Theoretical Mathematics & Applications, vol. 6, no. 2, 2016, 115-127
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2016

New inequalities of the real parts of the zeros of polynomials

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Abstract

In this paper, we find new inequalities of the real parts of the zeros of polynomials, and a new inequalities of the zeros and critical points of polynomials.

Mathematics Subject Classification: 15A18; 15A60; 26C10
Keywords: Real Parts; Zeros of Polynomials; critical points

1 Introduction

The problem of finding regions that contain some or all eigenvalues of matrices or zeros of polynomials has a long history. For example Cauchy gave an easily-calculated circular bound for complex coefficient polynomial zeros. Very recently research on this problem was based on Cauchy's work. For matrix eigenvalues, Gerschgorin theorem is very powerful tool for improving existing

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Article Info: *Received:* October 12, 2015. *Revised:* November 18, 2015.
Published online : April 5, 2016.

bounds or computing new ones. It can be used to find disks whose union contains all eigenvalues of a complex matrix. Such results may be applied to analyze the stability or the relative stability of discrete-time systems. They may also be applied to determine the eigenvalue distribution in certain disks for continuous-time systems, by shifting the origin of the s-plane. In recent years, numerous papers and comprehensive books have been published, for finding circular bounds of polynomial zeros which have real or complex coefficients.

2 Preliminary Notes

Definition 2.1 [2] *Let* f *be a polynomial of degree* $n \geq 3$, *with complex coefficients, and let* $z_1, z_2, ..., z_n$ *be the zeros of* f.

Let $D = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_{n-1} \end{bmatrix}$, *I , and J be the identity matrix of order (n-1) and the (n-1) × (n-1) matrix with all entries equal to 1, respectively. Then the (n-1) × (n-1) derivative companion matrix of* f *is given by*

$$C(f') = D\left(I - \frac{1}{n} J \right) + \frac{z_n}{n} J,$$

which is called a D-companion matrix of f.

Theorem 2.1 [3]. *If* $A, B, C \in M_n(\mathbb{C})$, *then*

$$\|ABC\|_p \leq \|A\| \|B\|_p \|C\|.$$

Lemma 2.1 [4].

If z any zero of $P(z) = z^n + a_n z^{n-1} + ... + a_2 z + a_1$, *then*

$$|z| \le \cos\frac{\pi}{n+1} + \frac{1}{2}\left(|a_n| + \sqrt{\sum_{j=1}^{n} |a_j|^2} \right).$$

Lemma 2.2 [5].

For $j = 1, 2, ..., n$, *we have*

$$\frac{1}{2}\left(-\operatorname{Re} a_n - \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2} \right) + \cos\frac{n\pi}{n+1} \le \operatorname{Re} z_j$$

$$\le \frac{1}{2}\left(-\operatorname{Re} a_n + \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2} \right) + \cos\frac{\pi}{n+1}$$

Lemma 2.3 [5]. *If the zeros of* $P(z) = z^n + a_n z^{n-1} + ... + a_2 z + a_1$ *are arranged in such a way that is* $\operatorname{Re} z_1 \ge \operatorname{Re} z_2 \ge ... \ge \operatorname{Re} z_n$, *then*

$$\sum_{j=1}^{k} \operatorname{Re} z_j \le \frac{1}{2}\left(-\operatorname{Re} a_n + \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} |a_j|^2} \right) + \frac{1}{2}\left(\frac{\sin\left(\frac{2k+1}{2n+2}\right)\pi}{\sin\frac{\pi}{2n+2}} \right) - \frac{1}{2}$$

for $k = 1, 2, ..., n-1$ *and*

$$\sum_{j=1}^{n} \operatorname{Re} z_j = \operatorname{Re} a_n .$$

Lemma 2.4 [7].

Let $A, B \in M_n(\mathbb{C})$ *be a conjugate normal matrices. Then*

$$\left(\operatorname{Re}\overrightarrow{\left(\mu(A)\right)}\right) + \left(\operatorname{Re}\overrightarrow{\left(\mu(B)\right)}\right) \succ_w \operatorname{Re}\left(\mu\left(A+B\right)\right).$$

Lemma 2.5 [2].

Let $A = [\mathrm{a}_{ij}] \in M_n(\mathbb{C})$. *if* $0 < p \le 2$, *then* $\sum_{j=1}^{n} s_j^p(\mathrm{A}) \le \sum_{i,j=1}^{n} |a_{ij}|^p$.

