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Model of warfare

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Abstract

The aim of the present work is the calculation of the surviving combat units of three tactical engagements of homogeneous forces. For this purpose, we describe the deterministic mathematical models for tactical engagements of heterogeneous forces between regular military forces. It should be noted that a homogeneously armed military power is held by a type of weapon system, e.g. tanks. These models assume that there is a positive number expressing the firepower of the military units, as well as how the two forces have equivalent efficacy in the use of weapons systems. These models are of theoretical interest and describe ancient battles, while failing to reflect modern military conflicts. Instead, heterogeneously forces have different attrition rates, which are defined by the type of weapons, the types of enemy targets and various factors.

Keywords: model of warfare; Lanchester's equation; differential equations

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1 Introduction

Strategy and tactic of opponents can be considered as two main characteristic elements of a general combat. The first general approach (military, political, etc) is designed to achieve an exhaustion of the enemy, while the second refers to more specific approaches in order to achieve specific objectives, which serve the general strategy. The purpose of this paper is the calculation of the surviving combat units of three tactical engagements of homogeneous forces at time t as well as the description of the deterministic mathematical models describing the battles between tactical engagements of heterogeneous forces. It should be mentioned that a tactical engagement of homogeneous force is one that holds a type weapon system, e.g. tanks. Mathematical models of tactical engagements of homogeneous forces assume that a positive number exists representing the firepower of military units, as well as how the forces have equivalent efficacy in the use of their weapon systems. These models, mainly of theoretical interest, are considered suitable for describing battles of antiquity, and usually fail to reflect the modern military conflicts. Conversely, tactical engagements of heterogeneous forces have different attrition rates, determined by the type of weapon systems, the type of enemy targets and various other factors ([25]).

A pioneer in the study of the theory of combat is Frederick Lanchester (1868-1946). The deterministic mathematical modeling of a battle between tactical engagements of heterogeneous forces based on the approach of attrition rates, are considered to be real functions of both the weapon characteristics, and the fires to the opponent force. The direct solutions of the deterministic differential equations focus on the determination of the force strengths at any time t of the battles when the initial force strengths condition, attrition coefficients and reinforcement schedule are specified. The attrition rates are estimated difficult (see [7]). Lanchester's model was first developed as a description of *air combat*, in which each side was essentially composed of a single type of combat element. Force strength was then considered a simple matter of counting the number of aircraft in a side. Modern applications of Lanchester's ideas to *land combat* run into the problem that each side consists of a number of types of combat element (infantry, artillery, tanks etc) each of which interacts differently with each of the opposing sides' combat types. The development of heterogeneous combat models is a central issue in most current military combat simulations. It is important to remember that Lanchester's equations are not a model of combat, only a model for combat attrition. The equations alone, therefore, cannot be expected to capture other effects such as the movement of engaged forces. This is frequently forgotten. There have been numerous attempts to compare historical combat data with the behaviour expected from Lanchester's equations, including the work of Helmbold and Hartley (see [14], [15], [16]). Hartley also includes a comprehensive review of the effort to validate combat attrition laws using historical analysis. A work by the author has also investigated the ability of Lanchester's equations to describe patterns observed in the casualty statistics using Hartley's database of historical battles. This includes an examination of the inclusion of a fractal model of spatial dispersion on casualty values and the distribution of casualties when Lanchester's equations are modeled as stochastic processes.

According to the simplest Lanchester type models, various assumptions are usually made (see [10] for details). However, these are not the only conditions, under which these attrition laws can be obtained. For example, it has been shown by Brackney (see [5]), that when the area held is varied so that the density of the forces in the area remains constant, attrition rates of the square-law type also result. Helmbold claimed that "victory in battle is primarily determined by factors other than numerical superiority, and challenges the ability of any model of combat which concentrates almost exclusively on numerical force size to yield a practically useful predictor of victory in battle" (see [16]). Willard reached similar conclusions and raised an additional theory on attrition rates (see [29]). The relevance of differential equations to modern warfare is a subject of much controversy (see [2]). This can be explained by the fact that Lanchester's equations do not contain any information about the spatial distribution of armies or their movement. Location matters greatly in the evolution and state of a struggle, whether some entity is fighting a war, defining its marketing campaign or its vaccinations programs. The capacity of modeling different spatial settings in a consistent and stable manner is crucial at the time of adopting an army strategy, since it allows the modeler to take into account local battles and disaggregated allocation of resources, but at the same time to keep in mind the global strategy (see [13], [30]).

