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The Weibull Length Biased Exponential Distribution: Statistical Properties and Applications.

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Abstract

When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. Length biased distribution is thus a special case of the more general form, known as weighted distribution. In this paper, a new three parameter probability distribution called the Weibull Length Biased Exponential distribution is proposed, its statistical properties are studied in minute details. Expansions of the density function of the WLBE Distribution, reliability analysis, asymptotic behavior, moments etc. are discussed in minute details. Maximum likelihood estimation method is employed to determine the estimate of the parameters of the proposed distribution. Simulation studies and application to two life time data is performed to determine the flexibility of the model in modeling lifetime data.

Keywords: Weibull generalized family of distribution, length biased exponential distribution, moments, maximum likelihood, fitted models.

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1. Introduction

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weighted function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. Length biased distribution is thus a special case of the more general form, known as weighted distribution. The concept of length-biased distribution find various applications in biomedical area such as family history and disease survival and intermediate events and latency period of AIDS due to blood transfusion [1]. The study of human families and wildlife populations was the subject of an article developed by Patil and Rao [2]. Patil, et al. [3] presented a list of the most common forms of the weight function useful in scientific and statistical literature as well as some basic theorems for weighted distributions and length-biased as special case. They arrived at the conclusion that the length biased version of some mixture of discrete distributions arises as a mixture of the length biased version of these distributions. Gupta R.D and Kundu D. [4] studied a new class of weighted exponential distribution which has applications in many fields such as: ecology, social and behavioral sciences and species abundance studies. Gupta R.C and Kirmani S. [1], studied the role of weighted distributions in stochastic modeling. Much work was done to characterize relationships between original distributions and their length biased version. A table for some basic distributions and their length biased forms is given by Patil and Rao [2] such as lognormal, Gamma, Pareto, Beta distribution. Khatree [5] presented a useful result by giving a relationship between the original random variable X and its length biased version Y. Recently Mudasir and S.P. Ahmad [6] studied the length biased Nakagami distribution.
Recall that the exponential distribution is given as,

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad x \geq 0, \theta > 0 \tag{1}$$

If the random variable X has distribution function $f(x,\gamma)$, with unknown parameter γ, then the corresponding length-biased distribution is of the form:

$$f^*(x,\theta) = \frac{x^c f(x,\theta)}{\mu_c^{'}} \tag{2}$$

Where,

$$\mu_c^{'} = \int x^c f(x,\gamma) dx \; for\ continous\ series$$

$$and\ \ \mu_c^{'} = \sum x^c f(x,\gamma)\ \ for\ discrete\ series$$

When $c = 1,$ we get length biased distribution
When $c = 2,$ we get area biased distribution
A length biased Weibull distribution is obtained by applying the weights x^c, where

c =1 to the exponential distribution.

A continuous random variable X is said to be exponentially distributed with parameter θ if as in equation (1),

$$f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, \qquad x \geq 0, \theta > 0$$

$$\mu_1^{'} = \int_0^{\infty} xf(x;\theta) = \int x\frac{1}{\theta}e^{-\frac{x}{\theta}}dx = \theta$$

$$\mu_2^{'} = \int_0^{\infty} x^2 f(x;\theta) = \int x^2\frac{1}{\theta}e^{-\frac{x}{\theta}}dx = \theta^2$$

$$f^*(x,\theta) = \frac{x^c f(x,\theta)}{\mu_c^{'}} = \frac{x\frac{1}{\theta}e^{-\frac{x}{\theta}}}{\theta} = \frac{x}{\theta^2}e^{-\frac{x}{\theta}}$$

$$f(x,\theta) = \frac{x}{\theta^2}e^{-\frac{x}{\theta}} \tag{3}$$

This is the required probability density function of the length biased exponential distribution, and its cumulative distribution is given as,

$$F(x,\theta) = 1 - \left(1 + \frac{x}{\theta}\right)e^{-\frac{x}{\theta}} \tag{4}$$

To this regard, our focus in this paper is to present an extension of the Length biased exponential distribution using the Weibull generalized family of distribution [7], the resulting distribution is called the Weibull generalized Length biased exponential distribution, various statistical properties will be looked at. The method of maximum likelihood is discussed for estimating the model parameter, simulation studies and application to two dataset is performed to determine the flexibility of the model in modeling lifetime data.