Lemma 2.6 [6].

Let $A = [\mathrm{a}_{ij}] \in M_n(\mathbb{C})$. *Then for* $p > 0$, $\sum_{j=1}^{n} |\lambda_j(\mathrm{A})|^p \le \sum_{j=1}^{n} s_j^p(\mathrm{A})$.

Lemma 2.7 [1].

Let $z_1, z_2, \ldots, z_n$ *be the zeros of polynomials f of degree* $n \geq 3$ *and* $w_1, w_2, \ldots, w_{n-1}$ *be the critical points of f. Then for* $p \geq 1$, *we have*

$$\left(\sum_{i=1}^{n-1} |w_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n-1} |z_i|^p\right)^{\frac{1}{p}} + \left(\frac{n-1}{n}\right)|z_n|, \tag{1}$$

$$\left(\sum_{i=1}^{n-1} |w_i|^p\right)^{\frac{1}{p}} \leq \left((n-2) + \frac{1}{n^p}\right)^{\frac{1}{p}} \max_{1\leq i\leq n-1}\{|z_i|\} + \left(\frac{n-1}{n}\right)|z_n|, \tag{2}$$

$$\left(\sum_{i=1}^{n-1} |w_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n-1} |z_i|^p\right)^{\frac{1}{p}} + \sqrt{\frac{n-1}{n^2}\sum_{i=1}^{n-1} |z_n - z_i|^2}. \tag{3}$$

3 Main Results

Theorem 3.1 *If z is any zero of* $p(z) = z^n + r\,z^{n-1} + \ldots. + r\,z + r$, *for any scaler r. Then*

$$|z| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(|r| + \sqrt{n\,r^2}\right)$$

Proof.

The companion matrix of p(z) is $C(p) = \begin{bmatrix} -r & -r & -r & \cdots & -r \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

By Lemma 1 we have $|z| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(|a_n| + \sqrt{\sum_{j=1}^{n} |a_j{}^2|}\right)$, and hence

$$|z| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left(|r| + \sqrt{n\,r^2}\right).$$ □

Example 2.1

If $p(z) = z^4 + r\,z^3 + r\,z^2 + r\,z + r$, then $|z| \le \cos\frac{\pi}{5} + \frac{3}{2}|r|$.

Theorem 3.2 *If z is any zero of* $p(z) = z^n + z^{n-1} + ... + z + 1$. *Then*

$$\frac{1}{2}\left(-1-\sqrt{1+(n-1)}\right)+\cos\frac{n\pi}{n+1} \le \operatorname{Re} z_j \le \frac{1}{2}\left(-1+\sqrt{1+(n-1)}\right)+\cos\frac{\pi}{n+1}$$

Proof.

$$\operatorname{Re} C(\mathrm{p}) = \begin{bmatrix} -1/2 & 0 & -1/2 & \cdots & -1/2 \\ 0 & 0 & 1/2 & \cdots & 0 \\ -1/2 & 1/2 & 0 & \cdots \ddots & 1/2 \\ \vdots & \vdots & \vdots\ddots & \ddots & 1/2 \\ -1/2 & 0 & 0\cdots & 1/2 & 0 \end{bmatrix}.$$

Thus, $\operatorname{Re} C(\mathrm{p}) = \mathrm{S}_n + T_n$, where S_n is partitioned matrix

$$S_n = \begin{bmatrix} -1 & x^* \\ x & 0 \end{bmatrix}$$

with $x = \left[-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}\right]^t$, and T_n is the n× n tridiagonal matrix

$$T_n = \begin{bmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 \\ & & & \ddots & \vdots \\ \vdots & \vdots & \vdots\ddots & \ddots & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}.$$

