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On Expandability in Bitopological Spais

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Abstract

In this paper we define pairwise expandable spaces and study their properties and their relations with other bitopological spaces several exam - ples are discussed and many will known theorems are generalized concerning pairwise expandable spaces and we shall investigate subspaces of pairwise expandable space and also bitopological spaces which are related to pairwise expandability.

Keywords : Pairwise expandable, Pairwise–normal, Pairwise paracompact, Bitopological.

1. Introduction

In [3],Kelly introduced the notion of a bitopological space, i.e. a triple (X, τ_1, τ_2) where X is a non-empty set and τ_1, τ_2 are two topologies on X. He also defined pairwise regular ($P-$regular), pairwise normal ($P-$normal), and obtained generalization of several standard results such as Urysohn′s lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim in [4] and Fletcheret in [1] . Also Fletcher in [1] gave the definitions of $\tau_1\tau_2-$open and $P-$open covers in bitopological spaces. Katetove in [5] obtained a necessary and sufficient condition under which every locally-finite collection of closed subsets of a space X can be expanded to a locally-finite collection of open subsets of X . L. Krajewski in [6] called such a space to be expandable , and he defined this space as a topological space X is called m-expandable if for every locally finite $\tilde{F} = \{F_\alpha : \alpha \in \triangle\}$ of subset of X with $|\triangle| \leq m$, there exist a locally. finite collection of open subsets $\tilde{G} = \{G_\alpha : \alpha \in \Lambda\}$ such that $F_\alpha \subseteq G_\alpha$, for all $\alpha \in \triangle$, where m be an infinite cardinal. If X is m-expandable for each m , then X will be called expandable . Since if $\tilde{F} = \{F_\alpha : \alpha \in \triangle\}$ is locally finite collection , then $CL(\tilde{F}) = \{CL(F_\alpha) : \alpha \in \triangle\}$. is also locally finite then a sufficient condition for X to be expandable is to show that for every closed collection of locally finite subsets of X there exists an open locally finite expansion .

Definition 1.1. Let m be an infinite cardinal, then a bitopological space (X, τ_1, τ_2) is called $\tau_\imath - m-$ expandable space if for every $\tau_\imath-$locally finite $\tilde{F} = \{F_\alpha : \alpha \in \triangle\}$ of subsets of X , there exist τ_j-locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \triangle\}$ of open subsets of X such that $F_\alpha \subseteq G_\alpha$, for all $\alpha \in \triangle$ and for $i \neq \jmath, i, \jmath = 1, 2$, in this case X is said to be $\tau_\imath - m-$expandable space with respect to τ_j for every

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cardinal m and $i \neq \jmath, i, \jmath = 1, 2$.
A bitopological space (X, τ_1, τ_2) is called a pairwise expandable space , proved that it is pairwise T_2- space and it is τ_1- expandable with respect to τ_2 and τ_2- expandable with respect to τ_1 .

Example 1.2. Let X be an infinite set , then $X = (X, \tau_{dis}, \tau_{dis})$ is obviously pairwise expandable , since each $\tau_{dis}-$ locally finite collection $\tilde{F} = \{F_\alpha : \alpha \in \triangle\}$ is expandable to an $\tau_{dis}-$ open locally finite collection in the obvious manner $F_\alpha \subseteq F_\alpha$, for all $\alpha \in \triangle$.

Definition 1.3. Let $X = (X, \tau_1, \tau_2)$ be a bitopological space . Then a collection $\tilde{F} = \{F_\alpha : \alpha \in \triangle\}$ of subsets of X is called pairwise locally finite collection if for each $x \in X$ there exist an τ_1- open set U containing x such that U intersects finitely many members of $\tilde{F}$,and there exist τ_2- open V containing x such that V intersects finitely many members of $\tilde{F}$.

Definition 1.4. A pairwise open cover $\tilde{U}$ of the bitopological space (X, τ_1, τ_2) is called pairwise point finite if each $x \in X$ is contained in a finite number of τ_1- open members of $\tilde{U}$ or it is contained in a finite number of τ_2-open member of $\tilde{U}$.

