqs. (A.14) and (A.18) as well as the representation of eq. (A.17), one has:

$$\begin{aligned}
&\left\{\forall s\in\delta\mathbb{N}^*\cap\left[t;t+\sqrt{\delta}\right],x(s)>y|x(t)=y\right\}\\
&\quad=\left\{\forall s\in\{\delta\mathbb{N}^*-t\}\cap\left[0;\sqrt{\delta}\right],x(s)>y|x(0)=y\right\}\\
&\quad=\Big\{\forall s\in\{\delta\mathbb{N}^*-t\}\cap\left[0;\sqrt{\delta}\right],\\
&\qquad\qquad x(0)e^{-\kappa s}+\theta\left(1-e^{-\kappa s}\right)+\frac{\sigma}{\sqrt{2\kappa}}W^{\mathbb{M}}\left(e^{2\kappa s}-1\right)e^{-\kappa s}>y|x(0)=y\Big\}\\
&\quad=\Big\{\forall s\in\{\delta\mathbb{N}^*-t\}\cap\left[0;\sqrt{\delta}\right],\\
&\qquad\qquad ye^{-\kappa s}+\theta\left(1-e^{-\kappa s}\right)+\frac{\sigma}{\sqrt{2\kappa}}W^{\mathbb{M}}\left(e^{2\kappa s}-1\right)e^{-\kappa s}>y|x(0)=y\Big\}\\
&\quad=\Big\{\forall s\in\{\delta\mathbb{N}^*-t\}\cap\left[0;\sqrt{\delta}\right],\\
&\qquad\qquad ye^{-\kappa s}+\theta\left(1-e^{-\kappa s}\right)+\frac{\sigma}{\sqrt{2\kappa}}W^{\mathbb{M}}\left(e^{2\kappa s}-1\right)e^{-\kappa s}>y|W^{\mathbb{M}}(0)=0\Big\}\\
&\quad=\Big\{\forall s\in\{\delta\mathbb{N}^*-t\}\cap\left[0;\sqrt{\delta}\right],\\
&\qquad\qquad ye^{-\kappa s}+\theta\left(1-e^{-\kappa s}\right)+\frac{\sigma}{\sqrt{2\kappa}}W^{\mathbb{M}}\left(e^{2\kappa s}-1\right)e^{-\kappa s}>y\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \forall s \in \{\delta \mathbb{N}^* - t\} \cap \left[0; \sqrt{\delta}\right], \frac{\sigma}{\sqrt{2\kappa}} W^{\mathbb{M}} \left(e^{2\kappa s} - 1\right) e^{-\kappa s} > (y - \theta) \left(1 - e^{-\kappa s}\right) \right\} \\
&= \left\{ \forall s \in \{\delta \mathbb{N}^* - t\} \cap \left[0; \sqrt{\delta}\right], W^{\mathbb{M}} \left(e^{2\kappa s} - 1\right) > \frac{\sqrt{2\kappa}}{\sigma} (y - \theta) \left(1 - e^{-\kappa s}\right) e^{\kappa s} \right\} \\
&= \left\{ \forall s \in \{\delta \mathbb{N}^* - t\} \cap \left[0; \sqrt{\delta}\right], W^{\mathbb{M}} \left(e^{2\kappa s} - 1\right) > \frac{\sqrt{2\kappa}}{\sigma} (y - \theta) \left(e^{\kappa s} - 1\right) \right\} \\
&= \left\{ \forall u \in e^{2\kappa \{\delta \mathbb{N}^* - t\}} \cap \left[1; e^{2\kappa \sqrt{\delta}}\right], W^{\mathbb{M}} (u - 1) > \frac{\sqrt{2\kappa}}{\sigma} (y - \theta) \left(\sqrt{u} - 1\right) \right\} \\
&\subseteq \left\{ \forall u \in e^{2\kappa \{\delta \mathbb{N}^* - t\}} \cap \left[1; e^{2\kappa \sqrt{\delta}}\right], W^{\mathbb{M}} (u - 1) > \frac{\sqrt{2\kappa}}{\sigma} (y - \theta) \left(e^{\kappa \sqrt{\delta}} - 1\right) \right\} \\
&= \left\{ \forall v \in \left\{e^{2\kappa \{\delta \mathbb{N}^* - t\}} - 1\right\} \cap \left[0; e^{2\kappa \sqrt{\delta}} - 1\right], W^{\mathbb{M}} (v) > \frac{\sqrt{2\kappa}}{\sigma} (y - \theta) \left(e^{\kappa \sqrt{\delta}} - 1\right) \right\}
\end{aligned}
$$