The eigenvalue of S_n are

$$\lambda_1(\mathrm{S}_n) = \frac{1}{2}\left(-1+\sqrt{1+(n-1)}\right),$$

$$\lambda_n(\mathrm{S}_n) = \frac{1}{2}\left(-1-\sqrt{1+(n-1)}\right),$$

and

$$\lambda_j(\mathrm{S}_n) = 0 \quad \text{for} \quad j = 1, 2, ..., n-1.$$

It is well known that the eigenvalue of T_n are

$$\lambda_j(\mathrm{T}_n) = \cos\frac{j\pi}{n+1} \quad \text{for} \quad j = 1, 2, ..., n.$$

and hence by Lemma 2 we have,

$$\frac{1}{2}\left(-1-\sqrt{1+(n-1)}\right)+\cos\frac{n\pi}{n+1} \le \operatorname{Re} z_j \le \frac{1}{2}\left(-1+\sqrt{1+(n-1)}\right)+\cos\frac{\pi}{n+1}$$

for $j = 1, 2, ..., n$. □

Example 3.2

If $p(z) = z^4 + z^3 + z^2 + z + 1$,then $-\frac{3}{2}+\cos\frac{4\pi}{5} \le \operatorname{Re} z_j \le \frac{1}{2}+\cos\frac{\pi}{5}$.

Theorem 3.3 *If* $A = diag(r, r, ..., r)$. *Then* $\sum_{j=1}^{n} s_j^2(A) \le n r^2$.

Proof . By Lemma 5, we have

$$\sum_{j=1}^{n} s_j^p(\mathrm{A}) \le \sum_{i,j=1}^{n} \left| a_{ij} \right|^p$$

Put $\left|a_{ij}\right| = |r|$, for i, j=1,2,…,n and p=2, we have $\sum_{j=1}^{n} s_j^2(A) \le n r^2$. □

Theorem 3.4 *Let* $A = \left[a_{ij} \right] \in M_n(\mathbb{C})$, *have two eigenvalues. Then*

$$\left|\lambda_1\right|^2 + \left|\lambda_2\right|^2 \le (\mathrm{s}_1 + \mathrm{s}_2)^2.$$

Proof. By Lemma 6, we have

$$\sum_{j=1}^{n} \left|\lambda_j(\mathrm{A})\right|^p \le \sum_{j=1}^{n} s_j^p(A).$$

By taking p=2, we have the result.

Theorem 3.5 *Let* $A=\left[a_{ij}\right]\in M_n(\mathbb{C}),$ *if* $0< p\le 2.$ *Then*

$$\sum_{j=1}^{n}\left|\lambda_j(\mathrm{A})\right|^p \le \sum_{i,j=1}^{n}\left|a_{ij}\right|^p .$$

Proof. By Lemma 5, we have

$$\sum_{j=1}^{n} s_j^p(\mathrm{A}) \le \sum_{i,j=1}^{n}\left| a_{ij}\right|^p .$$

and by Lemma 6, we have

$$\sum_{j=1}^{n}\left|\lambda_j(\mathrm{A})\right|^p \le \sum_{j=1}^{n} s_j^p(A).$$

Hence, the result is hold

$$\sum_{j=1}^{n}\left|\lambda_j(\mathrm{A})\right|^p \le \sum_{i,j=1}^{n}\left|a_{ij}\right|^p . \qquad \square$$

Corollary 2.1

Let $A=\left[a_{ij}\right]\in M_n(\mathbb{C}).$ *Then* $\sum_{j=1}^{n}\left|\lambda_j(\mathrm{A})\right|^2 \le \sum_{i,j=1}^{n}\left|a_{ij}\right|^2 .$

Proof . By Theorem 2.5, we have

$$\sum_{j=1}^{n}\left|\lambda_j(\mathrm{A})\right|^p \le \sum_{i,j=1}^{n}\left|a_{ij}\right|^p$$

By taking p=2 in Theorem 5, we get the result. $\square$

Theorem 3.6 *If the zeros of* $P(\mathrm{z})=\mathrm{z}^n+r\,\mathrm{z}^{n-1}+\ldots+r\,\mathrm{z}+r$ *for* $r=x+iy$ *is any complex number are arranged in such a way that is* $\operatorname{Re} z_1\le \operatorname{Re} z_2\le \ldots\le \operatorname{Re} z_n$, *then*

$$\sum_{j=1}^{n}\operatorname{Re} z_j \le \frac{1}{2}\left(-x+\sqrt{x^2+(n-1)(x^2+y^2)}\right)+\frac{1}{2}\left(\frac{\sin\left(\frac{2k+1}{2n+2}\right)\pi}{\sin\frac{\pi}{2n+2}}\right)-\frac{1}{2}$$

for k=1,2,..,n-1 and

$$\sum_{j=1}^{n} \operatorname{Re} z_j = x .$$

Proof. By Lemma 3 we have

$$\sum_{j=1}^{k} \operatorname{Re} z_j \le \frac{1}{2}\left(-\operatorname{Re} a_n + \sqrt{(\operatorname{Re} a_n)^2 + \sum_{j=1}^{n-1} \left| a_j \right|^2} \right) + \frac{1}{2}\left(\frac{\sin\left(\dfrac{2k+1}{2n+2}\right)\pi}{\sin \dfrac{\pi}{2n+2}} \right) - \frac{1}{2}$$

