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Spectral relationships of Volterra- Fredholm integral equations in some domains

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Abstract

In this work, the solutions of Volterra- Fredholm integral equations of the first and second kind in one, two and three dimensional are obtained in the space $L_2(\Omega) \times C[0,T]$, $T < 1$. The Fredholm integral term is measured with respect to position, where Ω is the domain of integration; while Volterra integral is measured with respect to time. The solutions are obtained, using two different methods. For the first method, we have a Volterra integral equation, while for the second method, we obtain a linear system of Fredholm integral equation. Several spectral relationships are obtained when the kernel of position takes a logarithmic form, Carleman function, elliptic integral form, potential function, generalized potential function, Macdonald kernel, and other interesting cases are discussed.

Mathematics Subject Classification: 45B05; 45R10

Keywords: Volterra-Fredholm integral equation (**V-FIE**); System of Fredholm integral equations; linear algebraic system; generalized potential function; Macdonald kernel

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1 Introduction

Many problems in mathematical physics, theory of elasticity, viscodynamics fluid and mixed problems of mechanics of continuous media reduce to an integral equation of the second kind, in different dimensions, with continuous or discontinuous; see [1-6]. For these applications, the authors have established many different analytic and numeric methods for solving the different kinds of integral equations with different kernels; see [7-11].

One of the important methods for solving the singular integral equations is the orthogonal polynomials method. Using this method, the authors can obtain the spectral relationships for the integral equations of the first kind. Then, using the results in solving the singular Fredholm integral equations of the second kind. More information for the spectral relations can be found in Refs.[12-17], and for the orthogonal polynomials in [18-21]

Consider the V-FIE of the second kind

$$\mu\,\Phi(x,t)-\int_0^t\int_\Omega F(t,\tau)k(x,y)\Phi(y,\tau)dy\,d\tau=\left[\gamma(t)+\beta(t)x-f_*(x)\right]=f(x,t) \quad (1.1)$$

$$(x=\bar{x}(x_1,x_2,x_3),=\bar{y}(y_1,y_2,y_3);\ (x,y)\in\Omega, t\in[0,T],T<1)$$

under the conditions

$$\int_\Omega \Phi(x,t)dx=N(t), \qquad \int_\Omega x\,\Phi(x,t)dx=M(t), \quad (1.2)$$

The integral equation (1.1) can be investigated from the contact problem of a rigid surface (G, υ) having an elastic material occupying the domain Ω, where $f_*(x)$ describing the surface of the stamp. This stamp is impressed into an elastic layer surface (plane) by a variable known force N(t) whose eccentricity of application e(t), and a variable known moment M(t), $t\in[0,T]$, $T<1$; that case rigid displacements γ(t) and β(t) respectively. Here G is the displacement magnitude,υ is Poisson's coefficient, μ is a given finite constant defines the kind of integral equation. In addition, γ(t), β(t) and $F(t,\tau)$ are given continuous functions belong to the class C[0,T], $t \in [0,T],T < 1$. The given function $f(x,t)$ belongs to L_2 (Ω) x C [0,T], where Ω is the domain integration. The discontinuous kernel of position $k(x,y)$, $(x = \bar{x}(x_1, x_2, x_3), = \bar{y}(y_1, y_2, y_3))$ is known function, where its formula depends on the surface used, and the contact domain of integration, while $F(t,\tau)$ is the continuous kernel of Volterra integral term in time represents the resistance force of the contact domain. The unknown function $\Phi(x,t)$, will be obtained, under certain conditions, in the space L_2 (Ω) x C [0, T].

We must realize that, the generalized potential and Macdonald kernels are obtained, if the modules of elasticity is changing in the layer surface according to the power law $\sigma_i = k_0 \varepsilon_i^{\upsilon}, i = 1,2,3, 0 \le \upsilon < 1$, where σ_i and ε_i are the stress and strain rate intensities respectively, while k_0 and υ are constants depending on the physical properties of the elastic layer; see [22-23].

In order to guarantee the existence of unique solution of (1.1), we assume the following conditions

(i) The kernel of position, in general, satsfies the discontinuity condition $\left\{ \iint_{\Omega} k^2(x,y)dxdy \right\}^{1/2} = D$. (D-constant) ,where Ω is the domain of integration.

(ii)The positive continuous function $F(t,\tau) \in C([0,T] X [0,T])$ and satisfies $F(t,\tau) < E < E$, (E-constant), for $\tau \in$ [0,T].

(iii) The given continuous functions γ(t) , β(t) belong to the class C[0,T] , while $f_*(x) \in L_2$ (Ω). This leads that, the continuous function $f(x,t) \in L_2$ (Ω) × C [0,T].

(iv) The unknown function $\Phi(x,t)$ satisfies Hölder condition with respect to time and Lipschitz condition with respect to position.

Theorem1 (without Proof): The V-FIE (1.1) has a unique solution, under the condition $|\mu| > TDE$, $\max t = T$.