2. The Weibull-G Length Biased Exponential (WLBE) Distribution

In this section, we express all findings on the Weibull length biased exponential distribution. The cumulative distribution function and probability density function of the Weibull generalized family of distribution is given as,

$$F(x;\alpha,\beta,\theta) = 1 - e^{-\alpha\left\{\frac{G(x;\theta)}{[1-G(x;\Theta)]}\right\}^{\beta}} \tag{5}$$

and

$$f(x) = \alpha\beta g(x)\frac{G(x,\theta)^{\beta-1}}{[1-G(x,\theta)]^{\beta+1}} e^{-\alpha\left\{\frac{G(x;\theta)}{[1-G(x;\Theta)]}\right\}^{\beta}} \tag{6}$$

Respectively.
$for\ x > 0, \alpha > 0, \beta > 0$
By substituting equation (4) into equation (5), we obtain the cdf of the Weibull length biased exponential (WLBE) distribution given by,

$$F_{WLBE}(x) = 1 - exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\} \quad (7)$$

The corresponding probability density function obtained by substituting equations (3) and (4) into equation (6) is given by,

$$f_{WLBE}(x) = \alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\} \quad (8)$$

$\alpha > 0, \beta > 0, \theta = 0$

3. Expansion of the density function of the WLBE Distribution

We describe density function of the WLBE model as a mixture representation in terms of power series expansion.

$$exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\} = \sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta_i}$$

Then the pdf of the WLBE distribution reduces to

$$f_{WLBE}(x) = \alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\} \quad (9)$$

Where,

$$\left[1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\beta(i+1)-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\beta(i+1)-1}{k}\left(1+\frac{x}{\theta}\right)^k e^{-kx/\theta}$$

Therefore, we can also write the pdf as follows:

$$f_{WLBE}(x) = \beta\frac{x}{\theta^2}e^{-x/\theta}\sum_{i=0}^{\infty}\frac{(-1)^{i+k}\alpha^{i+1}}{i!}\binom{\beta(i+1)-1}{k}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{k-(\beta(i+1)+1)} \quad (10)$$

Where

$$\left[\left(1+\frac{x}{\theta}\right)\right]^{k-\beta(i+1)-1} = \sum_{j=0}^{\infty}\binom{k-\beta(i+1)-1}{j}\left(\frac{x}{\theta}\right)^{j}.$$

Thus the mixture representation of the WLBE distribution is given by

$$f_{WLBE}(x) = \beta\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+k}\alpha^{i+1}}{i!}\binom{\beta(i+1)-1}{k}\binom{k-\beta(i+1)-1}{j}\frac{x^{j+1}}{\theta^{j+2}}e^{-\frac{x}{\theta}(k-(\beta(i+1))}$$

$$f_{WLBE}(x) = \beta\sum_{i,j,k=0}^{\infty}\delta_{ijk}\frac{x^{j+1}}{\theta^{j+2}}e^{-\frac{x}{\theta}(k-(\beta(i+1))} \tag{11}$$

Where

$$\delta_{ijk} = \sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+k}\alpha^{i+1}}{i!}\binom{\beta(i+1)-1}{k}\binom{k-\beta(i+1)-1}{j}$$

4. Reliability Analysis for the WLBE Distribution

Expression for the survival function and hazard function of the WLBE Distribution are clearly stated in this section.