By taking $a_j = r = x + iy$ for j=1,2,…,n , where $x, y \in \mathbb{R}$ and $\operatorname{Re}(x+iy) = x$. Hence, we get the result. □

Theorem 3.7 *If* $A \in M_n(\mathbb{C})$ *be conjugate normal matrix, then*

$$\operatorname{Re}\left(\overrightarrow{\mu(A)}\right) \succ_w \operatorname{Re}\left(\mu(\mathrm{A})\right).$$

Proof . By Lemma 4, we have

$$\left(\operatorname{Re}\overrightarrow{\left(\mu(A)\right)}\right) + \left(\operatorname{Re}\overrightarrow{\left(\mu(B)\right)}\right) \succ_w \operatorname{Re}\left(\mu(A+B)\right).$$

By taking B=A , we have.

$$\left(\operatorname{Re}\overrightarrow{\left(\mu(A)\right)}\right) + \left(\operatorname{Re}\overrightarrow{\left(\mu(A)\right)}\right) \succ_w \operatorname{Re}\left(\mu(2A)\right).$$

$$\left(2\operatorname{Re}\overrightarrow{\left(\mu(A)\right)}\right) \succ_w \operatorname{Re}\left(\mu(2A)\right).$$

But $\operatorname{Re}\left(\mu(2A)\right) = 2\operatorname{Re}\left(\mu(A)\right)$. Hence, we get the result. □

Theorem 3.8 *Let* $z_1, z_2, ..., z_n$ *be the zeros of polynomials f of degree* $n \ge 3$ *and* $w_1, w_2, ..., w_{n-1}$ *be the critical points of f. Then we have*

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right) \le \left(\sum_{i=1}^{n-1}|z_i|^2\right) + 2\left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}}\left(\frac{n-1}{n}\right)|z_n| + \left(\left(\frac{n-1}{n}\right)|z_n|\right)^2, \qquad (4)$$

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right) \le \left((\mathrm{n}-2)+\frac{1}{n^2}\right)\left(\max_{1\le i\le n-1}\{|z_i|\}\right)^2 +$$

$$2\left((\mathrm{n}-2)+\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\max_{1\le i\le n-1}\{|z_i|\}\right)\left(\frac{n-1}{n}\right)|z_n|$$

$$+\left(\left(\frac{n-1}{n}\right)|z_n|\right)^2, \qquad (5)$$

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right) \le \left(\sum_{i=1}^{n-1}|z_i|^2\right) + 2\left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}}\sqrt{\frac{n-1}{n^2}\sum_{i=1}^{n-1}|z_n-z_i|^2} +$$

$$\frac{n-1}{n^2}\sum_{i=1}^{n-1}|z_n-z_i|^2. \qquad (6)$$

*Proof.*By Lemma 7 for $p\ge 1$, we have

$$\|C(f')\|_2 = \left\|D\left(I-\frac{1}{n}J\right)+\frac{z_n}{n}J\right\|_2 \le \left\|D\left(I-\frac{1}{n}J\right)\right\|_2 + \left\|\frac{z_n}{n}J\right\|_2$$

$$\le \|D\|_2\left\|\left(I-\frac{1}{n}J\right)\right\| + \left\|\frac{z_n}{n}J\right\|_2$$

$$=\left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}} + \left(\frac{n-1}{n}\right)|z_n|.$$

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}} + \left(\frac{n-1}{n}\right)|z_n|.$$

By square both sides we get the results of (4).

For (5), we have

$$\|C(f')\|_2 = \left\|D\left(I-\frac{1}{n}J\right)+\frac{z_n}{n}J\right\|_2 \le \left\|D\left(I-\frac{1}{n}J\right)\right\|_2 + \left\|\frac{z_n}{n}J\right\|_2$$

$$\le \left\|\left(I-\frac{1}{n}J\right)\right\|_2\|D\| + \left\|\frac{z_n}{n}J\right\|_2$$

$$=\left((n-2)+\frac{1}{n^2}\right)^{\frac{1}{2}}\max_{1\le i\le n-1}\{|z_i|\} + \left(\frac{n-1}{n}\right)|z_n|.$$

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right)^{\frac{1}{2}} \le \left((n-2)+\frac{1}{n^2}\right)^{\frac{1}{2}} \max_{1\le i\le n-1}\{|z_i|\}+\left(\frac{n-1}{n}\right)|z_n|.$$

By square both sides we get the result of (5).