In the remainder part of this paper, the solutions]of the **V-FIE**, in one, two and three dimensional of the first and second kind ,in the space L_2 (Ω) x C [0,T], will obtain. Where, we use a series polynomials method to separate the variables and obtain three integral equations of Volterra integral equations of the second kind with continuous kernel. Then, a quadratic numerical method will use to obtain linear system of **FIEs** with discontinuous kernel, where the solution can obtain, under certain conditions.

2 Method of separation of variables

The importance of this method comes from its wide applications in mathematical physics problems, where the eigenvalues and eigenfunctions of the problems can be discussed and studied. Therefore, we state without proof the following:

The principal theorem 2 : (see [17] and [18])

For a symmetric and positive kernel of position $k(x, y)$, the integral operator

$$K\psi = \int_{\Omega} k(x,y)\psi(y)\,dy, \tag{2.1}$$

through the time intervel $0 \le t \le T < 1$, is compact and self-adjoint operator. So, we can write (2.1) in the following form $\alpha_n K\psi = \psi_n$,where α_n and ψ_n are the eigenvalues and eigenfunctions of the integral operator respectively.

In view of Theorem 2, we will seek the solution of equation (1.1), in the following form

$$\Phi(x,t) = \Phi_0(x,t) + \Phi_1(x,t) \tag{2.2}$$

where $\Phi_0(x,t)$), $\Phi_1(x,t)$ represent, respectively the complementary and particularly solution of (1.1).

Using (2.2) in (1.1), we have

$$\mu\Phi_j(x,t) - \int_0^t \int_\Omega F(t,\tau)k(x,y)\Phi_j(y,\tau)dyd\tau = \delta_j\left[\gamma(t) + \beta(t)x - f_*(x)\right] \tag{2.3}$$

Where, $\delta_0 = 0$ at j=0, and $\delta_1 = 1$at j=1.

For t = 0 , the formula (1.1) becomes

$$\mu\Phi(x,0) = \left[\gamma(0) + \beta(0)x - f_*(x)\right] \tag{2.4}$$

Using (2.2) in (2.4) and substracting the result from (2.3), we have

$$\mu\left[\Phi_j(x,t) - \Phi_j(x,0)\right] - \int_0^t \int_\Omega F(t,\tau)k(x,y)\varphi_j(y,\tau))dyd\tau = \delta_j\left[\gamma(t) - \gamma(0) + \left(\beta(t) - \beta(0)\right)x\right] \tag{2.5}$$

Under the conditions (1.2), assume the unknown function $\Phi(x,t)$ of (2.3) in the following form.

$$\Phi_j(x,t) = \sum_{k=1}^{\infty}\left[A_{2k}^{(j)}(t)\psi_{2k}(x) + A_{2k-1}^{(j)}(t)\,\psi_{2k-1}(x)\right] \tag{2.6}$$

Hence, Eq. (2.5), after using (2.6), becomes

$$\mu\left[\left(A_{2k}^{(j)}(t) - A_{2k}^{(j)}(0)\right)\psi_{2k}(x) + \left(A_{2k-1}^{(j)}(t) - A_{2k-1}^{(j)}(0)\right)\psi_{2k-1}(x)\right]$$
$$- \int_0^t \int_\Omega F(t,\tau)k(x,y)\left[A_{2k}^{(j)}(\tau)\,\psi_{2k}(y) - A_{2k-1}^{(j)}(\tau)\,\psi_{2k-1}(y)\right)\Big]dy\,d\tau$$
$$= \delta_j\left[\gamma(t) - \gamma(0) + x\left(\beta(t) - \beta(0)\right)\right];\ k = 0,1,...,\infty \tag{2.7}$$

In view of principle theorem2 and the conditions of (1. 2) we can write Eq. (2.7) as

$$\mu A_k^{(0)}(t) - \alpha_n \int_0^t F(t,\tau) A_k^{(0)}(\tau) d\tau = \mu A_k^{(0)}(0) \, , \; k \geq 1 \tag{2.8}$$

$$\mu A_{2k}^{(1)}(t) - \alpha_{2k} \int_0^t F(t,\tau) \, A_{2k}^{(1)}(\tau) d\tau = \alpha_{2k} \, b_{2k} \left[\gamma(t) - \gamma(0)\right] \tag{2.9}$$

$$\mu A_{2k-1}^{(1)}(t) - \alpha_{2k-1} \int_0^t F(t,\tau) \, A_{2k-1}^{(1)}(\tau) d\tau = \alpha_{2k-1} b_{2k-1} \left[\beta(t) - \beta(0)\right] \tag{2.10}$$

where

$$\sum_{k=1}^{\infty} b_{2k} \; \psi_{2k} = 1 \;\; , \sum_{k=1}^{\infty} b_{2k-1} \; \psi_{2k-1} = x \tag{2.11}$$

Equations (2.8)-(2.10) represent **VIEs** of the second kind that have the same continuous kernel F(t, τ) ∈ C ([0,T] × [0 , T])and each of them has a unique solution in the class C[0,T]. The value of $A_k^0(0)$ can be obtained, directly from (2.3) and (2.4) in the form of $A_k^0(0) = \alpha_k \, \gamma(0)$. Different methods in [24, 25] have been discussed to obtain the general solution of **VIEs** of the first and second kind.

