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Existence and some estimates of hypersurfaces of constant Gauss curvature with prescribed boundary

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Abstract

In [3], Guan and Spruck prove that if Γ in $\mathbb{R}^{n+1}$ ($n \geq 2$) bounds a suitable locally convex hypersurface Σ with Gauss curvature K_Σ, then Γ bounds a locally convex K-hypersurface whose Gauss curvature is less than $\inf K_\Sigma$. In this article we are particularly interested in K-hypersurfaces which are not global graphs and will extend several results in [3]. The first main result is to establish the estimate $K_M \geq (\operatorname{diam} M/2)^{-n}$ for the Gauss curvature K_M of a K-hypersurface M which satisfies **Condition A** below. The second main task is that, in case Σ above is not a global graph, we construct a K-hypersurface $\widetilde{M}$ whose Gauss curvature $K_{\widetilde{M}}$ is slighter greater than $\inf K_\Sigma$. If, in addition, the hypersurface Σ satisfies **Condition B** below, then for each number K, $0 < K \leq (\operatorname{diam}\Sigma/2)^{-n}$, we show that there exists a locally convex immersed hypersurface M_1 in $\mathbb{R}^{n+1}$ with $\partial M_1 = \Gamma$ and the Gauss curvature $K_{M_1} \equiv K$.

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1 Introduction

In the paper [3], Guan and Spruck are concerned with the problem of finding hypersurfaces of constant Gauss-Kronecker curvature (K-hypersurfaces) with prescribed boundary Γ in $\mathbb{R}^{n+1}$ ($n \geq 2$). They prove that if Γ bounds a suitable locally convex hypersurface Σ, then Γ bounds a locally convex K-hypersurface. Here a surface Σ in $\mathbb{R}^{n+1}$ is said to be locally convex if at every point $p \in \Sigma$ there exists a neighborhood which is the graph of a convex function $x_{n+1} = u(x)$, $x \in \mathbb{R}^n$, for a suitable coordinate system in $\mathbb{R}^{n+1}$, such that locally the region $x_{n+1} \geq u(x)$ always lies on a fixed side of Σ. More precisely, they proved:

Theorem 1.1. (cf. Theorem 1.1 in [3]) *Assume that there exists a locally convex immersed hypersurface Σ in $\mathbb{R}^{n+1}$ with $\partial\Sigma = \Gamma$ and the Gauss curvature K_Σ. Let $K_0 = \inf K_\Sigma$. Suppose, in addition, that, in a tubular neighborhood of its boundary Γ, Σ is C^2 and locally strictly convex. Then there exists a smooth (up to the boundary) locally strictly convex hypersurface M with $\partial M = \Gamma$ such that $K_M \equiv K_0$. Moreover, M is homeomorphic to Σ.*

Note that a locally convex hypersurface is necessarily of class $C^{0,1}$ in the interior. For a locally convex hypersurface Σ which is not C^2, we refer to [5] the definition of Gauss curvature in weak sense.

As noted in [3], Theorem 1.1 is a huge jump in generality from the previous results in, e.g., [3], for it deals with general immersed K-hypersurfaces and not just graphs. In this article we are particularly interested in K-hypersurfaces which are not global graphs. We will extend several results in [3]. The first main result is concerning an estimate for the Gauss curvature K_M of a K-hypersurface M, which satisfies **Condition A** below. We shall establish the estimate $K_M \geq (\operatorname{diam} M/2)^{-n}$ for such a K-hypersurface M. To introduce **Condition A**, let $\mathbf{p}_i$, $1 \leq i \leq k$, be the vertices of the hypersurface M. Let D_i be the maximal domain (i.e. the largest simply connected region) on M containing $\mathbf{p}_i$ which, as a hypersurface in $\mathbb{R}^{n+1}$, can be represented as the graph of a convex function u_i defined in a domain Ω_i, $1 \leq i \leq k$.

Condition A. There exists some number m, $1 \leq m \leq k$, such that the maximal domain D_m lies in the interior of M.

We shall establish the following theorem, which is an immediate consequence of the proof of Theorem 3.5 in [3].

Theorem 1.2. *Assume that M is a smooth locally strictly convex K-hypersurface and also fulfills* **Condition A**. *Then there holds*

$$K_M \geq (\operatorname{diam} M/2)^{-n}.$$

We may notice that this result does not hold for proper subsets of a hemisphere, which does not fulfill **Condition A**. Also notice that the graph of any function does not fulfill **Condition A**.