Definition 1.5. [2] : A bitopological space X is called pairwise $m-$paracompact (respectively , pairwise $m-$metacompact) , if every open pairwise cover $\tilde{U}$ of X, such that $\left|\tilde{U}\right| \leq m$, has a pairwise locally finite (respectively , pairwise point finite) open refinement . If $m = \omega_0$, then a space X is called pairwise countably paracompact or pairwise countably metacompact respectively . If the space X is pairwise $m-$paracompact for every m , then X is called pairwise paracompact .

2. Characterizations of expandability in bitopological spaces

In this section we study the relationship of expandibility with other bitopological spaces, obtain some characterization of paracompactness , metacompactness and countably compactness involving expandibility .

Theorem 2.1. *If bitopological spaces $X = (X, \tau_1, \tau_2)$ is pairwise $m-$ paracompact then X is pairwise $m-$ expandable .*

Proof. Let $i \neq j$, $i, j = 1, 2$ and m be an infinite cardinal. Let $\tilde{F} = \{F_\alpha : \alpha \in \triangle\}$ be a locally finite of τ_i- closed subsets of a pairwise $m-$ paracompact space X. Let Γ be collection of all finite subsets of Δ and define $V_\gamma = X - \cup\{F_\alpha : \alpha \notin \gamma\}, \gamma \in \Gamma$, so V_γ is τ_i- open sets , $V_\gamma$meats only finitely many elements of $\tilde{F}$, $V_\gamma : \gamma \in \Gamma\}$ is τ_i- cover of X and $|\Gamma| \leq m$. Now , since X is pairwise $m-$paracompact space , then there is τ_j- open , τ_j-locally finite refinement say $\tilde{W} = \{W_s : S \in \Delta\}$. Set $U_\alpha = st(F_\alpha, W) = \{W_s \in W : W_s \cap F_\alpha \neq \phi\}, \alpha \in \triangle$.Thus , $F_\alpha \subseteq U_\alpha$ and U_α is τ_j- open subsets of X for each $\alpha \in \triangle$ .It is enough to show $\tilde{U} = \{U_\alpha : \alpha \in \triangle\}$ is locally finite collection of τ_j- subsets of X .Let $x \in X$ and O be a τ_j- open set containing x which is meets only finitely many members of $\tilde{W}$ .Thus $O \cap U_\alpha \neq \phi$ if and only if $O \cap W_s \neq \phi$ and $W_s \cap F_\alpha \neq \phi$

for some $S \in \Delta$. But W_s contained in some V_α , so W_s meat only finitely many F_α.Therefore $\tilde{U} = \{U_\alpha : \alpha \in \triangle\}$ is locally finite collection of τ_j- subsets of X.

Corollary 2.2. *If bitopological space $X = (X, \tau_1, \tau_2)$ is pairwise paracompact then X is pairwise expandable space .*

The proof follows immediately from theorem (2.1) the following example shows that the converse of the above corollary need not be true .

Example 2.3. Let w_1 denote the first uncountable ordinal and let $X = [0, w_1)$ with the usual order topology τ . Then $X = (X, \tau, \tau)$ is pairwise expandable as we shall show later, but it is not pairwise paracomact since the collection $\{[0, \alpha) : \alpha < w_1 \}$ is an τ_1- open cover of X which has no τ_2- open , τ_2- locally finite refinement .

Theorem 2.4. *: A bitopological space $X = (X, \tau_1, \tau_2)$ is pairwise w_o- expandable space if and only if it is pairwise countably paracompact .*

