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Modification of He's Variational Iteration Method and He's Homotopy Perturbation Method for Finding Approximate Solution of Nonlinear Fractional Integro-Differential Equations

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Abstract

Modification Variational iteration method and Homotopy perturbation method have been employed to obtain approximate solution nonlinear fractional integro-differential equations. Numerical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

Mathematics Subject Classification: 65K10; 45K05; 26A33

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1 Introduction

In recent few decades, the fractional integro-differential equations attracted attention of the scientific community because of its play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electro statistics [5].

There are many of techniques for the solution of fractional integro-differential equations, since it is relatively a new subject in mathematics, for example, Homotopy Perturbation Methods ([17],[19]), Variational iteration method ([18],[20]), Adomian decomposition method [13], Collection method [16], Legendre Wavelet method [19].

We will consider fractional order integro-differential equations of the form:

$$D^{\alpha} y(x) = \Phi(x) + \lambda \int_{0}^{x} k(x,t) F(y(t)) \tag{1.1}$$

and

$$D^{\alpha} y(x) = \Phi(x) + \lambda \int_{0}^{1} k(x,t) F(y(t)) \tag{1.2}$$

with the initial condition

$$y(0)=\beta\,, \quad n-1<\alpha\le n\,,\ n\in N \tag{1.3}$$

for $x,t\in[0,1]$, λ is a numerical parameter, where the function $\Phi(x)$, $k(x,t)$ are known and $y(x)$ is the unknown function, D^{α} is Caputo's fractional derivative and α is a parameter describing the order of the fractional derivative and $F(y(x)) = f(y(t))^{q}$, $q>1$, is a nonlinear continuous function.

The Homotopy perturbation method was established in 1998 by He ([7],[9-12]). The method is a powerful and efficient technique to find the solutions of nonlinear equations. The coupling of the perturbation and homotopy methods is called homotopy perturbation method. This method can take the advantages of the conventional perturbation method while eliminating its restrictions. In this method the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solutions.

The Variational iteration method was first proposed 1998 by He ([1-3], [6, 8], [14-15], [20]) and has found a wide application for the solution of linear and nonlinear differential equations, and was been worked out over a number of years by many authors. This method has been shown to effectively, easily and accurately solve a large class of nonlinear problems. Meanwhile, the Variational iteration method has been modified by many authors [1].

In this Paper, we will find approximate solution to the nonlinear fractional integro-differential equations by using modified of He's Variational Iteration Method and He's Homotopy Perturbation Methods. It will show these methods are a useful and simplify tools to solve nonlinear fractional integro-differential equations as used in other fields.

2 Preliminaries

In this section we present some basic definitions and properties of the fractional calculus theory, which are utilized in this paper [4, 19].

Definition 2.1 A real function $y(x), x > 0$, is said to be in the space C_μ , $\mu \in R$ if there exists a real number $p > \mu$, such that $y(x) = x^p y_1(x)$ where $y_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^k if $y^k \in R_\mu$, $k \in N$.

Definition 2.2 The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $y \in C_\mu$, $\mu \geq -1$ is defined as:

$$I^\alpha y(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\chi \dfrac{y(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0 \text{ , } t > 0 \\ y(t) \text{ , } \alpha = 0 \end{cases} \tag{2.1}$$

for β>0 and $\gamma > -1$, some properties of the operator I^α

- $I^\alpha I^\beta y(x) = I^{\alpha+\beta} y(x)$

- $I^{\alpha}I^{\beta}y(x)=I^{\beta}I^{\alpha}y(x)$
- $I^{\alpha}x^{y}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}x^{\gamma+\alpha}$

Definition 2.3 The Caputo fractional derivative of $y(x)\in C_{-1}^{k}$, $k\in N$ is defined as:

$$I^{\alpha}y(x)=\begin{cases}\frac{1}{\Gamma(k-\alpha)}\int_{0}^{\chi}\frac{y^{(k)}(t)}{(x-t)^{\alpha-k+1}}dt, & 0\le k-1<\alpha\le k\\ \frac{d^{k}y(x)}{dx^{k}} & ,\ \alpha=k\in N\end{cases} \tag{2.2}$$

for $\beta\ge 0$ and $\gamma\ge -1$, some properties of the operator D^{α}

- $D^{\alpha}D^{\beta}y(x)=D^{\alpha+\beta}y(x)$
- $D^{\alpha}D^{\beta}y(x)=D^{\beta}D^{\alpha}y(x)$
- $D^{\alpha}x^{y}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}x^{\gamma-\alpha} \quad , \quad \gamma\ge\alpha$