As a consequence of Theorem 1.2, we obtain:

Corollary 1.3. *Assume that M is a smooth locally strictly convex K-hypersurface and there holds*

$$K_M \leq (\operatorname{diam} M/2)^{-n},$$

then M does not satisfy **Condition A**; *that is, each maximal domain $\overline{D}_i$, $1 \leq i \leq k$, meets ∂M.*

The second main task of this paper is to prove that, if Σ satisfies the hypotheses in Theorem 1.1, and if we assume, in addition, that Σ cannot globally be represented as the graph of any function, then we are able to construct a K-hypersurface $\widetilde{M}$ whose Gauss curvature $K_{\widetilde{M}}$ is slighter greater than $\inf K_\Sigma$. In order to prove this, it suffices, in view of Theorem 1.1, to establish Proposition 1.4 below. To put precisely, we let $\widehat{\mathbf{p}}_\ell \in \Sigma$, $\ell = 1, 2, \cdots, \widehat{k}$, be those vertices where K_Σ achieves the minimum value, i.e. $K(\widehat{\mathbf{p}}_\ell) = \inf_\Sigma K$, $1 \leq \ell \leq \widehat{k}$. Also, we let $\widehat{D}_\ell$ be the maximal domain on Σ which, as a hypersurface in $\mathbb{R}^{n+1}$, can be represented as the graph of the convex function $\widehat{u}_\ell$ defined in the domain $\widehat{\Omega}_\ell$, $1 \leq \ell \leq \widehat{k}$.

Proposition 1.4. *Suppose the hypersurface Σ satisfies the hypotheses of Theorem 1. Assume Σ is not a global graph and and K_Σ is not constant inside $\widehat{D}_\ell$ for any ℓ, $1 \leq \ell \leq \widehat{k}$. Then there exists a locally convex immersed hypersurface Σ_1 in $\mathbb{R}^{n+1}$ with $\partial\Sigma_1 = \Gamma$ and Gauss curvature $K_{\Sigma_1} > \inf K_\Sigma$ everywhere. Moreover, in a tubular neighborhood of its boundary Γ, Σ_1 is C^2 and locally strictly convex.*

From Proposition 1.4 and Theorem 1.1 we obtain the following result.

Theorem 1.5. *Suppose the hypersurface* Σ *satisfies the hypotheses of Proposition 1.4. Then there exists a number* $K_1 > \inf K_\Sigma$ *such that, for each number* $0 < K < K_1$*, there exists a smooth (up to the boundary) locally strictly convex hypersurface* M *with* $\partial M = \Gamma$ *and* $K_M \equiv K$*; moreover,* M *is homeomorphic to* Σ*.*

We will further improve Theorem 1.1 in case Σ satisfies **Condition B** below. We introduce:

Condition B. For each ℓ, $1 \leq \ell \leq \widehat{k}$, the maximal domain $\widehat{D}_\ell$ lies in the interior of M.

We shall show the following.

Proposition 1.6. *If the hypersurface* Σ *satisfies the hypotheses in Proposition 1.4 and* **Condition B***, then there exists a locally convex immersed hypersurface* Σ_2 *in* $\mathbb{R}^{n+1}$ *with* $\partial \Sigma_2 = \Gamma$ *and* $\inf K_{\Sigma_2} > \min_{1 \leq \ell \leq \widehat{k}} (\mathrm{diam} \partial \widehat{D}_\ell / 2)^{-n}$*. Moreover, in a tubular neighborhood of its boundary* Γ*,* Σ_2 *is* C^2 *and locally strictly convex.*

From this and Theorem 1.1 we obtain:

Theorem 1.7. *Suppose the hypersurface* Σ *satisfies the hypotheses in Theorem 1 and* **Condition B***. Then for each number* K*,* $0 < K \leq (\mathrm{diam}\Sigma/2)^{-n}$*, there exists a locally convex immersed hypersurface* M_1 *in* $\mathbb{R}^{n+1}$ *with* $\partial M_1 = \Gamma$ *and the Gauss curvature* $K_{M_1} \equiv K$*. Moreover, in a tubular neighborhood of its boundary* Γ*,* M_1 *is* C^2 *and locally strictly convex.*

The key observation in proving Proposition 1.4 and Proposition 1.6 is that along $\partial \widehat{D}_\ell \setminus \Gamma$, the tangent hyperplane to Σ is vertical to the plane where $\widehat{\Omega}_\ell$ lies, and hence replacing $\widehat{D}_\ell$ by a graph "below" it while keeping $\Sigma \setminus \widehat{D}_\ell$ fixed we obtain another locally convex hypersurface.

2 Proofs of Theorems

2.1 Proof of Theorem 1.2

We may first observe:

Lemma 2.1. *If M is a compact K-surface without boundary, then there holds*

$$K_M \geq (\operatorname{diam} M/2)^{-n}.$$

Indeed, let $\mathbf{a}$ and $\mathbf{b}$ be the points on M with $d :=\operatorname{dist}(\mathbf{a}, \mathbf{b}) =\operatorname{diam} M$. Let $\mathbf{0}$ be the midpoint of the segment $\overline{\mathbf{ab}}$. Consider the ball $B := B_{d/2}(\mathbf{0})$ centered at $\mathbf{0}$ and of radius $d/2$, of which the segment $\overline{\mathbf{ab}}$ is a diameter. Then the sphere ∂B and the hypersurface M meet tangentially at the points $\mathbf{a}$ and $\mathbf{b}$. We treat two cases separetely.