Proof. The right hand side of this theorem follows directly by corollary (2.2) and theorem (2.1) For the left hand side . Let $i \neq j$, $i, j = 1, 2$ and let $\tilde{R} = \{R_k : k \in IN\}$ be a τ_i- open , τ_i- countable cover of X . Put $S_k = \cup\{R_t : t = 1, 2, ..., k\}$ Let $A_1 = S_1$ and $A_k = S_k - S_{k-1}$ for each $k = 2, 3, ...$ .Clearly $A_k \subset R_k$ for each $k \in IN$. Since R be a τ_i- cover of X then for each $x \in X$ there is $k \in IN$ such that $x \in R_k$ .Let k_x is the smallest such k , then $x \in A_{k_x}$ and hence $A = \{A_k : k \in IN\}$ is τ_j-refinement of R .Since $R_k \cap A_m = \phi$ for $m > k, m \in IN$ then A is a τ_i-countable locally finite collection of subset of X , but X is pairwise w_0- expandable space , so there is τ_j- open , τ_j- locally finite collection $\tilde{G} = \{G_k : k \in IN\}$ of subsets of X such that $A_k \subset G_k$ for each $k \in IN$. Let $U_k = R_k \cap G_k, k \in IN$. Since A is a refinement of R , then for each $x \in X$ there is $k \in IN$ such that $x \in A_k$. But $A_k \subset R_k$ and $A_k \subset G_k$ so $x \in A_k \subset R_k \cap G_k = U_k \subseteq G_k$. Thus $U = \{U_k : k \in IN\}$ is a τ_j- open, τ_j-refinement of R. Moreover , if $N(x)$ be a neighborhood of $x \in X$ then $N(x) \cap G_k \neq \phi$ because $\tilde{G}$ is a τ_j- locally finite collection subsets of X,and since $U_k = G_k \cap R_k$ for all $k \in IN$.Thus $\tilde{U}$ is τ_j- locally finite collection of subsets of X and hence $\tilde{U}$ is a τ_j- open , τ_j- locally finite refinement of $\tilde{R}$. Thus $X = (X, \tau_1, \tau_2)$ is pairwise countably paracompact space.

Corollary 2.5. *An pairwise expandable space is pairwise countably paracompact. The proof follows immediately from theorem (2.4).*

Remark 2.6. It is clearthat a bitopological space $X = (X, \tau_1, \tau_2)$ will be expandable if it has the property that every τ_i- locally finite collection is finite for each $i = 1, 2$.

Thus we have the following theorems :

Theorem 2.7. *The following are equivalent for a bitopological space $X = (X, \tau_1, \tau_2)$*

(1) *X is pairwise countably compact*
(2) *Every τ_i- locally finite collection subsets of X is finite for each $i = 1, 2$*

(3) *Every τ_i- locally finite disjoint collection of subsets of X is finite , for each $i = 1, 2$.*
(4) *Every τ_i-locally finite countable collection of subsets of X is finite,for each $i = 1, 2$.*
(5) *Every τ_i- locally finite countable disjoint collection of subsets of X is finite , for each $i = 1, 2$.*

Proof. The only parts that need proof are (1) implies (2) and (5) implies (1) the others follow immediately. Now we shall prove that (1) implies (2) by contradiction. Let $X = (X, \tau_1, \tau_2)$ be a pairwise countably compact space and suppose that there exist a τ_i- locally finite collection $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X such that $|\Delta| = W_0$, in other words $\tilde{F} = \{F_k : k = 1, 2, ...\}$. Let $H_m = \bigcup_{k=m}^{\infty} \overline{F_k}$, for $m = 1, 2, ..$ then $H_{m+1} \subseteq H_m$ and H_m is τ_i- close subset of X for each $m = 1, 2, ..$, and for some $i = 1, 2$.
Since X is pairwise countably compact , then we must have $\bigcap_{m=1}^{\infty} H_m \neq \phi$, because the collection $\tilde{H} = \{H_k : k = 1, 2, ...\}$ is decreasing family of τ_i- close subsets of X for some $i = 1, 2$. But this contradicts the assumption ,because if there exist $x \in \bigcap_{m=1}^{\infty} H_m$ then x belongs to infinitely many members $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$. But $\tilde{F}$ is a τ_i- locally finite collection and by the fact says , every locally finite collection of τ_i-subsets of X is point finite , then $\tilde{F}$ is point finite . therefore , $F^{\sim}$ must be finite.
(5) implies (1). Suppose that every τ_i-locally finite countable disjoint collection of subsets of X is finite , for each $i = 1, 2$, we shall prove (1) by contradiction. Suppose there exist a τ_i- countable infinite subset A of X , and A has no τ_i- cluster point (i, e A is τ_i- close subset of X) , for some $i = 1, 2$. Since $A = \cup\{\{x\} : x \in A\}$ and this collection is τ_i- discrete , so by the fact says every τ_i-discrete collection of subset of X is τ_i- bounded locally finite collection. Therefore by assumption it should be finite and reach a contradiction.