Lemma If $k-1<\alpha\le k$, $k\in N$, $y\in C_{\mu}^{k}$, $\mu>-1$ then the following two properties hold

- $D^{\alpha}I^{\alpha}y(x)=y(x)$
- $I^{\alpha}D^{\alpha}y(x)=y(x)-\sum_{k=0}^{n-1}y^{(k)}(0^{+})\frac{x^{k}}{k!}$

3 Analysis of the Modified Variational Iteration Method

For solving nonlinear fractional integro-differential equations withe initial conditions by constructing an initial trial-function without unknown parameters, we consider the following fractional functional equation

$$Ly+Ry+Ny=g(x) \tag{3.1}$$

where L is the fractional order derivative, R is a linear differential operator, and g is the source term. By using the inverse operator L_{x}^{-1} to both sides of (3.1), and using the given conditions, we obtain

$$y = f - L_x^{-1}[Ry] - L_x^{-1}[Ny] \tag{3.2}$$

where $L_x^{-1} = l^a$, and the function f represents the terms arising from integrating the source term *g* and from using the given conditions, all are assumed to be prescribed.

The basic character of He's method is the construction of a correction functional for (3.1), which reads

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(i)[Ly_n(i) + R\tilde{y}_n(i) + N\tilde{y}_n(i) - g(i)]di \tag{3.3}$$

Where λ is a Lagrange multiplier which can be identified optimally via variational theory [20], u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{y}_n = 0$: to solve (3.1) by He's VIM, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. Then the successive approximations $y_n(x); n \geq 0;$ of the solution y(x) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions, with determined λ; then several approximations $y_n(x); n \geq 0;$ follow immediately.

Consequently, the exact solution may be obtained by using

$$\lim_{n\to\infty} y_n(x) = y(x) \tag{3.4}$$

In summary, we have the following variational iteration formula for (3.2)

$$\begin{cases} y_0(x) \text{ is an arbitrary initial guess} \\ y_n(x) = y_n(x) + \int_0^x \lambda(i)[Ly_n(i) + R\tilde{y}_n(i) + N\tilde{y}_n(i) - g(i)]di \end{cases} \tag{3.5}$$

or equivalently, for (3.2), according to [6]:

$$\begin{cases} y_0(x) \text{ is an arbitrary initial guess} \\ y_n(x) = f(x) - L_x^{-1}[Ry_n] - L_x^{-1}[Ny_n] \end{cases} \tag{3.6}$$

where the multiplier Lagrange λ, has been identified.

It is important to note that He's VIM suggests that the y_0 usually defined by a suitable trial-function with some unknown parameters or any other function that satisfies at least the initial and boundary conditions. This assumption made by He ([2],[20]) and others will be slightly varied, as will be seen in the discussion.

4 Analysis of the Homotopy perturbation method

We consider the following nonlinear differential equation

$$A(y) - f(r) = 0 \ , \quad r \in \Omega \tag{4.1}$$

with boundary conditions

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0 \ , \quad r \in \Gamma \tag{4.2}$$

where A is a general differential operator, B is a boundary operator, y is a known analytical function, and Γ is the boundary of the domain Ω and A(y) is defined as follows:

$$A(y)=L(y)+N(y) \tag{4.3}$$

where L is linear, while N is nonlinear. Therefore (4.1) can be rewritten as follows

$$L(y)+N(y)-f(r)=0 \tag{4.4}$$

Homotopy-perturbation structure is shown as:

$$H(v,p) = (1-p)[L(v) - L(y_0)] + p[A(v) - f(r)] = 0 \tag{4.5}$$

where $r \in \Gamma$ and $p \in [0,1]$ is an embedding parameter, y_0 is an initial approximation of (4.1), which satisfies the boundary conditions. By (4.5), it easily follows that

