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Exponentially fitted collocation approximation method for the numerical solutions of Higher Order Linear Fredholm Integro-Differential-Difference Equations

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Abstract

This paper is concerned with the application of exponentially fitted collocation approximation method for the numerical solutions of Higher Order Linear Fredholm Integro-Differential-Difference Equations. Our approach entails substituting an assumed approximate solutions (Chebyshev and Legendre Polynomials as bases functions) into a slightly perturbed form of the given problem and then fitted the given mixed conditions with an exponential, having one free-tau parameter. Thus, the resulting equation is then collocated at equally spaced interior points of given intervals. Thus, resulted into algebraic linear system of equations which are combined with the exponentially fitted given mixed conditions. All together, these equations and then sovled using modification of MAPLE 13. The method is applied

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to a wider class of problems. The numerical results obtained for selected problems show that the approximate solution by Chebyshev Polynomials as basis function performed better than that of Legendre Polynomials as basis function in terms accurancy achieved, computational time and cost.

Keywords: Exponentially Fitted; Chebyshev and Legendre; bases function; Perturbed; accuracy

1 Introduction

The problem stated in equation (1) together with the mixed conditions given in equation (2) below have been recently considered by Abbas and Mehdi (2010) and solved equations (1) and (2) by Taylor Series.

$$\sum_{k=0}^{n} P_k(x)y^{(k)}(x)+\sum_{r=0}^{t} P_r^*(x)y^{(r)}(x-\tau)= f(x)+\int_a^b K(x,t)y(t-\tau)\,d\tau\,;\tau\geq 0 \qquad (1)$$

with the mixed conditions

$$\sum_{k=0}^{n-1}[a_{ik}\ y^{(k)}\ (a)+\beta_{ik}\ y^{(k)}\ (b)+\gamma_{ik}\ y^{(k)}\ (\eta)]=\mu_i\,;\quad i=0,1,...,s-1 \qquad (2)$$

In the method, Abbas and Mehdi, reported that the variable coefficients problems proved difficult to solve by Taylor Series especially when the variables are in the form of transcedentials or exponentials . Abbas and Mehdi (2010), used an approximate solutions by Taylor Series and transform the equation and the given mixed condition into matrix equation. By solving the system of algebraic equations, the Taylor’s coefficients of the solution function are obtained. Also, Taylor’s method has been extended to solve the Fredholm Integro-Differential-Difference Equation. The drawback of the Taylor Series method reported in Abbas and Mehdi (2010) is that the higher derivatives involved prove at times difficult to obtain. Our approach assumed an approximate solution in terms of Chebyshev Polynomials and Legendre Polynomials as the bases functions. The assumed approximate solution in terms of

Chebyshev Polynomials is substituted into a slightly perturbed given problem and exponential is fitted with one free tau-parameter to the mixed boundary conditions of the given problem. Thus, after simplification, the residual problem and the mixed boundary condition, lead to algebraic linear system of equations which are solved to obtain the unknown free tau-parameters and the unknown constants that appeared in the assumed solution.

2 Problem considered

In this work, the nth-order Linear Fredholm Integro-Differential-Difference Equation with variable coefficients given as

$$\sum_{k=0}^{n} P_k(x)y^{(k)}(x)+\sum_{r=0}^{t} P_r^*(x)y^{(r)}(x-\tau)= f(x)+\int_a^b K(x,t)y(t-\tau)\,d\tau\,;\tau\geq 0 \quad (3)$$

with the mixed conditions

$$\sum_{k=0}^{n-1}[a_{ik}\ y^{(k)}(a)+\beta_{ik}\ y^{(k)}(b)+\gamma_{ik}\ y^{(k)}(\eta)]=\mu_i\ ;\ \ i=0,1,...,s-1 \quad (4)$$

where, $P_k(x), P_r^*(x), k(x,t)$ and $f(x)$ are known smooth functions. Here, the real coefficients $\alpha_{ik}, \beta_{ik}, \gamma_{ik}, a, b, \eta$ and μ_i are given constants.