Let n_0 be the smallest integer such that $\delta n_0 - t \geq \delta$. Consequently, $\delta(n_0 - 1) - t < \delta$ and $\delta n_0 - t < 2\delta$. Define $\iota = e^{2\kappa(\delta n_0 - t)} 2\kappa\delta$ and $s_k = e^{2\kappa(\delta n_0 - t + \delta k)} - 1$ for $k = 1, 2, \cdots, N(\delta) = \lfloor \frac{1}{2\sqrt{\delta}} \rfloor$, hence:

$$
\begin{aligned}
s_k &= e^{2\kappa(\delta n_0 - t + \delta k)} - 1 = e^{2\kappa(\delta n_0 - t)} e^{2\kappa\delta k} - 1 \\
&\leq e^{2\kappa 2\delta} e^{2\kappa\delta \frac{1}{2\sqrt{\delta}}} - 1 \leq e^{2\kappa(2\delta + \frac{\sqrt{\delta}}{2})} - 1 \leq e^{2\kappa\sqrt{\delta}} - 1 \quad \text{since} \quad \delta \leq \frac{1}{16}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
s_{k+1} - s_k &= e^{2\kappa(\delta n_0 - t + \delta(k+1))} - e^{2\kappa(\delta n_0 - t + \delta k)} = e^{2\kappa(\delta n_0 - t)} e^{2\kappa\delta k} \left(e^{2\kappa\delta} - 1\right) \\
&\geq e^{2\kappa(\delta n_0 - t)} \left(e^{2\kappa\delta} - 1\right) \geq e^{2\kappa(\delta n_0 - t)} 2\kappa\delta = \iota
\end{aligned}
$$

Note that $0 \leq \delta n_0 - t + \delta k$ and $\delta n_0 - t + \delta k \leq 2\delta + \frac{\sqrt{\delta}}{2} \leq \sqrt{\delta}$ since $\delta \leq \frac{1}{16}$. Moreover, since $\delta \leq \frac{\ln(2)^2}{4\kappa^2}$, one obtains that $0 \leq \delta n_0 - t + \delta k \leq \sqrt{\delta} \leq \frac{\ln(2)}{2\kappa}$. But for $0 \leq x \leq \frac{\ln(2)}{2\kappa}$, it can be shown that $2\kappa x \leq e^{2\kappa x} - 1 \leq \frac{2\kappa}{\ln(2)} x$, therefore:

$$
\begin{aligned}
2\kappa\delta(k+1) &\leq 2\kappa \left(\delta n_0 - t + \delta k\right) \leq s_k = e^{2\kappa(\delta n_0 - t + \delta k)} - 1 \\
&\leq \frac{2\kappa}{\ln(2)} \left(\delta n_0 - t + \delta k\right) \leq \frac{2\kappa}{\ln(2)} \delta(k+2)
\end{aligned}
$$

Consider the integer $n^*(\delta)$ such that $\left(\lceil \frac{2}{\ln(2)} \rceil \left(n^*(\delta)^6 + 1\right)\right)^{n^*(\delta)} \leq N(\delta)$ and define the sequences $k_i = \left(\lceil \frac{2}{\ln(2)} \rceil \left(n^*(\delta)^6 + 1\right)\right)^i$ for $i = 1, 2, \cdots, n^*(\delta)$ and

$t_i = s_{k_i}$, hence:

$$\frac{t_{i+1}}{t_i} \geq \frac{2\kappa\delta\left(k_{i+1}+1\right)}{\frac{2\kappa}{\ln(2)}\delta\left(k_i+2\right)} \geq \frac{k_{i+1}}{\frac{2}{\ln(2)}k_i} = \frac{\left\lceil\frac{2}{\ln(2)}\right\rceil}{\frac{2}{\ln(2)}}\left(n^*(\delta)^6+1\right) \geq n^*(\delta)^6+1$$