In view of Eqs (2.8)-(2.10) the general solution of Eq. (1.1) under the conditions (1.2) can be adapted in the form

$$\Phi_n(x,t) = \sum_{k=1}^{n} \left[A_k^0(t) + A_k^{(t)} \right] \psi_k(x) \tag{2.12}$$

where $A_k^{(0)}(t)$ and $A_k^{(1)}(t)$ must satisfy the inequality

$$\left[\sum_{k=1}^{n} \left| A_k^{(0)}(t) + A_k^{(0)}(t) \right|^2 \right]^{1/2} < \delta \; , \; (n \to \infty, \delta < 1 \, , 0 \leq t \leq T) \tag{2.13}$$

from the above discussion, we can state the following:

Theorem 3: The inequality (3.13) tells us that, the series (2.12) is uniformly convergent in the space $L_2(\Omega) \times C[0, T]$, $T< 1$, $n \to \infty$. Therefore, the solution of **V- FIE** of (1.1) can be obtained in a series from of (2.12).

Theorem 4: Under the conditions (i)-(iv) and equation (2.13), we have $\|\Phi(x,t)-\Phi_n(x,t)\| \to 0$ *as* $n \to \infty$ where $\Phi(x,t)$ represents the unique solution of (1.1), and its approximate solution $\Phi_n(x,t)$ is given by (2.12).

Theorem 5: The error of the approximate method used can be calculated as $E_n = \|\Phi(x,t)-\Phi_n(x,t)\|$, $t \in [0,T]$, where $E_n \to 0$ as $n \to \infty$.

3 System of Fredholm integral equations

In this section, we use quadratic method, see [8,10], to transform the **V-FIE** to **SFIEs**. The importance of this method comes from its wide applications in the applied sciences especially in the theory of elasticity, mixed problems in mechanics and in contact problems. For this, we divide the interval [0 ,T], $0 \le t \le T < 1$ as:

$0 = t_0 < t_1 < t_2 < \ldots < t_\ell = T$, when $t = t_k$, $k = 0, 1, \ldots.. \ell$.

Hence, the integral term of (1.1), in this case, becomes

$$\int_0^t \int_\Omega k(x,y)\, F(t,\tau)\Phi(y,\tau)\, dy\, d\tau =$$

$$= \sum_{j=0}^{k} u_j\, F(t_k, t_j) \int_\Omega k(x,y)\, \Phi(y,t_j)\, dy + 0(h_k^{\tilde{p}+1}) \quad (h_k \to 0\ ,\ \tilde{p} > 0) \tag{3.1}$$

$$(h_k = \max_{0 \le j \le k} h_j\ ,\ h_j = t_{j+1} - t_j\ )$$

The values of u_j and the constant $\tilde{p}$ depend on the number of derivatives of F(t, τ) with respect to t, for all values of τ for example, if $F(t,\tau) \in C^4([0,T] \times [0,T])$ then, $\tilde{p} = 4$, $\tilde{p} = k$ and $u_0 = \frac{1}{2} h_0$, $u_4 = \frac{1}{2} h_4$, $u_i = h_i$, $i = 1,2,3$. More information for the characteristic points and quadrature coefficients are found in [8, 10].

Using (31) in (1.1), we have

$$\mu \Phi_k(x) - \sum_{j=0}^{k} u_j F_{j,k} \int_{\Omega} k(x,y)\Phi_j(y)dy = f_k(x) \tag{3.2}$$

where, we used the notations $\Phi(x,t_k) = \Phi_k(x), F(t_k,\tau_j) = F_{k,j}, f(x,t_k) = f_k(x)$.

In addition, the boundary conditions (1.2) become

$$\int_{\Omega} \Phi_k(x)dx = N_k \;,\; \int_{\Omega} x\Phi_k(x)dx = M_k \tag{3.3}$$

The formula (3.2) represents linear **SFIEs** of the second kind. When μ = 0, in Eq. (3.2), we have linear **SFIEs** of the first kind.

4 Spectral relations of SFIEs of the first kind

Our attention now, is obtaining the spectral relations for the following equation

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{\Omega} k(x,y)\Phi_j(y)dy = f_k(x) \tag{4.1}$$

4.1 One Dimensional Integral Equation

(1) Let, in (4.1) $k(x,y)=\ln|x-y|$, $\Omega=[-1,1]$ then, we have **SFIEs** of the first kind with logarithmic kernel.