2.1 Chebyshev polynomials

The well known Chebyshev Polynomials $T_n(x)$ are defined in interval [-1,1] as

$$T_n(x)=Cos\{nCos^{-1}x\}; \quad (5)$$

and determined with the aid of the following recurrence formula.

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)\,;n=1,2,\ldots$$

Few terms of Chebyshev polynomials valid in [-1,1] are listed

$$T_0(x)=1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

…etc

The analytic form of the Chebyshev polynomials $T_n(x)$ of degree n is given by

$$T_n(x) = n\sum_{i=0}^{[\frac{n}{2}]} (-i)^i \, 2^{n-2i-1} \frac{(n-i-1)}{(i)!(n-2i)!} x^{n-2i},$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.

The orthogonality condition is

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & for \quad i = j = 0; \\ \frac{\pi}{2} & for \quad i = j \neq 0; \\ 0 & for \quad i \neq j. \end{cases}$$

2.2 Legendre polynomials

Legendre polynomials denoted by $P_n(x)$ in the domain [-1,1] is defined by

$$P_{n+1}(x) = \frac{1}{n+1}\{(2n+1)xP_n(x) - nP_{n-1}(x)\}$$

and

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n; \; n = 0,1,2,......$$

The first few terms of the Legendre polynomials are given below for $-1 \le x \le 1$

$$P_0(x) = 1.$$

$$P_1(x) = x.$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

…etc

3 Literature review

The theory of integro-differential-difference equtaions, the method used, and its wide applications have advanced beyond the adolescent stage to occupy a central position in applicable analysis. In fact, in the last 12 years, the proliferation of the subject has been witnessed by hundreds of research articles, several monographs, many works have been done by several autrhors in the theory of integro-differential-difference equations. Stefanini and Bede (2008) solved the above mentioned approach under strongly generalized differentiability of integro-differential-difference equations.In his case, the derivative exists and the solution of integro-differential-difference equations may have decreasing length of the support, but the uniqueness is lost. Chebyshev finite difference method [Dehghan and Saadatmandi (2008)], Legendre Tau method [Dehghan and Saadatmandi of equation (2010)] and Variational Iteration Method (VIM) [Biazar and Gholami Porshokouhi (2010), Bessel matrix method [Yuzbas et al (2011). Among them are Taiwo and Adebisi (2012), Taiwo and Alimi (2014) and Taiwo and Raji (2014). The authors reported above have used collocation approximation method by power series method and canonical polynomials approximate solutions. Homotopy Analysis Method (HAM) was also introduced by Liao (2014) to obtain series solutions of various linear and nonlinear problems of this type.

4 Description of exponentially fitted Tau-method

In this section, we discuss the exponentially fitted collocation tau-method for the solutions of Linear Fredholm Integro-Differential-Difference Equations. In this method, we assumed an approximate solution of the form

$$y_N(x)=\sum_{r=0}^{N}a_rL_r(x)\,; \quad a\le x\le b \tag{6}$$

where $a_r(r\ge 0)$ are unknown constants to be determined and $L_r(x)$ are the Legendre polynomials defined above. Thus, equation (3) is substituted into slightly perturbed equation (1) to obtain,

$$\begin{aligned}\sum_{r=0}^{N}\sum_{k=0}^{S}P_k(x)a_rL_r^{(k)}(x)+\sum_{r=1}^{N}\sum_{i=0}^{t}P_i^*(x)a_rL_r^{(i)}(x-\tau)\\ = f(x)+\int_a^b K(x,t)\left\{\sum_{r=0}^{N}a_rL_r(t-\tau)\right\}dt+H_N(x), \quad \tau\ge 0\end{aligned} \tag{7}$$

where $H_N(x)$ is the perturbation term given as

$$H_N(x)=\tau_2T_N(x)+\tau_3T_{N-1}(x)+\tau_4T_{N-2}(x)+...$$

and $\tau_2,\tau_3,\tau_4,...\tau_n$ are $n-1$ free tau parameters to be determined along with $a_n(n\ge 0)$.