Caes 1. M contacts ∂B from the inner side of $\overline{B}$ at $\mathbf{a}$ or $\mathbf{b}$; i.e. an open nighborhood of $\mathbf{a}$ or $\mathbf{b}$ on M lies in the inner side of $\overline{B}$. Therefore the Gauss curvature of M at $\mathbf{a}$ or $\mathbf{b}$ is greater than that of ∂B at $\mathbf{a}$ or $\mathbf{b}$, which is $(\operatorname{diam} M/2)^{-n}$.

Case 2. An open subset D_0 of M whose closure $\overline{D}_0$ contains $\mathbf{a}$ lies outside B. Since $d :=\operatorname{dist}(\mathbf{a}, \mathbf{b}) =\operatorname{diam} M$, we know that some nonempty open subset of M lies in the interior of B. Therefore D_0 is included in a region D_0^* whose boundary ∂D_0^* is an $(n-2)$-dimensional closed subset of ∂B without boundary. A part of the region D_0^* and a part of ∂B including $\mathbf{p}$ can be respectively represented as the graphs of u_0 and a function u over a domain Ω_0^* such that $u_0 = u$ along $\partial\Omega_0^*$ and $u_0 < u$ in Ω_0^*. Were the Gauss curvature of D_0^* less than that of ∂B, the maximum principle would imply that $u_0 > u$ in Ω_0^*, which would not be the case. Therefore over some point $q \in \Omega_0^*$ the Gauss curvature of D_0 at $(q, u_0(q))$ is greater than that of ∂B at $(q, u(q))$. Thus again we conclude that $K_M \geq (\operatorname{diam} M/2)^{-n}$.

This result will not be used in the rest of this article. However, the reasoning which leads to this result will be used in the proofs of Lemma 2.4 below, Proposition 1.4 in **2.2** and Proposition 1.7 in **2.3**.

Next we observe that the following result is essentially proved in the last paragraph of the proof of Theorem 3.5 in [3].

Proposition 2.2. *Assume that M is a smooth locally strictly convex K-hypersurface. Denote by $\kappa_{\max}[M]$ the maximum of all principal curvatures of M. If $\kappa_{\max}[M]$ is achieved at an interior point $\mathbf{p}$ of M, and we choose coordinates in $\mathbb{R}^{n+1}$ with origin at $\mathbf{p}$ such that the tangent hyperplane at $\mathbf{p}$ is given by $x_{n+1} = 0$ and M locally is written as a strictly convex graph $x_{n+1} = u(x_1, \cdots, x_n)$, then*

$$\kappa_{\max}(\mathbf{p}) \leq C_0 K, \tag{1}$$

with

$$C_0 = (x^0_{n+1})^{n-1}; \tag{2}$$

here $\mathbf{x}^0 = (x^0_1, \cdots, x^0_n, x^0_{n+1}) \in \mathbb{R}^{n+1}$ is so chosen that the function $\widehat{\rho} := |\mathbf{x} - \mathbf{x}^0|$, $\mathbf{x} \in M$, achieves its local maximum value at $\mathbf{p}$.

Indeed, in the last paragraph of the proof of Theorem 3.5 in [3], this estimate of $\kappa_{\max}$ is obtained at a local maximum point of the function κe^{ρ}, the maximum being taken for all the normal curvatures κ over M, where $\rho = |\mathbf{x} - \mathbf{x}^0|^2$, $\mathbf{x} \in M$ and $\mathbf{x}^0 \in \mathbb{R}^{n+1}$ is a fixed point. However, in order to obtain an estimate of $\kappa_{\max}(\mathbf{p})$, the point $\mathbf{x}^0$ has to be so chosen that the function $\widehat{\rho} = |\mathbf{x} - \mathbf{x}^0|$, $\mathbf{x} \in M$, achieves its local maximum value at $\mathbf{p}$. Using the argument in [3] we are able to derive

$$0 \geq 2n\Big(\frac{\kappa_{\max}(\mathbf{p})}{K}\Big)^{\frac{1}{n-1}} - 2nx^0_{n+1},$$

from which follows (1). We notice that, in the fourth and fifth lines from the bottom in page 295 of [3], we should append the number n before the parentheses.

We are now able to formulate the following.

Corollary 2.3. *Under the hypotheses of Proposition 1 on M and $\mathbf{p}$, we have*

$$K = K(\mathbf{p}) \geq C_0^{-n/(n-1)},$$

where C_0 is the constant introduced in (2).

Indeed, from Proposition 2.2, we have

$$K(\mathbf{p}) = \kappa_1 \kappa_2 \cdots \kappa_n \leq (C_0 K(\mathbf{p}))^n,$$

from which we obtain Corollary 2.3.

Instead of obtaining an estimate of the constant C_0, we make the following observation, from which and Corollary 2.3 we obtain Theorem 1.2.