$$H(v,0) = L(v) - L(y_0) = 0 \tag{4.6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{4.7}$$

and the changing process of p from zero to unity is just that of H(v,p) from $L(v) - L(y_0)$ to $A(v) - f(r)$. In topology, this is called deformation, $L(v) - L(y_0)$ and $A(v) - f(r)$ are called homotopic. The embedding parameter p is introduced much more naturally, unaffected by artificial factors. Furthermore, it can be considered as a small parameter for $0 < p \le 1$. By applying the perturbation technique used in [12], we assume that the solution of (4.5) can be expressed as:

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{4.8}$$

Therefore, the approximate solution of (4.1) can be readily obtained as follows:

$$y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{4.9}$$

Equation (4.9) is the solution of equation (1) obtained by Homotopy perturbation method.

5 Numerical examples

In this section, two examples are presented. The examples are nonlinear volterra integro-differential equations that using HPM and the results are compared with the exact solutions.

Example 5.1

Consider the following nonlinear fractional integro-differential equation:

$$D^{\frac{2}{3}} y(x) = \Phi(x) + \lambda \int_0^x xt\,(y(t))^3\,dt \tag{5.1.1}$$

where $\Phi(x) = \dfrac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{7}{12}\right) x^{\frac{5}{12}}} - \dfrac{4}{11} x^{\frac{15}{4}}$ with the initial condition y(0)=0, and exact solution $y(x) = \sqrt[4]{x}$.

The solution according to (MVIM)

$$D^{\frac{2}{3}}y(x)=\frac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{7}{12}\right)x^{\frac{5}{12}}}-\frac{4}{11}x^{\frac{15}{4}}+\int_0^x xt(y(t))^3\,dt \tag{5.1.2}$$

We take the operator $I^{\frac{2}{3}}$ on both sides of equation (5.1.1) we obtain:

$$D^{\frac{2}{3}}y(x)=y(0)+I^{\frac{2}{3}}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{7}{12}\right)x^{\frac{5}{12}}}-\frac{4}{11}x^{\frac{15}{4}}+\int_0^x xt(y(t))^3\,dt\right) \tag{5.1.3}$$

According to the original VIM (3.3) and corresponding the recursive scheme (3.5), we obtain:

$$f(x)=f(x_0)+f_1(x)=I^{\frac{2}{3}}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{7}{12}\right)x^{\frac{5}{12}}}-\frac{4}{11}x^{\frac{15}{4}}\right) \tag{5.1.4}$$

$$f(x)=\sqrt[4]{x}-0.1316930145x^{\frac{53}{12}} \tag{5.1.5}$$

by assuming

$f_0(x)=\sqrt[4]{x}$ and $f_1(x)=-0.1316930145x^{\frac{53}{12}}$

with starting of the initial approximation, $y_0(x)=f_0(x)=\sqrt[4]{x}$, we obtain,

$$y_1(x)=\sqrt[4]{x}-0.1316930145x^{\frac{53}{12}}+L_x^{-1}\left(y_0(x)\right) \tag{5.1.6}$$

$$y_1(x)=\sqrt[4]{x}-0.1316930145x^{\frac{53}{12}}+I^{\frac{2}{3}}\left(\int_0^x xt(\sqrt[4]{x})^3\,dt\right)=\sqrt[4]{x}$$

$$y_{n+1}(x)=\sqrt[4]{x}-0.1316930145x^{\frac{53}{12}}+L_x^{-1}\left(y_n(x)\right)=\sqrt[4]{x}\quad,\quad n\geq 1$$

in similarly view equation (5.1.6) it is obtained $y(x)=\sqrt[4]{x}$ where it is the exact solution of equation (5.1.1).