Now for (6) we have,

$$C(f')=D+\frac{1}{n}(z_nJ-DJ)=D+\frac{1}{n}\begin{bmatrix} z_n-z_1 & z_n-z_1 & \cdots & z_n-z_1 \\ z_n-z_2 & z_n-z_2 & \cdots & z_n-z_2 \\ \vdots & \vdots & \ddots & \vdots \\ z_n-z_{n-1} & z_n-z_{n-1} & \cdots & z_n-z_{n-1} \end{bmatrix}=D+\frac{1}{n}E,$$

where

$$E=\begin{bmatrix} z_n-z_1 & z_n-z_1 & \cdots & z_n-z_1 \\ z_n-z_2 & z_n-z_2 & \cdots & z_n-z_2 \\ \vdots & \vdots & \ddots & \vdots \\ z_n-z_{n-1} & z_n-z_{n-1} & \cdots & z_n-z_{n-1} \end{bmatrix}.$$

Note that rank(E) ≤ 1

$$\left(\sum_{i=1}^{n-1}|w_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n-1}|z_i|^p\right)^{\frac{1}{p}}+\left(\frac{n-1}{n}\right)|z_n|,$$

$$\left(\sum_{i=1}^{n-1}|w_i|^p\right)^{\frac{1}{p}} \le \left((n-2)+\frac{1}{n^p}\right)^{\frac{1}{p}} \max_{1\le i\le n-1}\{|z_i|\}+\left(\frac{n-1}{n}\right)|z_n|,$$

$$\left(\sum_{i=1}^{n-1}|w_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n-1}|z_i|^p\right)^{\frac{1}{p}}+\sqrt{\frac{n-1}{n^2}\sum_{i=1}^{n-1}|z_n-z_i|^2}.$$

By Definition 1 and Theorem 1 and taking p=2, we have

$$\|C(f')\|_2=\left\|D\left(I-\frac{1}{n}J\right)+\frac{z_n}{n}J\right\|_2 \le \left\|D\left(I-\frac{1}{n}J\right)\right\|_2+\left\|\frac{z_n}{n}J\right\|_2$$

$$\le \|D\|_2\left\|\left(I-\frac{1}{n}J\right)\right\|+\left\|\frac{z_n}{n}J\right\|_2$$

$$=\left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}}+\left(\frac{n-1}{n}\right)|z_n|.$$

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}} + \left(\frac{n-1}{n}\right)|z_n|.$$

By square both sides we get the results of (4).

For (5), we have

$$\begin{aligned}\|C(f')\|_2 = \left\|D\left(I-\frac{1}{n}J\right)+\frac{z_n}{n}J\right\|_2 &\le \left\|D\left(I-\frac{1}{n}J\right)\right\|_2 + \left\|\frac{z_n}{n}J\right\|_2 \\ &\le \left\|\left(I-\frac{1}{n}J\right)\right\|_2 \|D\| + \left\|\frac{z_n}{n}J\right\|_2 \\ &= \left((n-2)+\frac{1}{n^2}\right)^{\frac{1}{2}} \max_{1\le i\le n-1}\{|z_i|\} + \left(\frac{n-1}{n}\right)|z_n|.\end{aligned}$$

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right)^{\frac{1}{2}} \le \left((n-2)+\frac{1}{n^2}\right)^{\frac{1}{2}} \max_{1\le i\le n-1}\{|z_i|\} + \left(\frac{n-1}{n}\right)|z_n|.$$

By square both sides we get the result of (5).