Lemma 2.4. *Under the hypotheses of Proposition 2.2 on M and $\mathbf{p}$ and under* **Condition A** *with $\mathbf{p}_m = \mathbf{p}$, we have either*

$$C_0 \leq (\text{ diam M}/2)^{n-1}, \tag{3}$$

or

$$K_M \geq (\text{ diam M}/2)^{-n}. \tag{4}$$

Proof. As indicated in **Condition A**, $D_m \subset M$ is the maximal domain on M which can be represented as the graph of a convex function u_m defined in a domain Ω_m. Let P_m be the plane where Ω_m lies. We notice that the tangent hyperplane to M along ∂D_m is orthogonal to the plane P_m.

Let $\mathbf{a}$ and $\mathbf{b}$ be the points on ∂D_m such that $d_0 :=$dist$(\mathbf{a}, \mathbf{b})$=diam$\,\partial D_m$. Let $\mathbf{0}$ be the midpoint of the segment $\overline{\mathbf{ab}}$, $d_1 :=$ dist$(\mathbf{0}, \mathbf{p}_m)$ and $d := \max(d_1, d_0/2)$. Consider the ball $B := B_d(\mathbf{0})$ centered at $\mathbf{0}$ and of radius d. We treat two cases separately.

Case 1. If $d = d_0/2 \geq d_1$, then the segment $\overline{\mathbf{ab}}$ is a diameter of the ball B. Since the tangent hyperplane to M along ∂D_m is vertical to the plane P_m, we know that the sphere ∂B and the hypersurface M meet tangentially at the points $\mathbf{a}$ and $\mathbf{b}$. Since $d_0 =$(diam$\,\partial D_m)/2 \geq d_1 :=$dist$(\mathbf{0}, \mathbf{p}_m)$, an open subset of the boundary ∂D_m, together with the vertex $\mathbf{p}_m$, lies inside the ball $\overline{B}$. The reasoning leading to Lemma 2.1 can be applied here to conclude that one of the following holds:

(i) M contacts ∂B from the inner side of $\overline{B}$ at $\mathbf{a}$ or $\mathbf{b}$ and therefore (4) holds.

(ii) An open subset D_m^0 of D_m whose closure contains $\mathbf{a}$ lies outside B. Since $\mathbf{p}_m$ lies inside B, we know that D_m^0 is included in a region D_m^* whose boundary ∂D_m^* is an $(n-2)$-dimensional subset of ∂B without boundary. The reasoning in **Case 2** in the proof of Lemma 2.1 again enables us to conclude (4).

Case 2. If $d = d_1 \geq d_0/2$, then the sphere ∂B meets the hypersurface M tangentially at the point $\mathbf{p}_m$. We shall treat two possibilities separately.

(i) If the function $\widehat{\rho}_0 := |\mathbf{x} - \mathbf{0}|$, $\mathbf{x} \in M$, achieves its local maximum value at $\mathbf{p}_m$, we are allowed to take $\mathbf{x}^0 = \mathbf{0}$ in Proposition 2.2, and then, setting the origin of the coordinate system to be at $\mathbf{p}_m$, we obtain $|\mathbf{x}^0| = d_1$, from which (3) follows.

(ii) If the function $\widehat{\rho}_0 = |\mathbf{x}-\mathbf{0}|$, $\mathbf{x} \in M$, fails to take its local maximum value at $\mathbf{p}_m$, then, since M meets ∂B tangentially, an open subset $\widehat{D}'_m$ of D_m whose closure contains $\mathbf{p}_m$ lies outside B. However, since $d_1 \geq d_0/2$, we know that some open subset of ∂D_m lies in the interior of B. Therefore $\widehat{D}'_m$ is included in a region $\widehat{D}''_m$ whose boundary $\partial\widehat{D}''_m$ is an $(n-2)$-dimensional subset of ∂B without boundary. The reasoning in **Case 2** in the proof of Lemma 2.1 again yields (4). □

2.2 Proof of Proposition 1.4 and Theorem 1.5

We first recall the approach taken in [3]. Namely, according to [1], if Σ is the graph of a locally convex function $x_{n+1} = u(x)$ over a domain Ω in $\mathbb{R}^n$, then $K_\Sigma = K$ if and only if u is a viscosity solution of the Gauss curvature equation

$$\det(u_{ij}) = K(1 + |\nabla u|^2)^{\frac{n+2}{2}} \quad \text{in } \Omega. \tag{5}$$

A major difficulty in proving Theorem 1.1 lies in the lack of global coordinate systems to reduce the problem to solving certain boundary value problem for this Monge-Ampère type equation. To overcome the difficulty, Guan and Spruck [3] adopted a Perron method to deform Σ into a K-hypersurface by solving the Dirichlet problem for the equation (5) locally. They consider a disk on Σ which can be represented as the graph of a function and use the following existence result to replace such a disk by another graph of less curvature.

Lemma 2.5. (Theorem 1.1. [2], Theorem 2.1 [3]) *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{0,1}$. Suppose there exists a locally convex viscosity subsolution $\underline{u} \in C^{0,1}(\overline{\Omega})$ of (5), i.e.*

$$\det(\underline{u}_{ij}) \geq K(1 + |\nabla \underline{u}|^2)^{\frac{n+2}{2}} \quad \text{in } \Omega, \tag{6}$$

where $K \geq 0$ is a constant. Then there exists a unique locally convex viscosity solution $u \in C^{0,1}(\overline{\Omega})$ of (5) satisfying $u = \underline{u}$ on $\partial\Omega$.