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-1}^{1} \ln|x-y|\Phi_j(y)dy = f_k(x) \tag{4.2}$$

For obtaining the spectral relations of (4.2),we use the orthogonal polynomial method. For this, Let $T_n(x) = \cos(n\cos^{-1}x)$, $x \in[-1,1]$, $n \geq 0.$ denotes the

Chebyshev polynomials of the first kind, while $U_n(x)=\dfrac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)}$, $n\ge 0$,

denotes the Chebyshev polynomials of the second kind. It is well known that $\{T_n(x)\}$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$, while $\{U_n(x)\}$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{\frac{1}{2}}$. It appears reasonable to attempt a series expansion to $\Phi_\ell(x)$ in Eq. (4.2) in terms of Chebyshev polynomials of the first kind. This choice is not arbitrary since one can identity a portion of the integral as the weight function associated with $T_n(x)$. For convenience, we use the orthogonal polynomials method with some well-known algebraic and integral relations associated with Chebyshev polynomials see [12,18]. Thus, in this aim, we represent $\Phi_\ell(x)$, $f_\ell(x)$ in the following forms

$$\Phi_\ell(x)=\frac{1}{\sqrt{1-x^2}}\sum_{n_\ell=0}^{\infty}a_{n_\ell}T_{n_\ell}(x),\quad f_\ell(x)=\sum\frac{f_{n_\ell}\,T_{n_\ell}(x)}{\sqrt{1-x^2}} \tag{4.3}$$

Hence, after substituting (4.3) in (4.2), and using the orthogonal polynomial of Chebyshev polynomials, the following spectral relationships, from (4.2), can be obtained

$$\sum_{j=0}^{k}u_jF_{j,k}\int_{-1}^{1}\frac{\ln|x-y|}{\sqrt{1-y^2}}T_{n_j}(y)\,dy=\begin{Bmatrix}\pi\ln 2\sum_{j}^{k}u_jF_{j,k} & n=0\\ \pi\left(\sum_{j=0}^{k}u_j\dfrac{F_{j,k}}{n_j}\right)T_{n_j}(x), n_j\ge 1\end{Bmatrix} \tag{4.4}$$

where $T_n(x)$ is the Chebyshev polynomial of the first type.

Differentiating Eq. (4.4) with respect to x, we have the spectral relations of **SFIEs** of the first kind with Cauchy kernel, in the form

$$\sum_{j=0}^{k}u_jF_{j,k}\int_{-1}^{1}\frac{T_{n_j}(y)\,dy}{(x-y)\sqrt{1-y^2}}=\pi\sum_{J=0}^{k}u_jF_{j,k}U_{n_j-1}(x)\quad(n_j\ge 1), \tag{4.5}$$

where $U_n(x)$ is the Chebyshev polynomial of the second type.

(2) If we let, in Eq. (4.1), k (x,y) = $|x-y|^{-\upsilon}$ $0\le\upsilon<1;\ \Omega=[-1,1]$, we have **SFIEs** with Carleman kernel. The importance of Carleman kernel came from the work of Artiunians [26], who has shown that, the contact problem of the nonlinear theory of plasticity, in its first approximation reduces to FIE of the first kind with Carleman kernel. Hence, we have

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-1}^{1} |x-y|^{-\upsilon} \phi_j(y)\,dy = f_k(x)\,,\ |x|\le 1 \qquad (4.6)$$

To obtain the solution of the formula (3.14), we represent the unknown and known functions, respectively in the following form, see [27,28]

$$\Phi_k(x) = \frac{1}{(1-x^2)^{(\frac{1-\nu}{2})}} \sum_{n=0}^{\infty} a_{nk}\, C_{2n_k}^{\frac{\nu}{2}}(x), \quad f_\ell(x) = \frac{1}{(1-x^2)^{(\frac{1-\nu}{2})}} \sum_{n=0}^{\infty} f_{n\ell}\, C_{2n_k}^{\frac{\nu}{2}}(x). \qquad (4.7)$$

Here, $C_{2n}^{\nu}(x)$ are Gegenbauer polynomials, a_{nk} are the unknown coefficients and $f_{n\ell}$ are the known coefficients. The term $(1-x^2)^{-(\frac{1-\nu}{2})}$ is called the weight function of the Gegenbauer polynomials.

Using the orthogonal polynomial method [18-19], and the following relations [28]

1. $n\, C_n^{\nu}(x) = 2\,\nu[x\ C_{n-1}^{\nu+1}(x) - C_{n-2}^{\nu+1}(x)],$

2. $\int_{-1}^{1} \left(1-x^2\right)^{\frac{1}{2}(\nu-1)} C_{2n}^{\nu}(x)\,dx = \dfrac{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}\ C_n^{\frac{\nu}{2}}\left(2x^2-1\right) \qquad \left(\operatorname{Re}\nu > 0\right)$

3. $\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} \left[C_n^{\nu}(x)\right]^2 dx = \dfrac{\pi\ 2^{1-2\nu}\ \Gamma(2\nu+n)}{n!\,(n+\nu)\left[\Gamma(\nu)\right]^2};\ (\operatorname{Re}\nu > -\dfrac{1}{2})$

where $\Gamma(x)$ is the Gamma function, we arrive to the following spectral relationships