We then fitted an exponential with one free tau-parameter into equation (2) to obtain

$$\sum_{r=0}^{N}\sum_{k=0}^{k-1}a_{ik}a_rL_r^k(a)+\beta_{ik}a_rL_r^k(b)+\gamma_{ik}`a_rL_r^k(\eta)+\tau_ne^{(a,b,\eta)}=\mu_i \tag{8}$$

$$i=0,1,...,n-1; \; a\le x\le b$$

Equation (4) is further simplified to obtain

$$\begin{aligned}&\left\{P_0(x)L_0^{(0)}(x)+P_0^*(x)L_0^{(0)}(x-\tau)-\int_a^b K(x,t)L_0(t-\tau)dt\right\}a_0\\&+\left\{P_1(x)L_1^{(1)}(x)+P_1^*(x)L_1^{(1)}(x-\tau)-\int_a^b K(x,t)L_1(t-\tau)dt\right\}a_1\\&+\left\{P_2(x)L_2^{(2)}(x)+P_2^*(x)L_2^{(2)}(x-\tau)-\int_a^b K(x,t)L_2(t-\tau)dt\right\}a_2\ +........+\dots\\&+\left\{P_n(x)L_N^{(n)}(x)+P_t^*(x)L_N^{(t)}(x-\tau)-\int_a^b K(x,t)L_N(t-\tau)dt\right\}a_N=f(x)+H_N(x)\end{aligned}\qquad(9)$$

According to Ortiz (1969), the numbers of $\tau_i\ (i\geq 1)$ introdurced are equivalent to the degrees or orders of the problems considered. In order to satisfy Ortiz (1969), the remaining τ -free parameter is then fixed to the mixed boandary conditions as shown in equation (5).

Hence, equation (6) is thus collocated at point $x=x_k$ to obtain

$$\begin{aligned}&\left\{P_0(x_k)L_0^{(0)}(x_k)+P_0^*(x_k)L_0^{(0)}(x_k-\tau)-\int_a^b K(x_k,t)L_0(t-\tau)dt\right\}a_0+\\&+\left\{P_1(x_k)L_1^{(1)}(x_k)+P_1^*(x_k)L_1^{(1)}(x_k-\tau)-\int_a^b K(x_k,t)L_1(t-\tau)dt\right\}a_1+\\&+\left\{P_2(x_k)L_2^{(2)}(x_k)+P_2^*(x_k)L_2^{(2)}(x_k-\tau)-\int_a^b K(x_k,t)L_2(t-\tau)dt\right\}a_2+\cdots+\\&+\left\{P_n(x_k)L_N^{(n)}(x_k)+P_t^*(x_k)L_N^{(t)}(x_k-\tau)-\int_a^b K(x_k,t)L_N(t-\tau)dt\right\}a_N-\\&-\sum_{i=1}^{N-1}\tau_i T_{N-i}(x_l)=f(x)\end{aligned}\qquad(10)$$

where for some obvious practical reason, we have chosen the collocation points to be

$$x_k=a+\frac{(b-a)k}{N+1};\quad k=1,2,...,N+1$$

Thus, we have $(N+1)$ collocation equations in $(N+n+1)$ unknowns $(a_0,a_1,\dots,a_{N-1})$ contants to be determined along with $n-1$ free-tau parameters. n extra equations are obtained by equation (5). Altogether, we have a total of $(N+n+1)$ algebraic linear system of equations in $(N+n+1)$ unknown constants. The $(N+n+1)$ linear algebraic system of equations are then solved by Gaussian elimination method to obtain the unknown constants $a_i (i \geq 0)$ which are then substituted back into the approximate solution in equation (3). Thus, equations (6) and (7) are then put in Matrix form of the form

$$A\underline{X} = \underline{B} \tag{11}$$

where,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2N} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \dots & a_{mN} \end{pmatrix}$$

$$\underline{X} = \left(a_0 \ a_1 \ a_2 \ a_3 \ \dots \ a_N \ \tau_1 \ \dots \ \tau_{N-1} \ \tau_N \right)^T$$

$$\underline{B} = \left(f(x_1) \ f(x_2) \ f(x_3) \dots \ f(x_{N+1}) \ \dots \ f(x_{N+S-1}) \right)^T$$