The survival function is mathematically expressed as,

$$S_{WLBE}(x) = 1 - F_{WLBE}(x)$$

$$S_{WLBE}(x) = exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\} \tag{12}$$

$\theta > 0, \alpha > 0, \beta = 0$

Also, the hazard or failure rate was derived using,

$$h_{WLBE}(x) = \frac{f_{WLBE}(x)}{S_{WLBE}(x)}$$

This implies that

$$h_{WLBE}(x) = \frac{\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}}{exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}} \tag{13}$$

5. Asymptotic Behavior of the WLBE Distribution

The behavior of the WLBE model in equation (9) is examined as $x \to 0$ and as $x \to \infty$

$$\lim_{x\to 0} f_{WLBE}(x) = \lim_{x\to 0}\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}$$
$$= 0$$
$$\lim_{x\to \infty} f_{WLBE}(x) = \lim_{x\to \infty}\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\} = 0$$

This clearly shows that the Weibull length biased exponential distribution is unimodal. A clear observation of figure 1 shows the WLBE model has only one peak. This supports our claim that the WLBE distribution has only one mode.

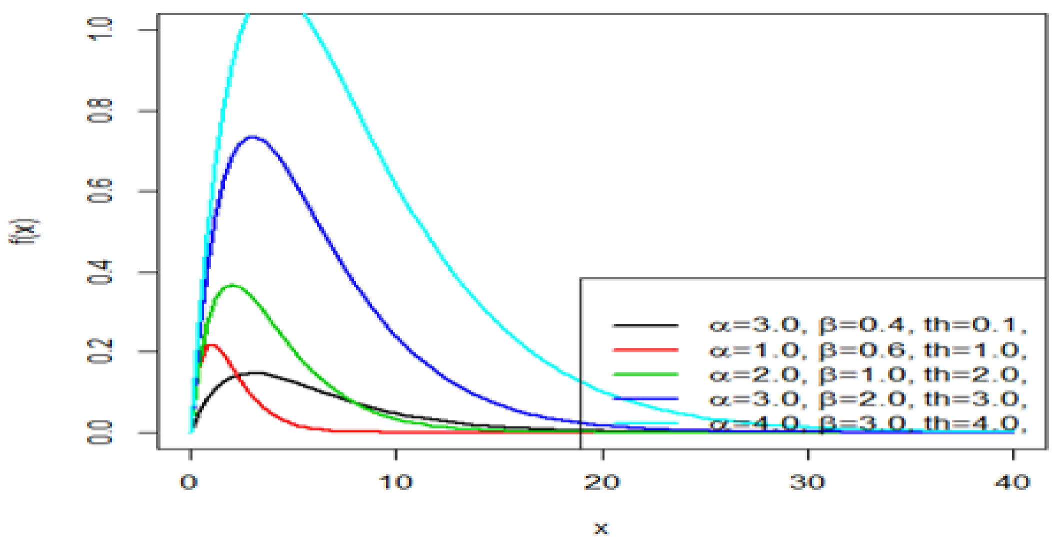

Figure 1: pdf of the Weibull Length Biased Exponential Distribution at selected parameter value

6. Moments

In this section, we derive the moments of the WLBE distribution.
Theorem: Let X be a random variable with WLBE distribution. Then the rth raw moment is

$$\mu_r' = \beta\sum_{i,j,k=0}^{\infty}\delta_{ijk}\frac{\beta^r\Gamma(r+j+2)}{(k-\beta(i+1))^{r+j+2}} \qquad (14)$$

Proof: The r^{th} moment of X of a distribution can be obtained using following integral

$$\mu_r' = \int_0^\infty y^r f(x;\alpha,\beta,\theta)dx$$

$$\mu_r' = \beta \sum_{i,j,k=0}^{\infty} \frac{\delta_{i,j,k}}{\theta^{j+2}} \int_0^\infty y^{r+j+1}\, e^{-\frac{x}{\theta}(k-(\beta(i+1))}dy \tag{15}$$

Let $Z = -\frac{x}{\theta}(k - (\beta(i+1))$. Then, we can write $y = \frac{z\beta}{(k-\beta(i+1))}$ and $dy = \frac{\beta}{(k-\beta(i+1))}dz$

The above equation reduces to

$$\mu_r' = \beta \sum_{i,j,k=0}^{\infty} \frac{\delta_{i,j,k}}{\theta^{j+2}} \int_0^\infty \frac{z^{r+j+1}\beta^{r+j+1}}{\left(k-\beta(i+1)\right)^{r+j+1}} e^{-z} \frac{\beta}{\left(k-\beta(i+1)\right)} dz$$

$$\mu_r' = \beta \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \int_0^\infty \frac{\beta^r}{\left(k-\beta(i+1)\right)^{r+j+2}} \int_0^\infty z^{r+j+1}\, e^{-z} dz \tag{16}$$