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-1}^{1} \frac{C_{m_j}^{\upsilon/2}(y)\,dy}{|x-y|^{\upsilon}(1-y^2)^{\omega}} = \frac{\pi}{\Gamma(\upsilon).\cos\frac{\upsilon\pi}{2}} \sum_{J=0}^{k} u_j\, F_{j,k}\, \Gamma(m_j+\upsilon)\, C_{m_j}^{\upsilon/2}(x); \qquad (4.8)$$

$$(m\ge 0\ ,\omega=\frac{1-\upsilon}{2};|x|\le 1)$$

4.2 Two and three Dimensional Integral Equations with finite domain

(3) If the modules of elasticity is changing in the layer surface according to the power law $\sigma_i = k_0\, \varepsilon_i^{\upsilon}, i = 1,2,3, \upsilon = 0.5$. In this case, the kernel of Eq. (4.1) takes a potential function form, and the contact domain Ω is represented as Ω = {(x, y, z) ∈ Ω: $\sqrt{x^2 + y^2} \le a$, $z = 0$ } , hence we have

$$\sum_{j=0}^{k} u_j\, F_{j,k} \iint_{\Omega} \frac{\Phi_j(\xi,\eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = f_k(x,y) \tag{4.9}$$

The formula (4.9) represents SFIEs with potential kernel.

Following the same way of [16], on noting the difference in notation, we arrive to the following

$$\sum_{J=0}^{k} u_j F_{j,k} \int_0^1 W_n(r,\rho)\, \psi_j(\rho)\, d\rho = g_k(r) \tag{4.10}$$

where, we used the notations of polar coordinates and we represent $W_n(r,\rho)$ as a formula of Weber-Sonien integral,

$$W_n(r,\rho) = \sqrt{r\rho} \int_0^{\infty} J_n(ur)\, J_n(u\rho)\, du; \quad (J_n(x) \text{ is a Bessel function of order n}) \tag{4.11}$$

The general solution of Eq. (4.11) leads us to obtain

Assume the solution of (4.10) in the form

$$Z_k^{(m)}(r) = \frac{1}{\sqrt{1-r^2}} \sum_{n_k=0}^{\infty} a_{n_k}^{(m)}\, P_{2n_k}^{(m)}\left(\sqrt{1 - r^2}\right), \quad (k = 0, 1, 2, \ldots, \ell) \tag{4.12}$$

where $P_{2n}(y)$ is the Legendre polynomial and $\sqrt{1-r^2}$ is called the weight function of the Legendre polynomial. Then, using orthogonal polynomials method [18.19], we obtain the following spectral relationships:

$$\sum_{j=0}^{k} u_j F_{j,k} \int_0^1 \frac{W_n(x,y) P_{m_j}^{(n_j,-\frac{1}{2})}\left(1-2y^2\right)dy}{\sqrt{1-y^2}} = x^n \sum_{J=0}^{k} \lambda_{m_j} u_j F_{j,k} P_{m_j}^{(n,-\frac{1}{2})}\left(1-2x^2\right) \quad (4.13)$$

where $\lambda_{m_j} = \frac{\Gamma^2\left(\frac{1}{2}+m_j\right)}{\left(2m_j\right)!\Gamma\left(1+m_j+n\right)}$ and $P_m^{(\alpha,\beta)}(x)$ is a Jacobi polynomial.

(4) If the modules of elasticity is changing in the layer surface according to the power law $\sigma_i = k_0 \varepsilon_i^{\upsilon}, i=1,2,3, 0 \le \upsilon < 1$. In this case, the kernel of Eq. (4.1) takes a generalized potential function form, and the following contact domain {Ω = {x, y, z) ∈ Ω : $\sqrt{x^2+y^2} \le a$, $z=0$ }. Hence, we have

$$\sum_{j=0}^{k} u_j F_{j,k} \iint_{\Omega} \frac{\Phi_j(\xi,\eta)\, d\xi d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\upsilon}} = f_k(x,y)\ , \qquad (0 \le \upsilon < 1) \qquad (4.14)$$

Following the same way of [16, 22], after using polar coordinates and noting the difference in notations, we have

$$\sum_{j=0}^{\infty} u_j F_{j,k} \int_0^1 W_n^{\upsilon}(r,\rho)\psi_j(\rho)d\rho = g_k(r) \qquad (4.15)$$

where

$$W_n^{\upsilon}(x,y) = \sqrt{xy}\int_0^{\infty} t^{\upsilon} J_n(tx) J_n(ty)\, dt. \qquad (4.16)$$

The formula (4.16) is called Weber-Sonien integral formula. The solution of the formula (4.15), after using orthogonal polynomials, leads us to the following

$$\sum_{j=0}^{\infty} u_j F_{j,k} \int_0^1 \frac{W_n^{\upsilon}(x,y)}{(1-y^2)^{\omega}} . P_{mj}^{(n,-\omega)}(1-2y^2)\, dy = x^n \sum_{j=0}^{\infty} u_j F_{j,k}\, \lambda_{mj}^{*}\, P_{mj}^{(n,-\omega)}(1-2x^2), \quad (4.17)$$

where $\lambda_{mj}^{*} = \frac{2^{-2\omega}\Gamma^2\left(\frac{1+\upsilon}{2}+m_j\right)}{\left(2m_j\right)!\Gamma\left(1+m_j+n\right)}$, $(0 \le \upsilon < 1,\ \omega = \frac{1-\upsilon}{2}; m \ge 0)$

The formula (4.17) represents eigenvalues and eigenfunctions for a linear system of FIE of the first kind with a generalized potential function.