$$a_{11} = P_0 \ (x_1) \ L_0^{(0)} \ (x_1) + P_0^* \ (x_1) \ L_0^{(0)} \ (x_1 - \tau) - \int_a^b (x_1, t) \ L_0 \ (t - \tau) \ dt$$

$$a_{12} = P_1 \ (x_1) \ L_1^{(0)} \ (x_1) + P_1^* \ (x_1) \ L_1^{(0)} \ (x_1 - \tau) - \int_a^b (x_1, t) \ L_1 \ (t - \tau) \ dt$$

$$a_{13} = P_2 \ (x_1) \ L_2^{(0)} \ (x_1) + P_2^* \ (x_1) \ L_2^{(0)} \ (x_1 - \tau) - \int_a^b (x_1, t) \ L_2 \ (t - \tau) \ dt$$

$$\vdots$$

$$a_{1N} = P_S \ (x_1) \ L_N^{(S)} \ (x_1) + P_t^* \ (x_1) \ L_N^{(t)} \ (x_1 - \tau) - \int_a^b K(x_1, t) \ L_N \ (t - \tau) \ dt$$

$$a_{21} = P_0 \ (x_2) \ L_0^{(0)} \ (x_2) + P_0^* \ (x_2) \ L_0^{(0)} \ (x_2 - \tau) - \int_a^b (x_2, t) \ L_0 \ (t - \tau) \ dt$$

$$a_{22} = P_1 \ (x_2) \ L_1^{(0)} \ (x_2) + P_1^* \ (x_2) \ L_1^{(0)} \ (x_2 - \tau) - \int_a^b (x_2, t) \ L_1 (t - \tau) dt$$

$$a_{23} = P_2 (x_2) L_2^{(0)} (x_2) + P_2^* (x - 2) L_2^{(0)} (x_2 - \tau) - \int_a^b (x_2, t) L_2 \ (t - \tau) \ dt$$

$$\vdots$$

$$a_{2N} = P_S \ (x_2) \ L_N^{(S)} \ (x_2) + P_t^* \ (x_2) \ L_N^{(t)} \ (x_2 - \tau) - \int_a^b (x_2, t) \ L_N \ (t - \tau) \ dt \tag{12}$$

The matrix equation (11) is then solved by Gaussian Elimination Method to obtain the unknown constants, which are then substituted into the approximate solution given in equation (6).

5 Illustrative examples

Exponentially Fitted Collocation Tau-Method By Chebyshev Polynomials Function

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and MAPLE 13 package is used to carry-out the calculation. The absolute errors used is defined as

$$| y(x) - y_N(x) |; \quad a \le x \le b$$

Numerical Example 1

Consider the first-order linear Fredholm integro-differential-difference equation [Abbas and Mehdi (2010)]

$$y'(x) - y(x) + xy'(x-1) + y(x-1) = x - 2\int_{-1}^{1}(x+t)y(t-1)dt, \tag{13}$$

with the mixed condition

$$y(-1) - 2y(0) + y(1) = 0\,. \tag{14}$$

The exact solution of the problem is $y(x) = 3x + 4$.

Numerical Example 2

Consider the second order linear Fredholm integro-differential-difference equation

$$\begin{aligned}(x+4)^2 y''(x) - (x+4)y'(x) + y(x-1) - y'(x-1) = \\ = \ln(x+3) - \frac{1}{x+3} + 3\ln(3) - 5\ln(5) + \int_{-1}^{1} y(t)dt\end{aligned} \tag{15}$$

with conditions

$$y(0) = ln(4), \;\; y'(0) = \frac{1}{4}, \tag{16}$$

and the exact solution of this problem is $y(x) = \ln(x+4)$.

Numerical Example 3

In this example, we consider a third-order linear Fredholm integro-differential-difference equation with variable coefficients given as

$$\begin{aligned} &y'''(x) - xy'(x) + y''(x-1) - xy(x-1) = \\ &\quad = -(x+1)(sin(x-1) + cos(x)) - cos2 + \int_{-1}^{1} y(t-1)dt \end{aligned} \tag{17}$$

with conditions

$$y(0) = 0,\ \ y'(0) = 1,\ \ y''(0) = 0, \tag{18}$$

and the exact solution is $y(x) = sin(x)$.