7. Moment Generating Function

We define the moment generating function of the WLBE distribution as

$$M_x(t) = \int_0^\infty e^{tx} f(x;\alpha,\beta,\theta)\, dx = \int_0^\infty \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x;\alpha,\beta,\theta)dx \tag{17}$$

$$M_x(t) = \beta \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{t^r}{r!} \int_0^\infty \frac{x^{r+j+1}}{\beta^{j+2}} e^{-\frac{x}{\theta}(k-(\beta(i+1))} dx$$

After thorough integration and simplification, we get the moment generating function of the Weibull Length biased exponential distribution as,

$$M_x(t) = \beta \sum_{i,j,k=0}^{\infty} \delta_{i,j,k} \frac{t^r \beta^r \Gamma(r+j+2)}{r!(k-\beta(i+1))^{r+j+2}} \tag{18}$$

8. Quantile Function and Random Number Generation

If U is a uniform random variable with (0,1), then $x = Q(U)$. The quantile function of X corresponding to the cdf of the WLBE distribution is,

$$x = F^{-1}(u) = \beta \left[\frac{e^{\frac{x}{\beta}}}{\left(1-\log(1-u)^{-\frac{1}{\alpha}}\right)^{-\frac{1}{\beta}}} - 1 \right] \tag{19}$$

Since it is a complex equation, then by iteration method, the equation provides the quantiles and random numbers of the WLBE distribution.

9. Order Statistics

We considered a random sample denoted by $X_1 \dots X_n$ from the densities of the WLBE distribution. Then,

$$f_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!} f_{WLBE}(x) F_{WLBE}(x)^{s-1}[1 - F_{WLBE}(x)]^{n-s} \tag{20}$$

The probability density function of the s^{th} order statistics for the WLBE distribution is given as,

$$f_{s:n}(x) = \frac{n!}{(s-1)!\,(n-s)!}\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}\left(1 - exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}\right)^{s-1}\left[1-\left(1-exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}\right)\right]^{n-s} \tag{21}$$

The WLBE distribution has minimum order statistics given as;

$$f_{1:n}(x) = n\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}\left[1-\left(1-exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}\right)\right]^{n-1} \tag{22}$$

And maximum order statistics given as;

$$f_{n:n}(x) = n\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\frac{\left(1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right)^{\beta+1}}\right]exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}\left(1-exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}{\left(1+\frac{x}{\theta}\right)e^{-x/\theta}}\right]^{\beta}\right\}\right)^{n-1} \tag{23}$$

10. Parameter Estimation

Here we express the maximum likelihood estimate of the parameters of the WLBE model. Let $X_1, \dots X_n$ indicate a random sample of the complete WLBE distribution data, and then the sample's likelihood function is given as,

$$L = \prod_{i=1}^{n} f(x_i; \alpha, \beta, \theta)$$

$$L = \prod_{i=1}^{n} \alpha\beta\theta^{-2} x_i e^{-x_i/\theta} \left[\frac{\left(1-\left(1+\frac{x_i}{\theta}\right)e^{-x_i/\theta}\right)^{\beta-1}}{\left(\left(1+\frac{x_i}{\theta}\right)e^{-x_i/\theta}\right)^{\beta+1}} \right] exp\left\{-\alpha\left[\frac{1-\left(1+\frac{x_i}{\theta}\right)e^{-x_i/\theta}}{\left(1+\frac{x_i}{\theta}\right)e^{-x_i/\theta}}\right]^{\beta}\right\} \quad (24)$$

The log likelihood function may be expressed as,

$$L(x) = nlog\alpha + nlog\beta + \sum log(x) - 2nlog(\theta) - \sum\frac{x}{\theta} + (\beta-1)\sum log\left\{1 - \left(1+\frac{x}{\theta}\right)e^{-\frac{x}{\theta}}\right\} - (\beta+1)\sum log\left(1+\frac{x}{\theta}\right)e^{-\frac{x}{\theta}} - \alpha\sum\left\{\frac{e^{\frac{x}{\theta}}}{\left(1+\frac{x}{\theta}\right)} - 1\right\}^{\beta} \quad (25)$$