4.3 Special cases

(a) Let in (4.17) $\upsilon = 0.5$, we have directly the spectral relationships of (4.16)

(b) Logarithmic kernel: Let in (4.17) $\upsilon = \frac{1}{2},\ m = \pm\frac{1}{2}$

(c) Carleman function Carleman kernel: Let in (4.17) $m = \pm\frac{1}{2}$

(d) The spectral relationships of SFIEs with elliptic integral kernel can be obtained, from (4.17) when υ = n = 0, to have

$$\sum_{j=0}^{k} u_j F_{j,k} \int_0^1 \frac{yK[\frac{\sqrt{2xy}}{x+y}]P_{2m_j}(\sqrt{1-y^2})dy}{\sqrt{1-y^2}} = \\ = \frac{\pi^2}{4}\sum_{j=0}^{k} u_j F_{j,k}\left[\frac{\left(2_{mj}-1\right)!}{(2_{mj})!}\right]^2 P_{2m_j}(\sqrt{1-x^2}) \tag{4.18}$$

where $K[\frac{\sqrt{2uv}}{u+v}]$ is called the elliptic integral from, and $P_m(z)$ is called Legendre polynomial function.

4.4 Semi- infinite interval

To obtain the spectral relationships of Eq. (4.13), (4.17) and (4.18) of the semi-infinite interval, we represents the Weber-Sonien integral in terms of Gauss hyper -geometric function, see formula 8 of [28] PP. 715, to deduce the following important property

$$W_n^{\upsilon}\left(x^{-1}, y^{-1}\right) = (xy)^{1+\upsilon}\, W_n^{\upsilon}(x, y) \tag{4.19}$$

Using, in (4.13), (4.17) and (4.18), the substitution $x = t^{-1}$, $y = v^{-1}$, and making use of property (4.19), the spectral relations of the semi-infinite interval for **SFIEs** with potential kernel, generalized potential kernel and elliptic kernel, respectively take the following forms

$$\sum_{j=0}^{k} u_j F_{j,k} \int_1^{\infty} \frac{W_n(t,v)\ P_{m_j}^{\left(n_j-\frac{1}{2}\right)}\left(1-v^{-2}\right)dv}{v^{\frac{1}{2}+n}\sqrt{v^2-1}} = \qquad (4.20)$$

$$= \sum_{j=0}^{k} \frac{u_j F_{j,k} \lambda_{m_j}\ P_{m_j}^{\left(n_j-\frac{1}{2}\right)}\left(1-2t^{-2}\right)}{t^{1+n}}\ ;\ (1 \le t < \infty)$$

$$\sum_{j=0}^{k} u_j F_{j,k} \int_1^{\infty} \frac{W_n^{\upsilon}(t,v)\ P_{m_j}^{(n_j-w)}\left(1-2v^{-2}\right)dv}{v^{n+w}\left(v^2-1\right)^{\omega}} = \qquad (4.21)$$

$$= \sum_{j=0}^{k} u_j F_{j,k} \frac{\lambda_{m_j}^{*}\ P_{m_j}^{\left(n_j-w\right)}\left(1-2t^{-2}\right)}{t^{1+\upsilon+n}};\ \left(1 \le u < \infty, w = \frac{1-\upsilon}{2}\right)$$

and

$$\sum_{j=0}^{k} u_j F_{j,k} \int_1^{\infty} \frac{K\left[\frac{2\sqrt{tv}}{t+v}\right] P_{2m_j}\left[\sqrt{1-v^{-2}}\right]dv}{(t+v)v^{3/2}\sqrt{v^2-1}} = \qquad (4.22)$$

$$= \frac{\pi^2}{4}\sum_{j=0}^{k} u_j F_{j,k}\left[\frac{\left(2_{mj}-1\right)!}{(2_m)!}\right]^2 P_{2m_j}\left(\sqrt{1-t^{-2}}\right).$$

where λ_m and λ^*_m are given by (4.13), (4.17) respectively.

5 Three Dimensional Integral Equations with infinite domain

5.1 If the domain of integration of equation (4.9) is defined as $\Omega = \{(x,y,z) \in \Omega : |y| < a, -\infty < x < \infty, -\infty < z < 0\}$, we have the following linear system of integral equations

$$\sum_{J=0}^{k} u_j F_{j,k} \int_{-\infty}^{\infty} \int_{-a}^{a} \frac{\Phi_j(\xi,\eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} = f_k(x,y) \qquad (5.1)$$

For this aim, we use the following Fourier transform

$$\Phi_{j,\alpha}(x) = \int_{-\infty}^{\infty} \Phi_j(x,y)\, e^{i\alpha y}\, dy; \qquad f_{j,\alpha}(x) = \int_{-\infty}^{\infty} f_j(x,y)\, e^{i\alpha y}\, dy \tag{5.2}$$

where α is a parameter of Fourier transform. Hence, equation (5.1) yields

$$\sum_{J=0}^{k} u_j F_{j,k} \int_{-\infty}^{\infty} \int_{-a}^{a} \frac{\Phi_{j,\alpha}(\zeta) e^{i\alpha y}\, dy d\zeta}{\sqrt{(x-\zeta)^2 + y^2}} = f_{j,\alpha}(x)$$