Consequently, the sequence $t_1, \cdots, t_{n^*(\delta)}$ satisfies the assumptions of lemma A.3 and $0 < \frac{\sqrt{2\kappa}}{\sigma}(\theta-y)\left(e^{\kappa\sqrt{\delta}}-1\right)$, therefore:

$$\begin{aligned}
&\mathbb{M}\left\{\max\left\{W^{\mathbb{M}}\left(t_1\right), \cdots, W^{\mathbb{M}}\left(t_{n^*(\delta)}\right)\right\} < \frac{\sqrt{2\kappa}}{\sigma}(\theta-y)\left(e^{\kappa\sqrt{\delta}}-1\right)\right\} \\
&\quad \leq \xi_{n^*(\delta)}\left(\frac{\frac{\sqrt{2\kappa}}{\sigma}(\theta-y)\left(e^{\kappa\sqrt{\delta}}-1\right)}{\sqrt{e^{2\kappa(\delta n_0-t)}2\kappa\delta}}\right) \\
&\quad \leq \xi_{n^*(\delta)}\left(\frac{\frac{\theta-y}{\sigma}\left(e^{\kappa\sqrt{\delta}}-1\right)}{e^{\kappa\delta}\sqrt{\delta}}\right) \\
&\quad \leq \xi_{n^*(\delta)}\left(\frac{\frac{\theta-y}{\sigma}\left(\frac{1}{\ln(2)}\kappa\sqrt{\delta}\right)}{e^{\kappa\delta}\sqrt{\delta}}\right) \\
&\quad \leq \xi_{n^*(\delta)}\left(\kappa\frac{\theta-y}{\sigma\ln(2)}\right)
\end{aligned}$$

Therefore, the probability $p\left(t, \sqrt{\delta}, y\right)$ is bounded by:

$$\begin{aligned}
&p\left(t, \sqrt{\delta}, y\right) \\
&\quad \leq \mathbb{M}\left\{\forall v \in \left\{e^{2\kappa\{\delta\mathbb{N}^*-t\}}-1\right\} \cap \left[0; e^{2\kappa\sqrt{\delta}-1}\right],\right. \\
&\qquad\qquad \left. W^{\mathbb{M}}(v) > \frac{\sqrt{2\kappa}}{\sigma}(y-\theta)\left(e^{\kappa\sqrt{\delta}}-1\right)\right\} \\
&\quad \leq \mathbb{M}\left\{\min\left\{W^{\mathbb{M}}\left(t_1\right), \cdots, W^{\mathbb{M}}\left(t_{n^*(\delta)}\right)\right\} > \frac{\sqrt{2\kappa}}{\sigma}(y-\theta)\left(e^{\kappa\sqrt{\delta}}-1\right)\right\} \\
&\quad = \mathbb{M}\left\{\max\left\{W^{\mathbb{M}}\left(t_1\right), \cdots, W^{\mathbb{M}}\left(t_{n^*(\delta)}\right)\right\} < \frac{\sqrt{2\kappa}}{\sigma}(\theta-y)\left(e^{\kappa\sqrt{\delta}}-1\right)\right\} \\
&\quad \leq \xi_{n^*(\delta)}\left(\kappa\frac{\theta-y}{\sigma\ln(2)}\right)
\end{aligned}$$

Consequently, the probability $\mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot) \geq \mathrm{T}(y)(\cdot)+\sqrt{\delta}\right\}$ is bounded by:

$$
\begin{aligned}
&\mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot) \geq \mathrm{T}(y)(\cdot)+\sqrt{\delta}\right\} \\
&\quad=\int_{0}^{\infty} \mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot) \geq t+\sqrt{\delta} | \mathrm{T}(y)(\cdot)=t\right\} \mathbb{M}\left\{\mathrm{T}(y)(\cdot) \in[t ; t+d t)\right\} \\
&\quad\leq \int_{0}^{\infty} \xi_{n^{*}(\delta)}\left(\kappa \frac{\theta-y}{\sigma \ln (2)}\right) \mathbb{M}\left\{\mathrm{T}(y)(\cdot) \in[t ; t+d t)\right\} \\
&\quad=\xi_{n^{*}(\delta)}\left(\kappa \frac{\theta-y}{\sigma \ln (2)}\right)
\end{aligned}
$$