Motivated by the approach taken in [3], we now proceed to establish Proposition 1.4. We consider a disk on Σ which can be represented as the graph of a function and contains a point at which the Gauss curvature takes the value $\inf K_\Sigma$ and then, instead of using Lemma 2.5, we shall replace such a disk by a graph whose Gauss curvature is everywhere greater than $\inf K_\Sigma$. Namely, as introduced before, we let $\widehat{\mathbf{p}}_\ell \in \Sigma$, $\ell = 1, 2, \cdots, \widehat{k}$, be those vertices where K_Σ achieves the minimum value, i.e. $K(\widehat{\mathbf{p}}_\ell) = \inf_\Sigma K$, $1 \le \ell \le \widehat{k}$, and let $\widehat{D}_\ell$ be the maximal domain on Σ which, as a hypersurface in $\mathbb{R}^{n+1}$, can be represented as the graph of a convex function $\widehat{u}_\ell$ defined in a domain $\widehat{\Omega}_\ell$, $1 \le \ell \le \widehat{k}$. Then the tangent hyperplane to M along $\partial\widehat{D}_\ell \setminus \Gamma$ is vertical to the plane P_ℓ.

For $1 \le \ell \le \widehat{k}$, let $\widehat{\Omega}_{\ell,\delta}$ be the tubular neighborhood with width δ along $\partial\widehat{\Omega}_\ell$, i.e.

$$\widehat{\Omega}_{\ell,\delta} = \{x \in \Omega_\ell : \operatorname{dist}(x, \partial\widehat{\Omega}_\ell) \le \delta\}.$$

We shall construct a convex function $\widetilde{u}_\ell$ defined over $\widehat{\Omega}_\ell$ with $\widetilde{u}_\ell = \widehat{u}_\ell$ along $\partial\widehat{\Omega}_\ell$ and $\widetilde{u}_\ell < \widehat{u}_\ell$ in $\widehat{\Omega}_{\ell,\delta} \setminus \partial\widehat{\Omega}_\ell$ for some $\delta > 0$. The graph of the function $\widetilde{u}_\ell$ over $\widehat{\Omega}_\ell$ is then a convex hypersurface $\widetilde{D}_\ell$. This naturally induces a $C^{0,1}$-diffeomorphism $\Psi_{\widetilde{\Sigma}} : \Sigma \to \widetilde{\Sigma} := \cup\widetilde{D}_\ell \cup (\Sigma \setminus \cup\widehat{D}_\ell)$ which is fixed on $\Sigma \setminus \cup\widehat{D}_\ell$. Since the tangent hyperplane to $\widetilde{D}_\ell$ along $\partial\widehat{D}_\ell \setminus \Gamma$ is vertical to the plane P_ℓ, $\widetilde{u}_\ell = \widehat{u}_\ell$ over $\partial\widehat{\Omega}_\ell$ and $\widetilde{u}_\ell < \widehat{u}_\ell$ in $\widehat{\Omega}_{\ell,\delta} \setminus \partial\widehat{\Omega}_\ell$, $1 \le \ell \le \widehat{k}$, we know that the hypersurface $\widetilde{\Sigma}$ is locally convex with $\partial\widetilde{\Sigma} = \partial\Sigma$.

In order to obtain the inequality $\inf K_{\widetilde{\Sigma}} > \inf K_\Sigma$, we choose the coordinate system with $\mathbf{p}_\ell = ((0, \cdots, 0), u_\ell(0, \cdots, 0))$, and then, letting $\widetilde{\mathbf{p}}_\ell = ((0, \cdots, 0), \widetilde{u}_\ell(0, \cdots, 0))$, we choose the function $\widetilde{u}_\ell$ to be strictly convex and to have $\inf K_{\widetilde{\Sigma}} = K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell) > K_{\mathbf{\Sigma}}(\mathbf{p}_\ell)$. For this, we observe that, since $K_\Sigma(\mathbf{p}_\ell) = \inf K_\Sigma < \sup_{\widehat{D}_\ell} K_\Sigma$, the equality in $\inf K_{\widetilde{\Sigma}} = K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell)$ can be achieved by choosing $v_\ell := \widehat{u}_\ell - \widetilde{u}_\ell$ defined over $\overline{\widehat{\Omega}_\ell}$ to be nonnegative and small enough. In order to obtain the strict convexity of $\widetilde{u}_\ell$, we make $v_\ell(x)$ strictly decreasing as the distance from x to $(0, \cdots, 0)$ increases. This also yields the inequality $K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell) > K_\Sigma(\mathbf{p}_\ell)$. Indeed, let $\mathbf{e}_{n+1}$ be the unit vector pointing in the direction of positive x_{n+1} axis and move the surface $\widetilde{D}_\ell$ in the direction of $\mathbf{e}_{n+1}$ and in the distance $v_\ell(0, \cdots, 0)$ to obtain the parallel surface $\widetilde{D}_\ell + v_\ell(0, \cdots, 0)\mathbf{e}_{n+1}$, which is the graph of the function $\widetilde{u}_\ell(x) + v_\ell(0, \cdots, 0)$ inside $\widehat{\Omega}_\ell$. Because v_ℓ achieves its maximum value at $(0, \cdots, 0)$, the surface $\widetilde{D}_\ell + v_\ell(0, \cdots, 0)\mathbf{e}_{n+1}$ meets the surface $\widehat{D}_\ell$ tangentially at $\mathbf{p}_\ell$ and $\widetilde{u}_\ell(x) + v_\ell(0, \cdots, 0) > u_\ell(x)$ inside $\widehat{\Omega}_\ell$. This yields $K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell) = K_{\widetilde{D}_\ell}(\widetilde{\mathbf{p}}_\ell) = K_{\widetilde{D}_\ell + v_\ell(0,\cdots,0)\mathbf{e}_{n+1}}(\mathbf{p}_\ell) > K_{\widehat{D}_\ell}(\mathbf{p}_\ell) = K_\Sigma(\mathbf{p}_\ell)$. We therefore obtain Proposition 1.4 by taking $\Sigma_1 = \cup\widetilde{D}_\ell \cup (\Sigma \setminus \cup\widehat{D}_\ell)$, from which follows Theorem 1.5.