Using the following famous relation [28]

$$\int_{0}^{\infty} \frac{\cos\alpha\, y\, dy}{\sqrt{(x-\zeta)^2 + y^2}} = K_0\left(|\alpha|\, |x-\zeta| \right),$$

We have

$$\sum_{J=0}^{k} u_j F_{j,k} \int_{-a}^{a} K_0\left(|\alpha|\, |x-\zeta| \right) \Phi_{j,\alpha}(\zeta)\, d\zeta = f_{j,\alpha}(x) \tag{5.3}$$

where $K_0(|t|)$ is the Macdonald kernel. Using orthogonal polynomial method, we have

$$\sum_{j=0}^{k} \int_{-a}^{a} \frac{C_{e_{nj}}[\cos^{-1}\frac{\xi}{a}, -q]q}{\sqrt{a^2-\xi^2}} K_0\left(|\xi-\eta|\right) d\xi = \\ = \pi \sum_{j=0}^{k} \frac{u_j F_{j,k}\, F_e K_{nj}(\theta j - q)}{F_e K_{nj}(\theta_j - q)}\, C_{e_{nj}}[\cos^{-1}\frac{\eta}{a}, -q]; \quad (q = \frac{a^2}{4}) \tag{5.4}$$

where F_eK_n (ℓ-q) $C_{e_n}(\theta_j\text{-}q)$ are called the Mathieu functions under the condition $0 \le \theta < 2\pi$, $0 < \ell < \infty$

5.2 If the domain of Eq. (5.1) is defined as $\Omega = \{(x,y,z) \in \Omega : -\infty < x, y < \infty, z < 0\}$, so, we have the following integral equation:

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\Phi_j(\xi,\eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = f_k(x,y) \tag{5.5}$$

The formula (5.5) represents system of Wiener-Hopf integral equations of the first kind. Using the Fourier transformation (5.2) , we have

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{0}^{\infty} K_0 \left|\xi-\eta\right| \Phi_{j,\alpha}(\xi)\, d\xi = f_{k,\alpha}(\eta) \tag{5.6}$$

which represents system of Wiener - Hopf integral equations with Macdonald Kernel. The general spectral relationships (5.6), after using orthogonal polynomial method and the properties of Chebyshev-Laguerre polynomials, are

$$\sum_{j=0}^{\infty} u_j F_{j,k} \int_{0}^{\infty} \frac{K_0\left(\left|t-\tau\right|\right) e^{-\tau} L_{m_j}^{-1/2}(2\tau)}{\sqrt{\tau}}\, d\tau = \frac{\pi}{\sqrt{2}} \sum_{j=0}^{\infty} u_j F_{j,k} \frac{(2m_j-1)!}{(2m_j)!} e^{-t} L_{m_j}^{-1/2}(2t) \ (t \ge 0) \tag{5.7}$$

where $L_m^{\alpha}(x)$ is the Chebyshev-Laguerre polynomials

5.3 If in Eq. (4.14), we define $\Omega = \{(x,y,z) \in \Omega : -\infty < x, y < \infty, -\infty < z < 0\}$, then we have

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\Phi_j(\xi,\eta)\, d\xi d\eta}{\left[\left(x-\xi\right)^2 + \left(y-\eta\right)^2\right]^{\frac{1}{2}+h}} = f_k(x,y), \quad (0 \le h \le \frac{1}{2}) \tag{5.8}$$

Using Fourier transformations method, we can get the following integral equation

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-\infty}^{\infty} \frac{K_n\left(\left|s\right| \left|\xi-\eta\right|\right) \psi_j(\eta)\, d\eta}{\left|\xi-\eta\right|^h} = \lambda g_k(\xi);$$

$$(\lambda = \frac{\Gamma\left(h+1/2\right)\left|s\right|^{-h}}{\sqrt{\pi}\, 2^{1-h}}, \ 0 \le h \le 1/2) \tag{5.9}$$

where s >0, is the coefficient of Fourier integral transformations, and $K_h(\left|.\right|)$ is the generalized Macdonald kernel, and $L_m^{\alpha}(2x)$ is the Chebyshev – Laguerre polynomial.

Here the following convergence expansion is considered

$$\frac{K_n\left(\left|x-y\right|\right)}{\left|x-y\right|^h} = \frac{\sqrt{\pi}}{2^{-h} e^{x+y}} \sum_{m=0}^{\infty} L_m^{h-1/2}(2x)\, L_m^{h-1/2}(2y), \ s=1 \tag{5.10}$$