When $\delta \rightarrow 0^{+}$, $N(\delta) \rightarrow \infty$, $n^{*}(\delta) \rightarrow \infty$ and $\xi_{n^{*}(\delta)}\left(\kappa \frac{\theta-y}{\sigma \ln (2)}\right) \rightarrow 0$, therefore:

$$
\lim_{\delta \rightarrow 0^{+}} \mathbb{M}\left\{\mathrm{T}(y)(\cdot) \leq \mathrm{T}^{\delta}(y)(\cdot) \leq \mathrm{T}(y)(\cdot)+\sqrt{\delta}\right\}=1
$$

Secondly, assume that $y>\theta$, $t>0$ and δ such that $0<\delta \leq \min \left\{\frac{1}{16}, \frac{\ln (2)^{2}}{4 \kappa^{2}}\right\}$. Consider the probability $p\left(t, \sqrt{\delta}, y\right)$ previously defined, one has:

$$
\begin{aligned}
p\left(t, \sqrt{\delta}, y\right) \leq \mathbb{M}\Big\{&\forall v \in\left\{e^{2 \kappa\left\{\delta \mathbb{N}^{*}-t\right\}}-1\right\} \cap\left[0 ; e^{2 \kappa \sqrt{\delta}}\right], \\
&W^{\mathbb{M}}(v)>\frac{\sqrt{2 \kappa}}{\sigma}(y-\theta)\left(e^{\kappa \sqrt{\delta}}-1\right)\Big\}
\end{aligned}
$$

Since the coefficients κ, θ and σ as well as δ are positive and $y>\theta$, one has:

$$
\frac{\sqrt{2 \kappa}}{\sigma}(y-\theta)\left(e^{\kappa \sqrt{\delta}}-1\right)>\frac{\sqrt{2 \kappa}}{\sigma}(\theta-y)\left(e^{\kappa \sqrt{\delta}}-1\right)
$$

therefore:

$$
\begin{aligned}
p\left(t, \sqrt{\delta}, y\right) \leq \mathbb{M}\Big\{&\forall v \in\left\{e^{2 \kappa\left\{\delta \mathbb{N}^{*}-t\right\}}-1\right\} \cap\left[0 ; e^{2 \kappa \sqrt{\delta}}\right], \\
&W^{\mathbb{M}}(v)>\frac{\sqrt{2 \kappa}}{\sigma}(\theta-y)\left(e^{\kappa \sqrt{\delta}}-1\right)\Big\}
\end{aligned}
$$

Using the same computations, one obtains:

$$\begin{aligned}
&p\left(t, \sqrt{\delta}, y\right) \\
&\quad \leq \mathbb{M}\left\{\forall v \in \left\{e^{2\kappa\{\delta\mathbb{N}^{*}-t\}} - 1\right\} \cap \left[0; e^{2\kappa\sqrt{\delta}}\right], \right. \\
&\qquad\qquad \left. W^{\mathbb{M}}(v) > \frac{\sqrt{2\kappa}}{\sigma}(\theta - y)\left(e^{\kappa\sqrt{\delta}} - 1\right)\right\} \\
&\quad \leq \mathbb{M}\left\{\min\left\{W^{\mathbb{M}}(t_1), \cdots, W^{\mathbb{M}}\left(t_{n^{*}(\delta)}\right)\right\} > \frac{\sqrt{2\kappa}}{\sigma}(\theta - y)\left(e^{\kappa\sqrt{\delta}} - 1\right)\right\} \\
&\quad \leq \xi_{n^{*}(\delta)}\left(\kappa\frac{y-\theta}{\sigma\ln(2)}\right)
\end{aligned}$$