2.3 Proof of Proposition 1.4 and Theorem 1.5

We now proceed to prove Proposition 1.4. It suffices to construct, for each ℓ, $1 \le \ell \le \widehat{k}$, a strictly convex hypersurface $\widetilde{D}_\ell$ with $\partial\widetilde{D}_\ell = \partial\widehat{D}_\ell$ and $\inf K_{\widetilde{D}_\ell} \ge (\mathrm{diam}\,\partial\widehat{D}_\ell/2)^{-n}$, for we can then take $\Sigma_2 = \cup\widetilde{D}_\ell \cup (\Sigma \setminus \cup\widehat{D}_\ell)$ to complete the proof of Proposition 1.4. For this purpose, we fix ℓ, $1 \le \ell \le \widehat{k}$. Let $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ be the points on $\partial\widehat{D}_\ell$ such that $d_\ell :=\mathrm{dist}\,(\mathbf{a}_\ell, \mathbf{b}_\ell) =\mathrm{diam}\,\partial\widehat{D}_\ell$. Let $\mathbf{0}_\ell$ be the midpoint of the segment $\overline{\mathbf{a}_\ell\mathbf{b}_\ell}$. Consider the ball $B_\ell := B_{d_\ell/2}(\mathbf{0}_\ell)$ centered at $\mathbf{0}_\ell$ and of radius $d_\ell/2$, of which the segment $\overline{\mathbf{a}_\ell\mathbf{b}_\ell}$ is a diameter. Since the tangent hyperplane to $\widehat{D}_\ell$ along $\partial\widehat{D}_\ell$ is vertical to the plane P_ℓ, the sphere ∂B_ℓ and the hypersurface $\widehat{D}_\ell$ meet tangentially at the points $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$. We claim

Lemma 2.6. *The whole $\partial\widehat{D}_\ell$ lies inside $\overline{B}_\ell$.*

Proof. It suffices to claim that each curve which is cut from $\partial\widehat{D}_\ell$ by a plane containing $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ lies in $\overline{B}_\ell$. Indeed, consider such a curve Γ_0. Since $d_\ell :=\mathrm{dist}\,(\mathbf{a}_\ell, \mathbf{b}_\ell) =\mathrm{diam}\,\partial\widehat{D}_\ell$, an open subset $\widetilde{\Gamma}_0$ of Γ_0 lies in B_ℓ. Suppose another open subset of Γ_0 does not lie in B_ℓ. We shall derive respective contradictions in two cases below and finish the proof.

Case i. Suppose the curvature of Γ_0 is increasing from $\mathbf{a}_\ell$ to a point $\mathbf{c} \in \Gamma_0$ and then decreasing from $\mathbf{c}$ to $\mathbf{b}_\ell$. Then near $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ the curvature of Γ_0 is less than $(\mathrm{diam}\,\partial\widehat{D}_\ell/2)^{-1}$, and hence this part of Γ_0 lies outside B_ℓ. Since $\widetilde{\Gamma}_0$ lies in B_ℓ, Γ_0 intersects ∂B_ℓ at points $\mathbf{c}_1$ and $\mathbf{c}_2$ such that $\mathbf{a}_\ell$ is nearer to $\mathbf{c}_1$ than $\mathbf{c}_2$. The maximum principle produces two points with curvature greater than $(\mathrm{diam}\,\partial\widehat{D}_\ell/2)^{-1}$ one of which is between $\mathbf{a}_\ell$ and $\mathbf{c}_1$, and the other is between $\mathbf{b}_\ell$ and $\mathbf{c}_2$. Therefore the part of Γ_0 between $\mathbf{c}_1$ and $\mathbf{c}_2$, which lies inside B_ℓ, has curvature greater than $(\mathrm{diam}\,\partial\widehat{D}_\ell/2)^{-1}$ everywhere, contradicting the maximum principle.