5.4 Also, the spectral relations for the **SFIEs** the first kind with Macdonald kernel, in the domain $\Omega=\{(x,y,z)\in\Omega:|y|<a,-\infty<x<\infty,-\infty<z<0\}$ and generalized potential function, in the form

$$\sum_{j=0}^{k}u_jF_{j,k}\int_{-a}^{a}K_n\left(|s||t-\xi|\right)\left(a^2-\xi^2\right)^{-\upsilon/2}\,P_{s_{nj}-\upsilon}^{\upsilon}(\xi-\theta)d\xi=$$
$$=(a^2-t^2)^{\upsilon/2}\sum_{j=0}^{k}u_jF_{j,k}\lambda_{n_j}\,P_{s_{nj}-\upsilon}^{\upsilon}(t-\theta),\quad\left(n=0,1,2,...,\upsilon=\frac{1}{2}-h,0\le h<\frac{1}{2}\right)\qquad(5.11)$$

where

$$\lambda_{n_j}=(-1)^{[n_j/2]}\alpha_{n_j}\sin(\pi h)\,q^{(n_j+h)/2}\left|2S\right|^{-h}2^{1-n_j}A_{n_j}^{\upsilon}(\theta)\,B_{n_j}^{\upsilon}(\theta)\left[E_{n_j}^{\upsilon}(\theta)\right]^{-1}\;;(q=\frac{a^2s^2}{4})$$

$$A_{n_j}^{\upsilon}(\theta)=\sum_{r=0}^{[n_j/2]}\frac{(-1)^r\,b_{n_j-\upsilon,-r}^{\upsilon}(\theta)}{r!\Gamma(n_j+h+1-r)}\left[\sum_{r=0}^{\infty}\frac{(-1)^r\,b_{n_j-\upsilon,-r}^{\upsilon}(\theta)}{r!\Gamma(1-n_j-h-r)}\right]^{-1}$$
$$B_{n_j}^{\upsilon}(\theta)=\left(-1\right)^{[n/2]}\sum_{r=-[\frac{n}{2}]}^{\infty}(-1)^r\,b_{n_j-\upsilon,-r}^{\upsilon}(\theta)-K_{n_j+2r+h}(2\sqrt{a})\qquad(5.12)$$

$$E_{n_j}^{\upsilon}(\theta)=\frac{1}{(n_j)!}\sum_{r=-[n_j/2]}^{\infty}(-1)^r\frac{\left(n_j+2r\right)!}{\Gamma(n_j+2h+2r)}b_{n_{j-\upsilon,r}}^{\upsilon}(\theta)+\frac{1}{\Gamma(n_j+2r)}\sum_{r=-\infty}^{-[n_j/2]-1}(-1)^r\,b_{n_{j-\upsilon,r}}^{-\upsilon}(\theta)$$

and

$$\alpha_{n_j}=\begin{cases}1\;\;, & n_j\quad even\\-1\;, & n_j\quad odd\end{cases}$$

The values of $b_{\ell,r}^{\mu}(\theta)$ can be determined from trinomial recursion relationships [27,28], such that $b_{\ell,r}^{\upsilon}(\theta)=b_{-\ell-1,0}^{\upsilon}(\theta)=b_{\ell,0}^{-\upsilon}(\theta)$; $b_{\ell,0}^{\upsilon}(\theta)=1$, $\ell=n_j-\upsilon$

Also, $b_{\ell,r}^{\upsilon}(\theta)=0$; $r\le-\left[\frac{n}{2}\right]-1$

and

$$b_{\ell,r}^{\upsilon}(\theta)=b_{-\ell-1,r}^{\upsilon}(\theta)=\frac{\Gamma(1+\upsilon+)\,\Gamma(1-\upsilon+2r+1)}{\Gamma(1-\upsilon+1)\;\;\Gamma(1+\upsilon+2r+1)}b_{\ell,r}^{-\upsilon}(\theta)\qquad(5.13)$$

The values of $K_{\ell}^{\upsilon}(\theta)$ in given by Eq. (29), p.175 of [27]

5.5 Finally, if we define the domain

$$\Omega = \{(x, y, z) \in \Omega : |y| > a, -\infty < x < \infty, -\infty < z < 0\},$$

we can have the following spectral relationships

$$\sum_{j=0}^{k} u_j F_{j,k} \int_{-a}^{a} \frac{K_n(|s||y-t|)}{|y-t|^{\upsilon} (a^2 - t^2)^{\upsilon/2}} P_{s_{nj}-\upsilon}^{\upsilon}(t-\theta) dt = \qquad (5.14)$$
$$= (sign\, y.(y^2 - a^2)^{\upsilon/2} \sum_{j=0}^{k} u_j F_{j,k} \gamma_{n_j} S_{nj-\upsilon}^{\upsilon(3)}(|y|, \theta)$$

where

$$\gamma_{n_j} = (-1)^{1+[n/2]} \beta_{n_j} \sin(\pi h) q^{[nj+h]/2} |2s|^{-h} 2^{1-nj} A_{n_j}^{\upsilon}(\theta) \left[E_{n_j}^{\upsilon}(\theta) \right]^{-1}$$

$$\beta_{n_j} = \begin{cases} \sqrt[4]{\pi^2 q} \quad e^{-i\frac{\pi\upsilon}{2}} & , \quad n_j \quad \text{even} \\ -\sqrt[4]{\pi^2 q} \quad e^{i\pi(1-\upsilon)/2} & , \quad n_j \quad \text{odd} \end{cases}$$

and $S_n^{\upsilon(2)}(y, \theta)$ are the spheroidal wave equations of the third kind.

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