The probability $\mathbb{M}\left\{\mathrm{T}^{\delta}(y)(\cdot) \geq \mathrm{T}(y)(\cdot) + \sqrt{\delta}\right\}$ is bounded by $\xi_{n^{*}(\delta)}\left(\kappa\frac{y-\theta}{\sigma\ln(2)}\right)$ as in the previous computations and:

$$\lim_{\delta\to 0^{+}} \mathbb{M}\left\{\mathrm{T}(y)(\cdot) \leq \mathrm{T}^{\delta}(y)(\cdot) \leq \mathrm{T}(y)(\cdot) + \sqrt{\delta}\right\} = 1$$

□

ACKNOWLEDGEMENTS. The authors thank Professors Gilberto Castellani, Claudio Pacati and Luca Passalacqua as well as the participants of Paris Financial Management Conference (PFMC-2014) and of the XVI Workshop on Quantitative Finance for their constructive remarks and comments. This research work was supported by Alef-Servizi spa and ECE Paris Graduate School of Engineering. The contents of this paper are of the authors' responsibility. They do not represent the views of their employers.

References

[1] O. Vasicek, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics*, **5**(2), (1977), 177-188.

[2] D. Brigo and F. Mercurio, *Interest Rate Models - Theory and Practice*, Springer, Berlin Heidelberg, 2006.

[3] J.C. Cox, J.E. Ingersoll and S.A. Ross, A Theory of the Term Structure of Interest Rates, *Econometrica*, **53**(2), (1985), 385 - 408.

[4] V. Gorovoi and V. Linetsky, Black's Model of Interest Rates as Options, Eigenfunction Expansions and Japanese Interest Rates, *Mathematical Finance*, **14**(1), (2004), 49 - 78.

[5] L. Krippner, Modifying Gaussian Term Structure Models When Interest Rates are Near the Zero Lower Bound *Working Paper DP2012/02 , Reserve Bank of New-Zealand*, (2012).

[6] J.H.E. Christensen and G.D. Rudebusch, Estimating Shadow-Rate Term Structure Models with Near-Zero Yields, *Journal of Financial Econometrics*, **13**(2), (2015), 226 - 259.

[7] A. Meucci and A. Loregian, Neither 'Normal' nor 'Lognormal': Modeling Interest Rates Across All Regimes, *Working Paper SYMMYS and Università degli Studi di Milano-Bicocca*, (2014).

[8] G. Castellani M. De Felice and F. Moriconi, *Manuale di finanza III. Modelli stocastici e contratti derivati*, Il Mulino, Bologna, 2006.

[9] C. Rogers, Gaussian Errors, *RISK*, January, (1996).

[10] G.R. Duffee and R.H. Stanton, Estimation of Dynamic Term Structure Models, *Quarterly Journal of Finance*, **2**(2), (2012).

[11] J.C. Hull and A.D. White, Pricing Interest-Rate-Derivative Securities, *Review of Financial Studies*, **3**(4), (1990), 573 - 592.

[12] R. Lochowski, Stopping Time for Ornstein-Uhlenbeck Process vs. Stopping Time for discrete AR(1) Process, *Working Paper Warsaw School of Economics*, (2007).

[13] S.H. Babbs and K.B. Nowman, Kalman Filtering of Generalized Vasicek Term Structure Models, *Journal of Financial and Quantitative Analysis*, **34**(1), (1999), 115 - 130.

[14] M. Broadie, P. Glasserman and S. Kou, A Continuity Correction for Discrete Barrier Options, *Mathematical Finance*, **7**(4), (1997), 325 - 349.

[15] V. Linetsky, Computing Hitting Time Densities for CIR and OU Diffusions: Applications to Mean-Reverting Models, *Journal of Computational Finance*, **7**(4), (2004), 1 - 22.

[16] V. Linetsky, Lookback Options and Diffusion Hitting Times: A Spectral Expansion Approach, *Finance and Stochastics*, **8**(3), (2004), 373 - 398.

[17] L. Alili, P. Patie and J. L. Pedersen, Representations of the First Hitting Time Density of an Ornstein-Uhlenbeck Process, *Stochastic Models*, **21**(4), (2005), 967 - 980.

[18] E. Di Nardo, On the First Passage Time for Autoregressive Processes, *Scientiae Mathematicae Japonicae (Online)*, (2008), 61 - 76.