Case ii. Suppose the curvature of Γ_0 is decreasing from $\mathbf{a}_\ell$ to a point $\mathbf{c}_0 \in \Gamma_0$ and then increasing from $\mathbf{c}_0$ to $\mathbf{b}_\ell$. We first claim that in this case near $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ the curve Γ_0 lies inside $\overline{B}_\ell$ and the curvatures of Γ_0 at $\mathbf{a}_\ell$ and

$\mathbf{b}_\ell$ are greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$. Indeed, would a part of Γ_0 between $\mathbf{a}_\ell$ and some point $\mathbf{c}_3$ lie outside B_ℓ, then the maximum principle would produce a point with curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$ in this part of Γ_0. Therefore the curvature at $\mathbf{a}_\ell$ would be greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$, contradicting the assumption that near $\mathbf{a}_\ell$ the curve Γ_0 lies outside B_ℓ. Hence near $\mathbf{a}_\ell$ the curve Γ_0 lies inside $\overline{B}_\ell$ and hence the curvature of Γ_0 at $\mathbf{a}_\ell$ is greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$. The behavior of the curve Γ_0 near $\mathbf{b}_\ell$ can be understood analogously.

If Γ_0 intersects ∂B_ℓ at some points $\mathbf{c}_4$ other than $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$, then the part of Γ_0 between $\mathbf{c}_4$ and some other point $\mathbf{c}_5$ lies outside B_ℓ, which provides us with a point with curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$ by the maximum principle. This implies that the part of Γ_0 between $\mathbf{a}_\ell$ and $\mathbf{c}_4$, which lies inside B_ℓ, has curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$ everywhere, contradicting the maximum principle. □

To proceed further, we consider two cases separately.

Case I. The point $\mathbf{p}_\ell$ lies inside $\overline{B}_\ell$.

We proceed to prove the following.

Lemma 2.7. *In* **Case I**, *the whole* $\widehat{D}_\ell$ *lies in* $\overline{B}_\ell$.

Proof. Consider the plane $\widetilde{P}_\ell$ containing $\overline{\mathbf{a}_\ell\mathbf{b}_\ell}$ and the point $\mathbf{p}_\ell$. Let $\Gamma_\ell := \widetilde{P}_\ell \cap B_\ell$ and $\widehat{\Gamma}_\ell := \widetilde{P}_\ell \cap \widehat{D}_\ell$. We first observe that in Case I the curve $\widehat{\Gamma}_\ell$ in $\overline{D}_\ell$ lies inside $\overline{B}_\ell$; in other words, $\widehat{\Gamma}_\ell$ situates "above" Γ_ℓ. Indeed, would some part of $\widehat{\Gamma}_\ell$ lie outside $\overline{B}_\ell$, then we would, analogously to the proof of Lemma 2.6, derive respective contradictions in two cases.

From this observation, Lemma 2.6 and the assumption that $\mathbf{p}_\ell \in \overline{B}_\ell$, we conclude that each curve in $\widehat{D}_\ell$ which is cut by a plane containing $\overline{\mathbf{0}_\ell\mathbf{p}_\ell}$ lies inside $\overline{B}_\ell$. This enables us to conclude that the whole $\widehat{D}_\ell$ lies in $\overline{B}_\ell$. □

In view of Lemma 2.7, it is easy to construct a $C^{0,1}$ convex surface $D_{0,\ell}$ passing through Γ_ℓ as well as $\partial\widehat{D}_\ell$, which situates "below" $\widehat{D}_\ell$ and "above" ∂B_ℓ in the sense that $D_{0,\ell}$ and a portion of ∂B_ℓ can be represented respectively as the graphs of functions $u_{0,\ell}$ and v_ℓ in $\widehat{\Omega}_\ell$ such that $v_\ell \le u_{0,\ell} \le \widehat{u}_\ell$ in $\widehat{\Omega}_\ell$. We may replace $\widehat{D}_\ell$ by $D_{0,\ell}$ while fixing $\Sigma \setminus \widehat{D}_\ell$. This provides us with a $C^{0,1}$ hypersurface $\widetilde{\Sigma}_0$. Since the tangent hyperplane to Σ along $\partial\widehat{D}_\ell$ is vertical to the plane P_ℓ, the hypersurface $\widetilde{\Sigma}_0$ is locally strictly convex. By approximation, we may assume without loss of generality that $D_{0,\ell}$ is C^2.

Let $\mathbf{p}_{0,\ell}$ be the "lowest" point of $D_{0,\ell}$. Each curve on $D_{0,\ell}$ which is cut by a plane containing $\overline{\mathbf{0}_\ell \mathbf{p}_{0,\ell}}$ lies in $\overline{B}_\ell$ and hence has the curvature at $\mathbf{p}_{0,\ell}$ greater than or equal to $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$. Therefore the hypersurface $\widetilde{\Sigma}_0$ has the Gauss curvature $K_{\widetilde{\Sigma}_0}(\mathbf{p}_{0,\ell}) \geq (\operatorname{diam}(\partial\widehat{D}_\ell))^{-n}$.