6 Table of results

In this section, we tabulated the results obtained for various values of N and the exact solution when evaluated at equally spaced interior intervals of consideration.

Table 1: Absolute Error for Example 1 (Case N=6)

x	Taylor Series [9]	Abbas and Mehdi (2010) Shifted Legendre Tau Method	Exponentially Fitted by Chebyshev Polynomial	Exponentially Fitted by Legendre Polynomial
-1.0	1.14×10^{-2}	6.15×10^{-5}	1.281×10^{-7}	1.532×10^{-7}
-0.9	7.40×10^{-3}	1.72×10^{-5}	1.420×10^{-7}	1.700×10^{-7}
-0.8	4.50×10^{-3}	8.99×10^{-6}	1.540×10^{-7}	1.840×10^{-7}
-0.7	2.40×10^{-3}	2.00×10^{-5}	1.640×10^{-7}	1.980×10^{-7}
-0.6	1.00×10^{-3}	2.05×10^{-5}	1.760×10^{-7}	2.120×10^{-7}
-0.5	2.00×10^{-4}	1.56×10^{-5}	1.840×10^{-7}	2.240×10^{-7}
-0.4	1.00×10^{-4}	9.46×10^{-6}	1.930×10^{-7}	2.370×10^{-7}
-0.3	1.00×10^{-4}	4.64×10^{-6}	2.030×10^{-7}	2.530×10^{-7}
-0.2	2.00×10^{-4}	1.91×10^{-6}	2.150×10^{-7}	2.690×10^{-7}
-0.1	1.00×10^{-4}	5.87×10^{-7}	2.290×10^{-7}	5.530×10^{-7}
0.0	-	-	1.000×10^{-8}	9.000×10^{-9}
0.1	-	-	2.590×10^{-7}	3.320×10^{-7}
0.2	-	-	2.770×10^{-7}	3.580×10^{-7}
0.3	-	-	2.950×10^{-7}	3.840×10^{-7}
0.4	-	-	3.140×10^{-7}	4.130×10^{-7}
0.5	-	-	3.310×10^{-7}	4.400×10^{-7}
0.6	-	-	3.420×10^{-7}	4.630×10^{-7}
0.7	-	-	3.500×10^{-7}	4.830×10^{-7}
0.8	-	-	3.450×10^{-7}	4.920×10^{-7}
0.9	-	-	3.280×10^{-7}	4.900×10^{-7}
1.0	-	-	2.910×10^{-7}	4.690×10^{-7}

Table 2: Absolute Error for Example 1 (Case N=7)

x	Taylor Series [9]	Abbas and Mehdi (2010) Shifted Legendre Tau Method	Exponentially Fitted by Chebyshev Polynomial	Exponentially Fitted by Legendre Polynomial
-1.0	7.80×10^{-3}	2.81×10^{-5}	1.570×10^{-6}	1.571×10^{-6}
-0.9	5.10×10^{-3}	2.51×10^{-5}	1.545×10^{-6}	1.546×10^{-6}
-0.8	3.10×10^{-3}	2.01×10^{-5}	1.502×10^{-6}	1.503×10^{-6}
-0.7	1.60×10^{-3}	1.46×10^{-5}	1.441×10^{-6}	1.442×10^{-6}
-0.6	8.00×10^{-4}	9.74×10^{-6}	1.359×10^{-6}	1.360×10^{-6}
-0.5	1.00×10^{-4}	5.89×10^{-6}	1.269×10^{-6}	1.269×10^{-6}
-0.4	0	3.23×10^{-6}	1.164×10^{-6}	1.164×10^{-6}
-0.3	1.00×10^{-4}	1.57×10^{-6}	1.053×10^{-6}	1.053×10^{-6}
-0.2	1.00×10^{-4}	6.15×10^{-7}	5.380×10^{-7}	5.390×10^{-7}
-0.1	0	1.33×10^{-7}	1.310×10^{-7}	1.320×10^{-7}
0.0	0	0	0	0
0.1	-	-	6.590×10^{-7}	6.590×10^{-7}
0.2	-	-	6.090×10^{-7}	6.090×10^{-7}
0.3	-	-	5.970×10^{-7}	5.970×10^{-7}
0.4	-	-	6.300×10^{-7}	6.300×10^{-7}
0.5	-	-	7.170×10^{-7}	7.170×10^{-7}
0.6	-	-	8.670×10^{-7}	8.660×10^{-7}
0.7	-	-	1.085×10^{-6}	1.084×10^{-7}
0.8	-	-	1.378×10^{-6}	1.377×10^{-6}
0.9	-	-	1.747×10^{-6}	1.746×10^{-6}
1.0	-	-	2.192×10^{-6}	2.191×10^{-6}