Now applying Homotopy perturbation method

$$D^{\frac{2}{3}}y(x)=\frac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{7}{12}\right)x^{\frac{5}{12}}}-\frac{4}{11}x^{\frac{15}{4}}+\int_0^x xt(y(t))^3\,dt$$

According to (1) we construct the following homotopy:

$$D^{\frac{2}{3}}y(x)=p\left(\Phi(x)+\int_0^x xt(y(t))^3\,dt\right) \tag{5.1.7}$$

$$p^0: D^{\frac{2}{3}}y_0(x)=0$$

$$p^1: D^{\frac{2}{3}}y_1(x)=\Phi(x)+\int_0^x xt(y_0(t))^3\,dt$$

$$p^2: D^{\frac{2}{3}}y_2(x)=\int_0^x xt[3y_0(t)^2y_1(t)]dt$$

$$p^3: D^{\frac{2}{3}}y_3(x)=\int_0^x xt[3y_0(t)^2y_2(t)+3y_0(t)y_1(t)^2]dt$$

$$p^4: D^{\frac{2}{3}}y_4(x)=\int_0^x xt[3y_0(t)^2y_3(t)+6y_0(t)+y_1(t)+y_2(t)+y_3(t)^2]dt, \ldots$$

by applying the operators $I^{\frac{2}{3}}$ to the above sets we obtain:

$$y_0(x)=0$$

$$y_1(x)=\sqrt[4]{x}-0.1316930145x^{\frac{51}{12}}$$

$$y_2(x)=0,\ y_3(x)=0,\ y_4(x)=0,\ \ldots$$

$y(x)=\sum_{i=0}^{\infty}y_i(x)$. Therefore the approximate solution of (5.1.1),

$$y(x)=\sqrt[4]{x}-0.1316930145x^{\frac{51}{12}}$$

Table 1: The error and numerical results of the example 5.1 by using MVIM and HPM $\alpha = \frac{2}{3}$

x	*Exact*	*Approximant by MVIM*	*Error by (MVIM)*	*Approximant by (HPM)*	*Error by (HPM)*
0.1	0.562341325	0.562341325	0	0.562336280	5.04541E-06
0.2	0.668740305	0.668740305	0	0.668632548	0.000107757
0.3	0.740082804	0.740082804	0	0.739436880	0.000645924
0.4	0.795270729	0.795270729	0	0.792969317	0.002301412
0.5	0.840896415	0.840896415	0	0.834730272	0.006166143
0.6	0.880111737	0.880111737	0	0.866316449	0.013795288
0.7	0.914691219	0.914691219	0	0.887438338	0.027252881
0.8	0.945741609	0.945741609	0	0.896589346	0.049152263
0.9	0.974003746	0.974003746	0	0.891311050	0.082692697
1	1	1	0	0.868306986	0.131693015

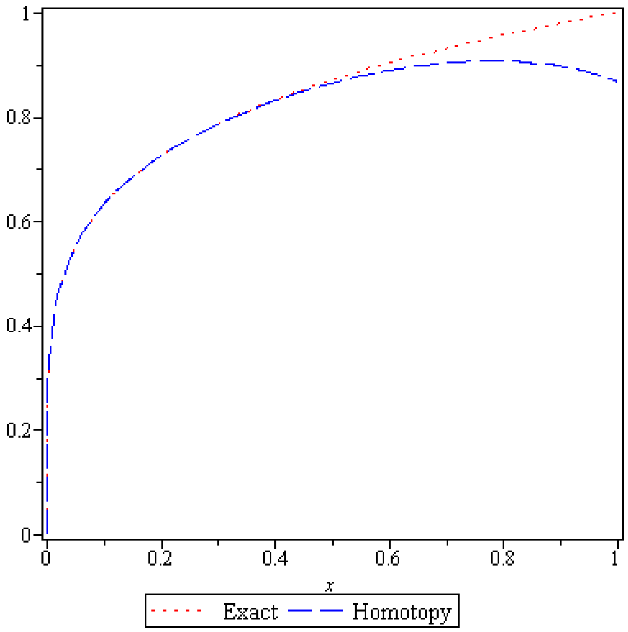

Figure 1: Comparison numerical results obtained by MVIM and HPM of example 5.1

Example 5.2

Consider the following nonlinear fractional integro-differential equation:

$$D^{0.8}y(x) = \Phi(x) + \lambda\int_0^1 (x-3t)(y(t))^2\,dt \tag{5.2.1}$$

where $\Phi(x) = \dfrac{125x^{\frac{11}{5}}\sin\left(\frac{\pi}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\pi} - \dfrac{36}{70}x + \dfrac{351}{400}$ with the initial condition $y(0) = 2$, $\lambda = \dfrac{1}{10}$, and exact solution $y(x) = 2 + x^3$.