Theorem 2.8. *A pairwise countably compact space is pairwise expandable .*

Proof. Follows from the remark (2.6) and theorem (2.7) part (1) .
The converse of theorem (2.8) need not be true by the following example.

Example 2.9. consider the bitopological space (IR, τ_s, τ_s) where τ_s denote the sorgenfrey topology, (i, e the real with the half-open interval topology) .
Since (IR, τ_s, τ_s) is pairwise paracompact space then by corollary (2.2) it is pairwise expandable space . But it is not pairwise countably compact because if so , then it would be pairwise compact and that would be contradiction.

Definition 2.10. A bitopological space $X = (X, \tau_1, \tau_2)$ is called pairwise semi-paracompact space, if every pairwise open cover $\tilde{U}$ of X has a pairwise $\sigma-$ locally finite pairwise open refinement i, e $\tilde{U}$ is a countable union a pairwise locally finite collections.

Theorem 2.11. *A bitopological space $X = (X, \tau_1, \tau_2)$ is pairwise paracompact if and only if X is pairwise w_0- expandable and pairwise semiparacompact .*

Proof. .The necessity of the condition is clear . We proceed to prove sufficiency. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a pairwise open cover of X and let $\tilde{V} = \bigcup_{i=1}^{\infty} V_i$ locally finite pairwise open refinement of $\tilde{U}$, where $V_i = \{\nu(\alpha, i) : \alpha \in \Delta\}$. Let $V_i^* = \cup\{\nu(\alpha, i) : \alpha \in \Delta_i\}$ and consider the collection $\{V_i^* : i = 1, 2, ...\}$ which is a pairwise countable open cover of X . Since X is pairwise w_0- expandable then by corollary (2.5) , X is pairwise countably paracompact . Therefore $\{V_i^* : i = 1, 2, ...\}$ has a pairwise locally finite pairwise open refinement, say $\tilde{G} = \{G_i : i = 1, 2, ...\}$, such that $G_i \subseteq V_i^*$ for each $i = 1, 2, ...$. Then $\{G_i \cap \nu(\alpha, i) : \alpha \in \Delta, i = 1, 2, ...\}$ is a pairwise locally finite pairwise open refinement of $\tilde{U}$. Hence X is pairwise paracompact .

Definition 2.12. A bitopological space $X = (X, \tau_1, \tau_2)$ is called pairwise $\theta-$refinable, if every pairwise open cover of X has a sequence of pairwise open refinements $\{U_i : i = 1, 2, ...\}$ satisfying the following :

(1) U_i is a pairwise open cover for each $i = 1, 2, 3, ...$.
(2) For each $x \in X$ there exist a positive integer n such that U_n is a pairwise point finite at X

Theorem 2.13. *A bitopological space $X = (X, \tau_1, \tau_2)$ is pairwise paracompact if and only if it is pairwise expandable and pairwise $\theta-$ refinable .*

Proof. We need only to prove sufficiency . suppose that X is pairwise expandable and pairwise $\theta-$ refinable . Let $\tilde{U} = \{u_\alpha : \alpha \in \Delta\}$ be any pairwis open cover of X , and let $\tilde{V} = \{V_i : i = 1, 2, ...\}$ be a sequence of pairwise open refinement satisfying (1) and (2) of definition (2.12) , where $V_i = \{V(\alpha, i) : \alpha \in \Delta\}$ for each $i = 1, 2, 3, ...$. We shall construct, for each i, a sequence $\{G(i, k) : k = 0, 1, ...\}$ of pairwise collections of open sets such that:

(1) $G(i, k)$ is a pairwise locally finite for each k
(2) Each element of $G(i, k)$ is a subset of some element of V_i .
(3) if a point $x \in X$ is an element of at most m elements of V_i , then $x \in \bigcup_{k=0}^{m} G^*(i, k)$.
(4) Each $x \in G^*(i, k)$ belongs to at least k elements of V_i , where $G^*(i, k) = \cup\{G \in G(i, k)\}$.