By taking the derivative with respect to $\alpha, \beta, and\ \theta$ and fixing the outcome to zero, we have,

$$\frac{\partial L(x)}{\partial\alpha} = \frac{n}{\alpha} - \sum\left\{\frac{e^{\frac{x}{\theta}}}{\left(1+\frac{x}{\theta}\right)} - 1\right\}^{\beta} \quad (26)$$

$$\frac{\partial L(x)}{\partial\beta} = \frac{n}{\beta} - \sum log\left\{1-\left(1+\frac{x}{\theta}\right)e^{-\frac{x}{\theta}}\right\} - \sum log\left\{\left(1+\frac{x}{\theta}\right)e^{-\frac{x}{\theta}}\right\} -\alpha\sum\left\{\frac{e^{\frac{x}{\theta}}}{\left(1+\frac{x}{\theta}\right)} - 1\right\}\left\{\frac{e^{\frac{x}{\theta}}}{\left(1+\frac{x}{\theta}\right)} - 1\right\}^{\beta} \quad (27)$$

$$\frac{\partial L(x)}{\partial\theta} = \frac{2n}{\theta} + \sum\frac{x}{\theta^2} - \alpha\beta\sum\left\{\frac{e^{\frac{x}{\theta}}}{\left(1+\frac{x}{\theta}\right)} - 1\right\}^{\beta-1}\left\{\frac{xe^{\frac{x}{\theta}}}{\theta^2\left(1+\frac{x}{\theta}\right)^2} - \frac{xe^{\frac{x}{\theta}}}{\beta^2\left(1+\frac{x}{\theta}\right)}\right\} +(\beta-1)\sum\frac{xe^{-\frac{x}{\theta}} - xe^{-\frac{x}{\theta}}\left(1+\frac{x}{\theta}\right)}{\theta^2\left[1-\left(1+\frac{x}{\theta}\right)e^{\frac{-x}{\theta}}\right]} - (\beta+1)\frac{\left[x\left(1+\frac{x}{\theta}\right)e^{\frac{-x}{\theta}} - xe^{-\frac{x}{\theta}}\right]}{\theta^2\left(1+\frac{x}{\theta}\right)^2} \quad (28)$$

Non linear equations are solved for $\alpha, \beta, and\ \theta$ by utilizing softwares MATHEMATICA (NMaximize) or R-Language.

11. Simulation using the WLBE Distribution

Now, we simulate n =30, 50, 100 and 300 times the WLBE distribution for $\theta = 0.6, 1, 2;\ \alpha = 0.5\ and\ \beta = 1.5$. We compute the ML estimates of parameters for each sample size. 10000 repetitions are obtained and then the Bias and MSE are computed. The values such obtained are used for comparison of the performance of ML estimators, for same values of 'α' and β and different values of β and are given in Table 1.

Table 1: Estimated bias and MSE for several values of the parameter

n		$\theta = 0.6$		$\theta = 1$		$\theta = 2$	
		$\alpha = 0.5$	$\beta = 1.5$	$\alpha = 0.5$	$\beta = 1.5$	$\alpha = 0.5$	$\beta = 1.5$
30	Bias	0.0888	0.1193	0.1152	0.2451	0.1076	0.0302
	MSE	0.3602	0.5236	0.401	0.4834	0.4123	0.3920
50	Bias	0.0293	0.0992	0.0301	0.2182	0.0394	0.0274
	MSE	0.2403	0.4452	0.2890	0.231	0.2742	0.2511
100	Bias	0.020	0.0723	0.0241	0.1092	0.0142	0.0192
	MSE	0.1832	0.3029	0.1342	0.1632	0.1720	0.2013
300	Bias	0.0156	0.0623	0.0118	0.0981	0.0092	0.0095
	MSE	0.0921	0.0251	0.0913	0.1024	0.092	0.1293

It is observed from table 1 that

- The Mean square error decreases as the sample size increases.
- Bias also decreases as sample size increases.