Table 3: Example 2: Exponentially Fitted Collocation Tau Method for Case N=6

x	Exact solution y(x)= In(x+4)	Yalcinbas & Akkaya N=6 y(x)	EFCP y(x)	EFLP y(x)	CM [14] E(x)	Yalcinbas & Akkaya E(x)	EFCP E(x)	EFLP E(x)
0.1	1.410986974	*	1.41098648	1.410986609	*	*	4.9400×10^{-7}	3.6500×10^{-7}
0.2	1.435084525	*	1.43508436	1.43508311	*	*	1.6400×10^{-7}	1.4150×10^{-6}
0.3	1.458615023	*	1.45861431	1.458613935	*	*	7.1300×10^{-7}	1.0880×10^{-6}
0.4	1.481604541	*	1.481602451	1.481596943	*	*	2.0900×10^{-6}	7.5980×10^{-6}
0.5	1.504077397	*	1.504076239	1.504076972	*	*	1.1580×10^{-6}	4.2500×10^{-7}
0.6	1.526056303	*	1.526052969	1.526054699	*	*	3.3300×10^{-6}	1.6040×10^{-6}
0.7	1.547562509	*	1.547560885	1.547559869	*	*	1.6240×10^{-6}	2.0400×10^{-6}

-0.8	1.568615918	*	1.568610399	1.568608969	*	*	5.5190×10^{-6}	6.9500×10^{-6}
0.9	1.589235205	*	1.589201616	1.589226901	*	*	3.3589×10^{-5}	8.3040×10^{-6}
1.0	1.608432622	*	1.609432622	1.609436856	*	*	5.2900×10^{-6}	1.0560×10^{-6}
0	1.386294361	1.386294	1.386294361	1.386294361	1.0000×10^{-6}	0	0	0
-0.1	1.360976553	1.360976	1.36097699	1.386294361	1.0000×10^{-6}	6.9300×10^{-7}	4.3300×10^{-7}	3.7000×10^{-7}
-0.2	1.335001067	1.334999	1.33500037	1.386294361	0	2.0000×10^{-6}	7.0100×10^{-7}	2.5180×10^{-6}
-0.3	1.30833282	1.308329	1.308331912	1.386294361	2.0000×10^{-6}	3.9700×10^{-6}	9.0800×10^{-7}	6.5670×10^{-6}
-0.4	1.280933845	1.280928	1.280932325	1.386294361	1.0000×10^{-6}	5.8400×10^{-6}	1.5200×10^{-6}	3.4534×10^{-5}
-0.5	1.252762968	1.252755	1.252759895	1.386294361	2.0000×10^{-6}	7.7400×10^{-6}	2.4730×10^{-6}	2.0850×10^{-6}
-0.6	1.223775432	1.223766	1.223766428	1.386294361	1.0000×10^{-6}	9.6500×10^{-6}	9.0040×10^{-6}	4.9270×10^{-6}
-0.7	1.193922468	1.193911	1.193899045	1.386294361	1.0000×10^{-6}	1.1500×10^{-5}	2.3423×10^{-5}	1.0740×10^{-6}
-0.8	1.16315081	1.163138	1.163147113	1.386294361	1.0000×10^{-6}	1.3200×10^{-5}	3.6970×10^{-6}	2.9396×10^{-5}
-0.9	1.131402111	1.131387	1.131399901	1.386294361	1.0000×10^{-6}	1.5000×10^{-5}	2.2100×10^{-6}	2.8480×10^{-6}
-1.0	1.098612289	1.098596	1.098608995	1.386294361	2.0000×10^{-6}	1.6600×10^{-5}	2.3440×10^{-6}	1.2824×10^{-5}