The solution according to (MVIM)

$$D^{0.8}y(x) = \frac{125}{11}\,\frac{x^{\frac{11}{5}}\sin\left(\frac{\pi}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\pi} - \frac{36}{70}x + \frac{351}{400} + \int_0^1 (x-3t)(y(t))^2\,dt \tag{5.2.2}$$

We take the operator $I^{0.8}$ on both sides of equation (5.2.1) we obtain:

$$y(x) = 2 + I^{0.8}\left(2.475282775x^{\frac{11}{5}} + 0.2775 - 0.1142857143x + \int_0^1 (x-3t)(y(t))^2\,dt\right)$$

According to the original VIM (3.3) and corresponding the recursive scheme (3.5), we obtain:

$$f(x) = f_0(x) + f_1(x) = 2 + I^{0.8}\left(2.475282775x^{\frac{11}{5}} + 0.2775 - 0.1142857143x\right),$$

$$f(x) = 2 + x^3 + 0.2979437785x^{\frac{4}{5}} - 0.06816960471x^{\frac{9}{5}} \tag{5.2.3}$$

by assuming $f_0(x) = 2 + x^3$ and $f_1(x) = 0.2979437785x^{\frac{4}{5}} - 0.06816960471x^{\frac{9}{5}}$ with starting of the initial approximation, $y_0(x) = f_0(x) = 2 + x^3$, we obtain,

$$y_1(x) = 2 + x^3 + 0.2979437785x^{\frac{4}{5}} - 0.06816960471x^{\frac{9}{5}} + L_x^{-1}(f_0(x)) \tag{5.2.4}$$

$$y_1(x) = 2 + x^3 + 0.2979437785x^{\frac{4}{5}} - 0.06816960471x^{\frac{9}{5}} +$$
$$+ I^{0.8}\left(\int_0^x (x-3t)(2+t^3)^2\,dt\right) = 2 + x^3$$

$$y_2(x) = 0\ ,\ y_3(x) = 0\ ,\ y_4(x) = 0\ ,\ \ldots$$

$$y_{n+1}(x) = 2 + x^3 + 0.2979437785x^{\frac{4}{5}} - 0.06816960471x^{\frac{9}{5}} + L_x^{-1}(y_n(x)) =$$
$$= 2 + x^3 \quad , \quad n \ge 1$$

Then $y(x) = \sum_{i=0}^{\infty} y_i(x)$ in similarly view equation (5.2.4) it is obtained $y(x) = 2 + x^3$, where it is the exact solution of equation (5.2.1).

Now applying Homotopy perturbation method

$$D^{0.8}y(x) = \frac{125}{11}\frac{x^{\frac{11}{5}}\sin\left(\frac{\pi}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\pi} - \frac{36}{70}x + \frac{351}{400} + \frac{1}{10}\int_0^1 (x-3t)(2+t^3)^2\,dt \tag{5.2.5}$$

According to (2) we construct the following homotopy:

$$D^{0.8}y(x) = p\left(\Phi(x) + \frac{1}{10}\int_0^1 (x-3t)(2+t^3)^2\,dt\right) \tag{5.2.6}$$

$$p^0 : D^{0.8}y_0(x) = 0$$

$$p^1 : D^{0.8}y_1(x) = \Phi(x) + \frac{1}{10}\int_0^1 (x-3t)(y_0(t))^2\,dt =$$
$$= \frac{125}{11}\frac{x^{\frac{11}{5}}\sin\left(\frac{\pi}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\pi} - \frac{36}{70}x + \frac{351}{400} + \frac{4}{10}x - \frac{6}{10} =$$
$$= 2.475282775x^{\frac{11}{5}} + 0.2775 - 0.1142857143x$$