12. Application of Weibull Length Biased Exponential (WLBE) Distribution to dataset.

Here, the importance and flexibility of the Weibull length biased exponential distribution by comparing the results of the model fit with some new and existing distribution. Two dataset were used in this study to compare between fits of the Weibull length biased exponential distribution (WLBE) with that of Beta Length Biased Exponential (BLBE), Exponentiated Generalized Length Biased Exponential (ELBE), Log Gamma Length Biased Exponential (LGALBE), Gamma Length Biased Exponential (GALBE), Length Biased Exponential (LBE), and Generalized Exponential (GE) distribution. DATA V was used.

<u>For DATA I</u>

The first data set represents the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile. The data set has been previously used by Oguntunde et al., [8], Obubu et al [9]. Apart from its application in discriminating between the Generalised Inverse Generalised Exponential distribution, Inverse Generalised Exponential distribution and the Inverse Exponential distribution, it has also been used to assess the superiority of the Exponentiated Generalised Exponential distribution over the Exponentiated Exponential distribution, Generalised Exponential distribution and Exponential distribution. The dataset consists of fifty-two (52) observations out of which twenty-one (21) are censored observations. The data set is as follows:

1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61*, 65, 67, 70, 72, 73, 74*, 77, 79*, 80*, 81*, 87*, 87*, 88*, 89*, 91, 93*, 96, 97*, 100, 101*, 104, 104*, 108*, 109*, 120*, 131*, 150*, 157, 167, 231*, 240*, 400*
NOTE: * denote censored observations

Table 2: Descriptive Statistics on the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis	Variance
0.550	1.375	1.590	1.507	1.685	2.240	0.999263	3.923761	0.1050575

From the above, we observed that the data is positively skewed with variance of 0.1050575

Table 3: MLEs, of parameters on the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile

Model	Estimates		
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
WLBE (Proposed Model)	2.963082621	0.003192285	19.832010734
BLBE (New)	5.063632	96318.220569	149.708914
ELBE	2.930833e+05	5.499483e+00	3.873680e+02
GALBE (New)	0.6616500	0.5374506	
LGALBE (New)	5.048596e+00	3.457120e+05	2.846806e+02
LBE	12.9249779	-	-
GE	31.34891	2.61157	-

For all competing distributions using the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile, Table 3 shows parameter estimate.

Table 4: Log-likelihood, AIC, AICC, and BIC values of models fitted for the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile

Model	Negative LL	AIC	BIC	CAIC
WLBE (Proposed Model)	15.27429	36.54859	42.97799	36.95537
BLBE (New)	20.88664	47.77327	54.20268	48.18005
ELBE	23.95317	53.90635	60.33575	54.31313
GALBE (New)	64.47559	134.952	139.2383	135.152
LGALBE (New)	20.86643	47.73286	54.16227	48.13964
LBE	30.08071	64.16143	68.4477	64.36143
GE	31.38347	66.76694	71.05321	66.96694

From Table 4, the WLBE has the highest log-likelihood values and the lowest AIC, CAIC, and BIC values; hence it is chosen as the most appropriate model amongst the considered distributions, implying that it provides a better fit than the Beta Length Biased Exponential (BLBE), Exponentiated Generalized Length Biased Exponential (ELBE), Log Gamma Length Biased Exponential (LGALBE), Gamma Length Biased Exponential (GALBE), Length Biased Exponential (LBE), and Generalized Exponential (GE) distribution.