Table 4: Example 2: Exponentially Fitted Collocation Tau Method for Case N= 7

x	Exact solution y(x)= In(x+4)	Yalcinbas & Akkaya N=7 y(x)	EFCP y(x)	EFLP y(x)	CM [14] E(x)	Yalcinbas & Akkaya E(x)	EFCP E(x)	EFLP E(x)
0.1	1.410986974	*	1.41098697	1.410986972	*	*	1.0000×10^{-9}	2.0000×10^{-9}
0.2	1.435084525	*	1.43508452	1.435084524	*	*	2.0000×10^{-9}	1.0000×10^{-9}
0.3	1.458615023	*	1.4586192	1.45861489	*	*	1.0300×10^{-7}	1.3300×10^{-7}
0.4	1.481604541	*	1.481604397	1.481603922	*	*	1.4400×10^{-7}	6.1900×10^{-7}
0.5	1.504077397	*	1.504075945	1.504076981	*	*	1.4520×10^{-6}	4.1600×10^{-7}
0.6	1.526056303	*	1.526056101	1.52605499	*	*	2.0200×10^{-7}	1.3130×10^{-6}
0.7	1.547562509	*	1.54755972	1.547560783	*	*	2.7890×10^{-6}	1.7260×10^{-6}
0.8	1.568615918	*	1.568612829	1.568614313	*	*	3.0890×10^{-6}	1.6050×10^{-6}
0.9	1.589235205	*	1.589229761	1.589224853	*	*	5.4440×10^{-6}	1.0352×10^{-5}
1	1.608432622	*	1.609429833	1.609429842	*	*	8.0790×10^{-6}	8.0700×10^{-6}
0	1.386294361	1.386294	1.386294361	1.38629436	1.0000×10^{-6}	0	0	1.0000×10^{-9}
-0.1	1.360976553	1.360976	1.360976553	1.360976552	1.0000×10^{-6}	0	0	1.0000×10^{-9}
-0.2	1.335001067	1.334999	1.335001067	1.335001066	$1{,}0000\times10^{-6}$	0	0	1.0000×10^{-9}
-0.3	1.30833282	1.308329	1.3083327	1.308332713	1.0000×10^{-6}	3.0000×10^{-6}	1.2000×10^{-7}	1.0700×10^{-7}
-0.4	1.280933845	1.280928	1.280933037	1.280936527	1.0000×10^{-6}	5.0000×10^{-6}	8.0800×10^{-7}	2.6770×10^{-6}

-0.5	1.252762968	1.252755	1.252759948	1.252761645	2.0000×10^{-6}	8.0000×10^{-6}	3.0200×10^{-6}	1.3230×10^{-6}
-0.6	1.223775432	1.223766	1.223775057	1.223788012	2.0000×10^{-6}	1.2000×10^{-5}	3.7500×10^{-7}	1.2580×10^{-5}
-0.7	1.193922468	1.193911	1.19393162	1.193933531	1.0000×10^{-6}	1.5000×10^{-5}	9.1520×10^{-6}	1.1063×10^{-5}
-0.8	1.16315081	1.163138	1.163146256	1.163139978	3.0000×10^{-6}	2.2000×10^{-5}	4.5540×10^{-6}	1.0832×10^{-5}
-0.9	1.131402111	1.131387	1.13141777	1.13138965	7.0000×10^{-6}	2.9000×10^{-5}	1.5659×10^{-5}	1.2461×10^{-5}
-1	1.098612289	1.098596	1.09860086	1.098596812	1.4000×10^{-5}	4.5000×10^{-5}	1.1429×10^{-5}	1.5477×10^{-5}

Note: (i) Exponentially Fitted by Chebyshev Polynomial (EFCP)

(ii) Exponentially Fitted by Ledendre Polynomial (EFLP)