Let $G(i, 0) = \phi$. Suppose that $G(i, 1)$, $G(i, 2)$, ..., $G(i, n)$ are constructed to satisfy $(i) - (i\nu)$. To construct $G(i, n + 1)$, let $\tilde{B}$ be the family of all $B \subseteq \Delta_i$ such that B has exactly $n + 1$ elements .Define $Y(B) = (X - \bigcup_{j=0}^{n} G^*(i, j)) \cap (X - \cup\{V(\alpha, i) \in V_i : \alpha \notin B\})$. Clearly $Y(B)$ is pairwise closed $(i, e$ $Y(B)$ is τ_1-closed with respect to τ_2 and conversely) and we claim that $\tilde{F} = \{Y(B) : B \in \tilde{B}\}$ is pairwise locally finite .
Let $x \in X$. Then we have two cases .
case(1): x belongs to $n + 1$ or more element of V_i . choose $n + 1$ elements say

$\alpha(1), \alpha(2), .., \alpha(n+1)$. Then $\cap_{j=1}^{n+1} V(\alpha(j), i)$ is a neighborhood of X which meets $Y(B)$ only if $B = \{\alpha(1), \alpha(2), ..., \alpha(n+1)\}$, which implies that $\tilde{F}$ is a pairwise discrete collection .
case(2): x belongs to less than $n+1$ elements of V_i. By (3) , $x \in \cup_{j=0}^{n} G^*(i,j)$ which is disjoint from each $Y(B)$. Then $\tilde{F}$ is a pairwise locally finite collection of closed subsets of X . Since X is pairwise expandable then there is a τ_r- locally finite collection of open sets $\tilde{H} = \{H(B) : B \in \tilde{B}\}$, Such that $Y(B) \subseteq H(B)$, for each $B \in \tilde{B}$. and for each $r = 1, 2$. But $Y(B) \subseteq V(\alpha, i)$, for each $\alpha \in B$. Let $T(B) = H(B) \cap (\cap\{V(\alpha, i) : \alpha \in B\})$. Then $Y(B) \subseteq T(B)$ for each $B \in \tilde{B}$. Define $G(i, n+1) = \{T(B) : B \in \tilde{B}\}$. Then $G(i, n+1)$ satisfies (1), (2), (3)and (4) above For (1) follows since $T(B) \subseteq \cap\{V(\alpha, i) : \alpha \in B\}$. For (3) , let $x \in X$ be such that x belongs to no more than $(n+1)$ elements of V_i . if $x \in \bigcup_{j=0}^{n} G^*(i,j)$, then the result follows . If $x \notin \bigcup_{j=0}^{n} G^*(i,j)$, then $x \in Y(B)$ for some $B \in \tilde{B}$. Now set $\tilde{G} = \bigcup_{i=1}^{\infty} (\bigcup_{j=1}^{\infty} G^*(i,j))$. Since $\tilde{V}$ satisfies (1) of definition (2.12) , $\tilde{G}$ is an pairwise open cover which refines $\tilde{V}$ from (1) above we have $\tilde{G}$ is pairwise $\sigma-$ locally finite so by theorem (2.11) , we get X is pairwise paracompact space.

Definition 2.14. A bitopological space $X = (X, \tau_1, \tau_2)$ is called pairwise metacompact if every pairwise open cover of the space (X, τ_1, τ_2) has a pairwise point finite parallel refinement.