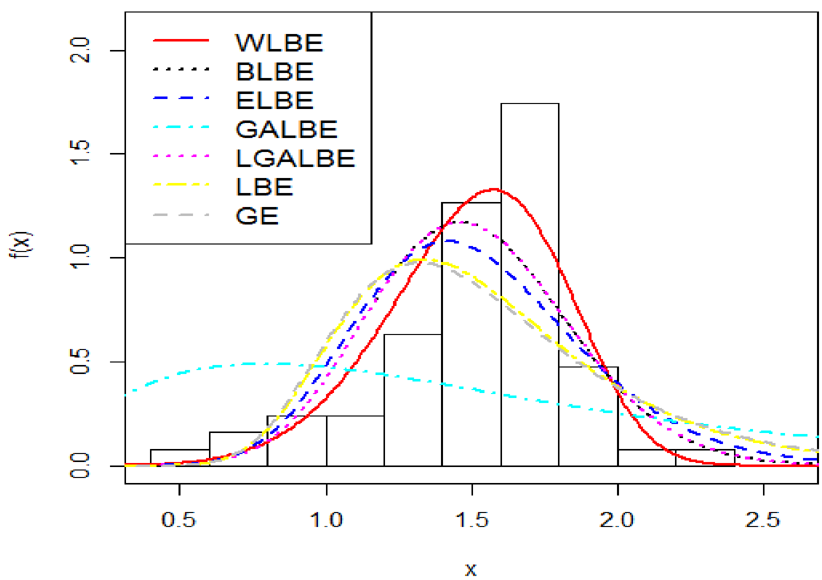

Figure 2: Histogram of the fitted distributions

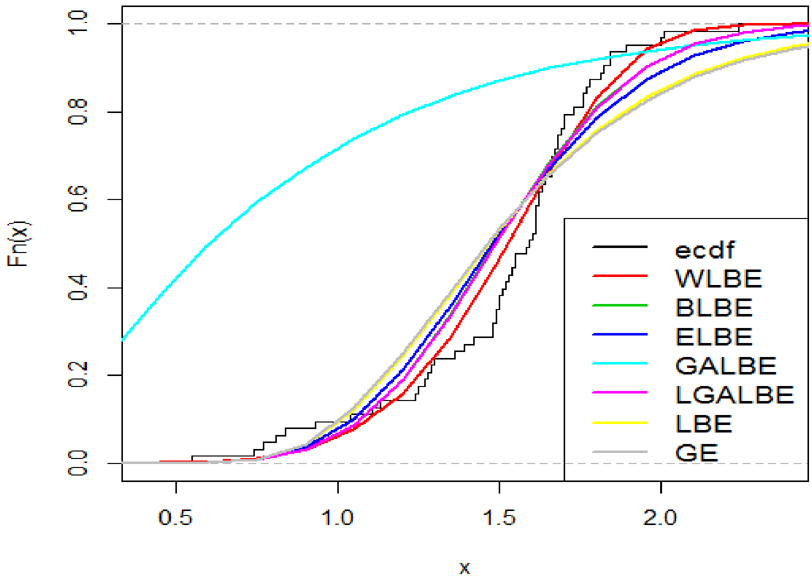

Figure 3: Empirical cdf of the fitted distributions

<u>For DATA II</u>

The sixth data set represents the time (in years) it took for 10 different countries to completely eliminate malaria. The data set was obtained from the World malaria report, 2019. The data set is as follows:

7, 3, 3, 3, 6, 3, 5, 4, 1, 3

Table 5: Descriptive Statistics on the times (in years) it took to eliminate malaria from 10 different countries in the world.

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis	Variance
1.00	3.00	3.00	3.80	4.75	7.0	0.4501384	2.545054	3.066667

From the above, we observed that the data is positively skewed with variance of 3.066667

Table 6: MLEs, of parameters on the times (in years) it took to eliminate malaria from 10 different countries in the world.

Model	Estimates		
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
WLBE (Proposed Model)	1.3825555	0.1022386	8.1333987
BLBE (New)	1.403846	13926.238947	290.527878
ELBE	42276.110180	1.430762	538.728872
GALBE (New)	0.6626756	1.3488160	
LGALBE (New)	1.43416	822.80909	68.57389
LBE	2.545748	-	-
GE	5.602514	0.620932	-

For all competing distributions using the times (in years) it took to eliminate malaria from 10 different countries in the world, Table 6 shows parameter estimate.