We now consider two possibilities separately.

(i) If $K_\Sigma(\mathbf{x}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at each point $\mathbf{x} \in \partial\widehat{D}_\ell$, then by choosing $\widehat{u}_\ell - u_{0,\ell}$ small enough, there still holds $K_{D_{0,\ell}}(\mathbf{x}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at each point $\mathbf{x} \in \partial\widehat{D}_\ell$. Then, since there holds also $K_{D_{0,\ell}}(\mathbf{p}_{0,\ell}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ and $D_{0,\ell}$ is C^2, we have $K_{D_{0,\ell}}(\mathbf{x}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at every point $\mathbf{x} \in D_{0,\ell}$. Therefore in this case we take $\widetilde{D}_\ell = D_{0,\ell}$ to complete the proof of Proposition 1.4.

(ii) Suppose $K_\Sigma(\mathbf{x}) \leq (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at some points $\mathbf{x} \in \partial\widehat{D}_\ell$. Then we consider a small neighborhood of $\partial\widehat{D}_\ell$ on Σ

$$D_{\ell,\delta} = \{\mathbf{x} \in \Sigma; \operatorname{dist}(\mathbf{x}, \partial\widehat{D}_\ell) < \delta\}$$

and replace $D_{\ell,\delta}$ by a C^2 hypersurfacce $\widetilde{D}_{\ell,\delta}$ with $K_{\widetilde{D}_{\ell,\delta}} > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ everywhere and $\partial\widetilde{D}_{\ell,\delta} = \partial D_{\ell,\delta}$, while keeping $\Sigma \setminus D_{\ell,\delta}$ fixed. Let $\widetilde{D}^*_{\ell,\delta}$ be the largest region in $\widetilde{D}_{\ell,\delta}$ which can be represented as the graph of some function and has $\partial D_{\ell,\delta} \cap \widehat{D}_\ell$ as one component of its boundary. We then apply the previous construction to the hypersurface $\widetilde{D}^*_{\ell,\delta} \cup (\widehat{D}_\ell \setminus \widetilde{D}_{\ell,\delta})$, instead of $\widehat{D}_\ell$, and obtain the desired hypersurface $\widetilde{D}_\ell$ to complete the proof of Proposition 1.4.

Case II. The point $\mathbf{p}_\ell$ lies outside B_ℓ.

In this case, to prove Proposition 1.4 it suffices to prove the following lemma and then take $\widetilde{D}_\ell = \widehat{D}_\ell$.

Lemma 2.8. *In* **Case II**, *the Gauss curvature* $K_\Sigma(\mathbf{p}_\ell)$ *of* Σ *at* $\mathbf{p}_\ell$ *is greater than* $(\operatorname{diam}\partial D_\ell/2)^{-n}$ *at* $\mathbf{p}_\ell$.

Indeed, in this case we choose the coordinate system whose origin 0 is at $\mathbf{p}_\ell$ and whose x_{n+1}-axis points in the normal direction of D_ℓ from $\mathbf{p}_\ell$ to

∂B_ℓ. Then a portion of $\widehat{D}_\ell$ and a portion of ∂B_ℓ can be represented as the graphs of functions $\widetilde{u}$ and $\widetilde{v}$ respectively over a neighborhood E of 0. Consider the nonnegative function $w := \widetilde{v} - \widetilde{u}$ over E. In view of Lemma 2.6, the function w achieves its maximum value at 0. We now use the reasoning used at the last paragraph in the proof of Proposition 1.3. Namely, Let $\mathbf{e}_{n+1}$ be the unit vector in the direction of the x_{n+1}-axis. By moving the hypesurface $\widehat{D}_\ell$ in the direction of $\mathbf{e}_{n+1}$ and in the distance of $w(0)$, we obtain the parallel hypersurface $\widehat{D}_\ell + w(0)\mathbf{e}_{n+1}$, which meets ∂B_ℓ tangentially at $\mathbf{p}_{0,\ell}$ and has greater curvature than ∂B_ℓ at $\mathbf{p}_\ell + w(0)\mathbf{e}_{n+1}$. That is, $K_\Sigma(\mathbf{p}_\ell) = K_{\widehat{D}_\ell + w(0)\mathbf{e}_{n+1}}(\mathbf{p}_\ell + w(0)\mathbf{e}_{n+1}) > K_{\partial B_\ell}(\mathbf{p}_\ell + w(0)\mathbf{e}_{n+1})$. Since $K_{\widehat{D}_\ell}(\mathbf{p}_\ell) = \inf K_{\widehat{D}_\ell}$, we conclude that $K_\Sigma \geq (\operatorname{diam} \partial \widehat{D}_\ell / 2)^{-n}$. $\square$

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