$$p^2 : D^{0.8}y_2(x) = \frac{1}{10}\int_0^x (x-3t)[2y_0(t)y_1(t)]dt$$
$$= 0.1564712113x - 0.3461629472$$

$$p^3 : D^{0.8}y_3(x) = \frac{1}{10}\int_0^x (x-3t)[2y_0(t)y_2(t) + y_1(t)^2]dt$$
$$= 0.06250756933 - 0.04252278025x$$

$$p^4 : D^{0.8}y_4(x) = \frac{1}{10}\int_0^x (x-3t)[2y_0(t)y_3(t) + 2y_1(t)y_2(t)]dt \quad , \ldots$$
$$= 0.02166583411 - 0.006819509148x$$

by applying the operators $I^{0.8}$ to the above sets we obtain: $y_0(x) = 2$,

$$y_1(x) = x^3 + 0.2979437785x^{\frac{4}{5}} - 0.0681696047x^{\frac{9}{5}}$$

$$y_2(x) = -0..3716652125x^{\frac{4}{5}} - 0.09333258181x^{\frac{9}{5}}$$
$$y_3(x) = 0.06711258159x^{\frac{4}{5}} - 0.02536415979x^{\frac{9}{5}}$$

$$y_4(x) = 0.02326198371x^{\frac{4}{5}} - 0.004067728375x^{\frac{9}{5}}$$

$y(x) = \sum_{i=0}^{\infty} y_i(x)$. Therefore the approximate solution of (5.2.1),

$$y(x) \cong 2 + x^3 + 0.01665313129x^{\frac{4}{5}} - 0.00426891106x^{\frac{9}{5}}$$

Table 2: The error and numerical results of the example 5.2 by using MVIM and HPM α= 0.8

x	*Exact*	*Approximant by MVIM*	*Error by (MVIM)*	*Approximant by (HPM)*	*Error by (HPM)*
0.1	2.001	2.001	0	2.003571686	0.0025717
0.2	2.008	2.008	0	2.012359766	0.0043598
0.3	2.027	2.027	0	2.032867327	0.0058673
0.4	2.064	2.064	0	2.071180594	0.0071806
0.5	2.125	2.125	0	2.133338790	0.0083388
0.6	2.216	2.216	0	2.225364552	0.0093646
0.7	2.343	2.343	0	2.353272702	0.0102727
0.8	2.512	2.512	0	2.523073743	0.0110737
0.9	2.729	2.729	0	2.740775540	0.0117755
1	3	3	0	3.012384220	0.0123842

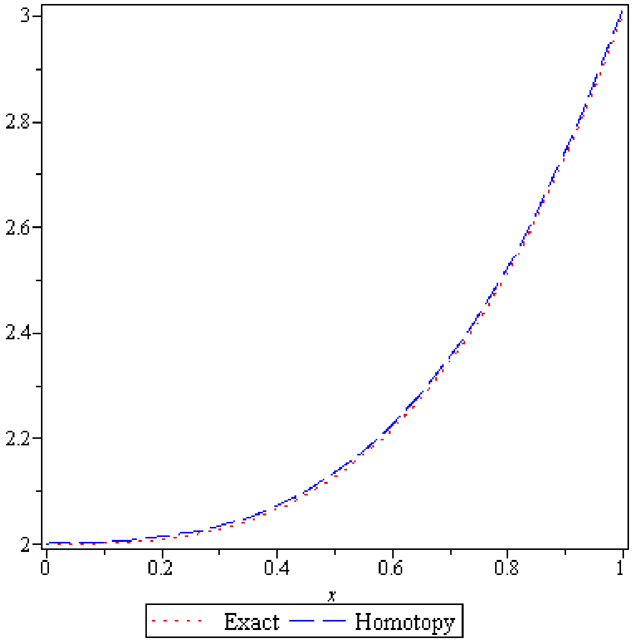

Figure 2: Comparison numerical results obtained by MVIM and HPM of example 5.2

6 Conclusions

In this work, we employed techniques MVIM and HPM to solve nonlinear fractional integro-differential equations successfully. The numerical results show that these methods have higher accuracy, good convergence with the exact solution and the results in MVIM are better than the results in HPM.

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