Table 7: Log-likelihood, AIC, AICC, and BIC values of models fitted for the times (in years) it took to eliminate malaria from 10 different countries in the world

Model	Negative LL	AIC	BIC	CAIC
WLBE (Proposed Model)	18.97721	43.76643	44.57218	47.56343
BLBE (New)	18.98361	43.96722	44.87498	47.96722
ELBE	18.98664	43.97327	44.88103	47.97327
GALBE (New)	20.52712	45.05424	45.65941	46.76853
LGALBE (New)	18.98646	43.97292	44.88067	47.97292
LBE	19.24976	42.49953	43.1047	44.21381
GE	19.34369	42.68737	43.29254	44.40166

From Table 7, the WLBE has the highest log-likelihood values and the lowest AIC, CAIC, and BIC values; hence it is chosen as the most appropriate model amongst the considered distributions, implying that it provides a better fit than the Beta Length Biased Exponential (BLBE), Exponentiated Generalized Length Biased Exponential (ELBE), Log Gamma Length Biased Exponential (LGALBE), Gamma

Length Biased Exponential (GALBE), Length Biased Exponential (LBE), and Generalized Exponential (GE) distribution.

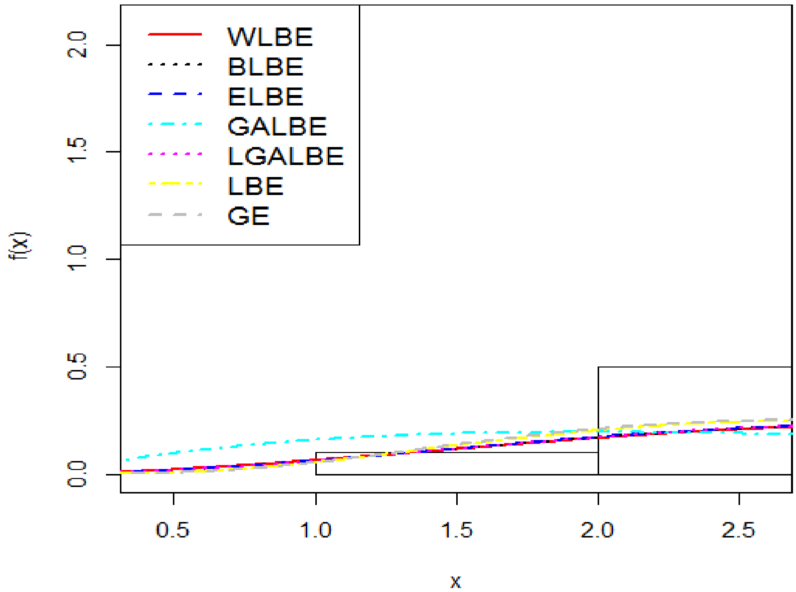

Figure 4: Histogram of the fitted distributions

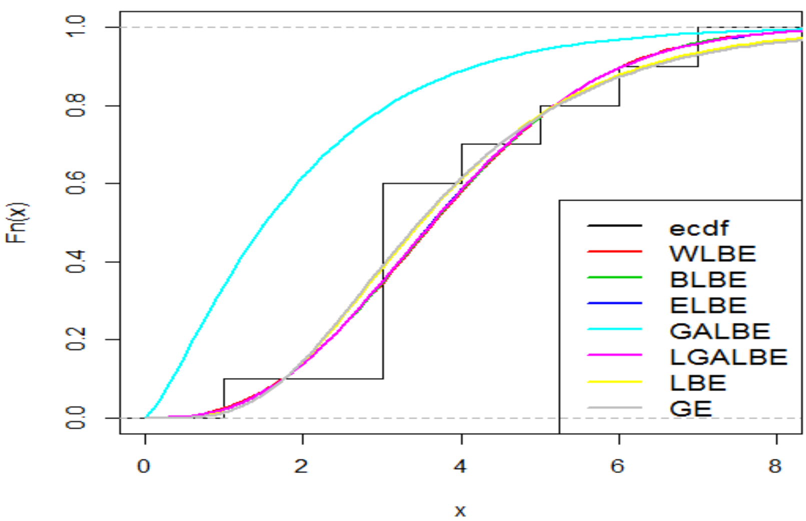

Figure 5: Empirical cdf of the fitted distributions

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