Table 5: Absolute Error for Example 3 (Case N=6)

x	Taylor Series [9]	Abbas and Mehdi (2010) Shifted Legendre Tau Method	Exponentially Fitted by Chebyshev Polynomial	Exponentially Fitted by Legendre Polynomial
-1.0	8.58×10^{-2}	3.84×10^{-2}	3.5265431×10^{-3}	3.6030331×10^{-2}
-0.8	3.93×10^{-2}	1.82×10^{-2}	9.7375499×10^{-3}	1.6853572×10^{-2}
-0.6	1.50×10^{-2}	7.00×10^{-3}	2.3146560×10^{-3}	7.2657253×10^{-3}
-0.4	4.12×10^{-3}	1.86×10^{-3}	2.5336547×10^{-3}	1.5524880×10^{-4}
-0.2	4.85×10^{-4}	2.04×10^{-4}	3.3284120×10^{-4}	1.3904090×10^{-4}
0.0	0	0	0	0
0.2	4.59×10^{-4}	1.48×10^{-4}	3.3094820×10^{-4}	2.3330743×10^{-3}
0.4	3.69×10^{-3}	9.67×10^{-4}	5.8281370×10^{-4}	5.8196320×10^{-4}
0.6	1.28×10^{-3}	2.55×10^{-3}	5.4082206×10^{-3}	3.6824940×10^{-4}
0.8	3.17×10^{-2}	4.44×10^{-3}	4.6462731×10^{-3}	3.2965807×10^{-3}
1.0	5.57×10^{-2}	5.76×10^{-3}	8.5542322×10^{-3}	6.5802160×10^{-4}

Table 6: Absolute Error for Example 3 (Case N=7)

x	Taylor Series [9]	Abbas and Mehdi (2010) Shifted Legendre Tau Method	Exponentially Fitted by Chebyshev Polynomial	Exponentially Fitted by Legendre Polynomial
-1.0	6.03×10^{-2}	5.05×10^{-3}	2.0457968×10^{-3}	3.6488756×10^{-3}
-0.8	2.28×10^{-2}	2.38×10^{-3}	4.0804920×10^{-4}	1.6961653×10^{-3}
-0.6	6.63×10^{-2}	9.14×10^{-4}	1.0899390×10^{-4}	6.9223360×10^{-4}
-0.4	1.20×10^{-3}	2.42×10^{-4}	2.3088280×10^{-4}	2.7500900×10^{-4}
-0.2	6.90×10^{-5}	2.65×10^{-5}	2.3803000×10^{-5}	3.1655200×10^{-5}
0.0	0	0	0	0
0.2	5.30×10^{-5}	1.19×10^{-5}	1.5948700×10^{-5}	3.0681600×10^{-5}
0.4	8.09×10^{-4}	1.25×10^{-4}	4.9193000×10^{-6}	1.1229900×10^{-5}
0.6	3.82×10^{-3}	3.30×10^{-4}	6.9982600×10^{-5}	1.7484580×10^{-4}
0.8	1.14×10^{-2}	5.78×10^{-4}	4.3726770×10^{-4}	3.6931340×10^{-4}
1.0	2.73×10^{-2}	7.53×10^{-4}	4.8805230×10^{-4}	6.1479170×10^{-4}

7 Conclusion and discussion of results

In this work, we solved Higher Order Linear Fredholm Integro-Differential-Difference Equations by exponentially fitted using two bases functions namely, Chebyshev and Legendre Polynomials. The results obtained by the proposed method for the solution of Higher Order Linear Fredholm Integro-Differential-Difference Equations are compared with other existing works in literature such as Taylor Series Approach and Shifted Legendre Tau Methods reported by Abbas and Mehdi (2010).

We observed from the tables of results presented that using Chebyshev

Polynomial as the basis function gave better results when compared with that of Legendre Polynomial. We also observed that where the solutions were known in Abbas and Mehdi (2010), the results of the proposed method are in good agreements. The proposed method does not require rigorous calculation and computation cost in terms of execution of results when compared with the work of Yalcinbas and Akkaya (2012).

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