Now for (6) we have,

$$C(f') = D+\frac{1}{n}(z_nJ-DJ) = D+\frac{1}{n}\begin{bmatrix} z_n-z_1 & z_n-z_1 & \cdots & z_n-z_1 \\ z_n-z_2 & z_n-z_2 & \cdots & z_n-z_2 \\ \vdots & \vdots & \ddots & \vdots \\ z_n-z_{n-1} & z_n-z_{n-1} & \cdots & z_n-z_{n-1}\end{bmatrix} = D+\frac{1}{n}E,$$

where

$$E = \begin{bmatrix} z_n-z_1 & z_n-z_1 & \cdots & z_n-z_1 \\ z_n-z_2 & z_n-z_2 & \cdots & z_n-z_2 \\ \vdots & \vdots & \ddots & \vdots \\ z_n-z_{n-1} & z_n-z_{n-1} & \cdots & z_n-z_{n-1}\end{bmatrix}.$$

Note that rank(E)≤ 1

$$E^*E = \begin{bmatrix} \overline{z_n-z_1} & \overline{z_n-z_2} & \cdots & \overline{z_n-z_{n-1}} \\ \overline{z_n-z_1} & \overline{z_n-z_2} & \cdots & \overline{z_n-z_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{z_n-z_1} & \overline{z_n-z_2} & \cdots & \overline{z_n-z_{n-1}}\end{bmatrix}\begin{bmatrix} z_n-z_1 & z_n-z_1 & \cdots & z_n-z_1 \\ z_n-z_2 & z_n-z_2 & \cdots & z_n-z_2 \\ \vdots & \vdots & \ddots & \vdots \\ z_n-z_{n-1} & z_n-z_{n-1} & \cdots & z_n-z_{n-1}\end{bmatrix}$$

$$=\begin{bmatrix} \sum_{i=1}^{n-1}|z_n-z_i|^2 & \sum_{i=1}^{n-1}|z_n-z_i|^2 & \cdots & \sum_{i=1}^{n-1}|z_n-z_i|^2 \\ \sum_{i=1}^{n-1}|z_n-z_i|^2 & \sum_{i=1}^{n-1}|z_n-z_i|^2 & \cdots & \sum_{i=1}^{n-1}|z_n-z_i|^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n-1}|z_n-z_i|^2 & \sum_{i=1}^{n-1}|z_n-z_i|^2 & \cdots & \sum_{i=1}^{n-1}|z_n-z_i|^2 \end{bmatrix}$$
$$=\left(\sum_{i=1}^{n-1}|z_n-z_i|^2\right)J.$$

Hence, $\sigma(E^*E)=\left\{(n-1)\sum_{i=1}^{n-1}|z_n-z_i|^2,0\right\}$, where 0 is of multiplicity n-2. So

$$s_1\left(\frac{1}{n}E\right)=\sqrt{\frac{n-1}{n^2}\sum_{i=1}^{n-1}|z_n-z_i|^2},\, s_j\left(\frac{1}{n}E\right)=0 \quad \text{for j=2,3,..,n-2}$$

Now,

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right)^{\frac{1}{2}} \le \|C(f')\|_2 = \left\|D+\frac{1}{n}E\right\|_2 \le \|D\|_2+\left\|\frac{1}{n}E\right\|_2$$
$$=\left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}}+\sqrt{\frac{n-1}{n^2}\sum_{i=1}^{n-1}|z_n-z_i|^2}.$$

$$\left(\sum_{i=1}^{n-1}|w_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n-1}|z_i|^2\right)^{\frac{1}{2}}+\sqrt{\frac{n-1}{n^2}\sum_{i=1}^{n-1}|z_n-z_i|^2}.$$

By square both sides we get the result of (6). □

References

[1] M. Adm, F. Kittaneh, Bounds and Majorization Relations for the Critical Points of Polynomials, *Linear Algebra and its Applications*, **436**, (2012), 2494-2503.

[2] W. Cheung, T. Ng, A companion matrix approach to the study of zeros and critical points of a polynomial, *J. Math. Anal. App*l., **319**, (2006) 690-707.

[3] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[4] M. Fujii and F. Kubo, Buzano's inequality and bounds for roots of algebraic equations, *Proc. Amer. Math. Soc.*, **117**, (1993), 359-361.

[5] F. Kittaneh, Bounds and a majorization for the real parts of the zeros of polynomials, *Proc. Amer. Math. Soc.*, **135**, (2007), 664-695.

[6] H. Linden, Bounds for the zeros of polynomials from eigenvalues and singular values of some companion matrices, *Linear Algebra Appl.*, **271**, (1998), 41-82.

[7] H.D. Sterck and Minghualin, Some Majorization Inequalities for Coneigevalues, *a publication of the International Linear Algebra Society*, **23**, (2012), 669-677.