Theorem 2.15. *Let $X = (X, \tau_1, \tau_2)$ be a bitopological space , if X is pairwise metacompact then X is pairwise $\theta-$refinable .*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a pairwise open cover of X , since X is pairwise metacompact then $\tilde{U}$ has pairwise point finite open parallel refinement $\tilde{V}$. Then the sequence $\{V_i : i = 1, 2, 3....\}$ where $V_i = V$ for each i satisfies the to conditions (1) and (2) in definition(2.12) Therefore X is pairwise $\theta-$refinable.

Corollary 2.16. *A bitopological space $X = (X, \tau_1, \tau_2)$ is pairwise paracompact if and only if X is pairwise expandable and pairwise metacompact.*

Proof. The proof follows using theorem (2.13) and theorem (2.15) .

3. Subspaces of pairwise expandable space and their relation with some bitopological spaces

In this section ,we shall investigate subspaces of pairwise expandable space and also bitopological spaces which are related to pairwise expandability .

Definition 3.1. Let $X = (X, \tau_1, \tau_2)$ be a bitopological space , a subset A of a space X is called pairwise closed in X it is $\tau_1 -$ closed and $\tau_2 -$ closed . and it is called pairwise open if $X - A$ is pairwise closed in X .

Lemma 3.2. *If A is a pairwise closed subset of bitopological space $X = (X, \tau_1, \tau_2)$, then any pairwise locally finite collection of subsets of A is a pairwise locally finite collection in X .*

Proof. Let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a pairwise locally finite collection of subsets of A , we claim that $\tilde{F}$ is a pairwise locally finite collection in X . Let $x \in X$, then either $x \in A$ or $x \notin A$. If $x \in A$, then there is a $\tau_i -$ open set U in A and a $\tau_i -$ open set V in X such that $U = V \cap A$ for some $i = 1, 2$ contains x , and U intersects finitely many members of $\tilde{F}$, and hence V dose so . on other hand , if $x \notin A$, then $U = X - A$ is $\tau_i -$ open contains x for some $i = 1, 2$ and intersects no member of $\tilde{F}$. Hence $\tilde{F}$ is pairwise locally finite collection in X .

Theorem 3.3. *Let $X = (X, \tau_1, \tau_2)$ be a pairwise expandable space . If A is pairwise closed subsets of X , then A is pairwise expandable .*

Proof. Let A be a pairwise closed subsets of a pairwise expandable X , and let $i \neq j$, $i, j = 1, 2$ and $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i −locally finite collection of closed subsets of A. By Lemma (3.2) $\tilde{F}$ is a τ_i −locally finite collection of closed subsets in X , and since X is pairwise expandable , there exist $\tau_j -$ open , τ_j −locally finite collection in X , say $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$, such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. Hence A is pairwise expandable.

But in general , any subsets of a pairwise expandable space need not be pairwise expandable , even though it is a pairwise housdorff pairwise compact space , as illustrated in the following example .

Example 3.4. Let X be the space $[0, w_1] \times [0, w_0]$ where w_1 is the first uncountable ordinal and w_0 is the first countable ordinal . Let τ be the order topology defined on X , then a bitopological spaces $X = (X, \tau, \tau)$ is pairwise housdorff , pairwise compact and pairwise expandable , but the Tychonoff plank $A = X - \{(w_1, w_0)\}$ be a subspace of X which is not pairwise expandable , since it is not w_0- expandable .

Theorem 3.5. *A bitopological space X is hereditary pairwise expandable if and only if every pairwise open subset is pairwise expandable .*

Proof. Only one implication needs a proof since the other is obvious . Let $i \neq j$, $i, j = 1, 2$ and suppose that each pairwise open subset of X is pairwise expandable. Let $B \subseteq X$ and $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i −locally finite collection of subset B . Define , $V = \{x \in X : x$ belongs to an $\tau_i -$ open set which intersects only finitely many member of $\tilde{F}\}$. then V is $\tau_i -$ open and $B \subseteq V$. By assumption V is pairwise expandable and $\tilde{F}$ is a τ_i −locally finite collection in V , so there is a τ_j −locally finite collection of open subsets of V, say $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$, such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. Therefore , the collection $\{G_\alpha \cap B : \alpha \in \Delta\}$ is the desired collection .

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