2 Lanchester-type equations with area fire

During World War I, Frederick W. Lanchester, a British engineer in the Royal Air Force, developed his theory, based on aircraft engagements, to explain why concentration of forces was useful in modern warfare (see [22], [23]). In 1916, he published a series of differential equations today known as *Lanchester's power laws* [26]. These equations were analyzed concurrently and independently by Osipov, who published a series of articles where he gives a remarkable analysis of the Lanchester's square law (see [17]). Suppose that Red army and Blue army engage each other in combat. Both sides use area fire and target acquisition times remain constant, independent of the force levels (a special case is when they are negligible). In any case, both sides tend to use a constant density defense. These two heterogeneous forces have a density of weapons units $M = M(t)$ and $K = K(t)$, respectively, as a function of time $t \geq 0$. Therefore, attrition is proportional to the product of army populations

$$\frac{dK(t)}{dt} = -\mu M(t)K(t), \qquad \frac{dM(t)}{dt} = -\kappa M(t)K(t), \tag{2.1}$$

where κ and μ correspond to the *attrition rates* of the armies $K(t)$ and $M(t)$, respectively, while they frame the efficiency of the firepower of the two armies and they are particularly assumed as constant and positive numbers. Concerning the area of fire equations (2.1), we obtain Lanchester's linear law

$$\frac{K_0 - K(t)}{M_0 - M(t)} = \frac{\mu}{\kappa}, \tag{2.2}$$

for all $t \geq 0$, where $K(0) = K_0 > 0$ and $M(0) = M_0 > 0$ constitute the initial conditions. Differential equations (2.1) follow the principle of mass action, as in the right part of the equations someone can find the product of the two involved armies $K(t)$ and $M(t)$. In case the outcome is typically a stalemate and there isn't any winner, then $K(t) = M(t) = 0$ and we have the *stalemate condition*

$$\Delta_1 = \kappa K_0 - \mu M_0 = -\mu \left(M_0 - \frac{\kappa}{\mu} K_0\right) = 0. \tag{2.3}$$

The equilibrium conditions between the two armies are:

(i) If $\left(M_0 - \frac{\kappa}{\mu} K_0\right) > 0$, i.e. if $\Delta_1 < 0$, then the army of Red will dominate and for $t \to +\infty$ we eventually have $M(t) \to \left(M_0 - \frac{\kappa}{\mu} K_0\right)$ and $K(t) \to 0$.

(ii) If $\left(M_0 - \frac{\kappa}{\mu} K_0\right) < 0$, i.e. if $\Delta_1 > 0$, then the army of Blue will dominate and for $t \to +\infty$ we have $M(t) \to 0$ and $K(t) \to \left(K_0 - \frac{\mu}{\kappa} M_0\right)$.

(iii) If $\left(M_0 - \frac{\kappa}{\mu} K_0\right) = 0$, then there wouldn't be any survivor and the outcome is a stalemate and the general solution is $M(t) = \frac{1}{\mu t + \frac{1}{M_0}}$, and $K(t) = \frac{1}{\kappa t + \frac{\kappa}{\mu M_0}}$.

3 Lanchester-type equations with aimed fire

The deterministic Lanchester-type equations with aimed fire are

$$\frac{dK(t)}{dt} = -\mu M(t), \qquad \frac{dM(t)}{dt} = -\kappa K(t), \tag{3.1}$$

where κ and μ are the *attrition rates* of the armies $K(t)$ and $M(t)$, respectively, assumed as constant and positive numbers, since their values aren't *a priori* known. The autonomous system of differential equations (3.1) can be written as

$$\frac{dX(t)}{dt} = \Theta X(t), \tag{3.2a}$$

where

$$X(t) = [K(t), M(t)]^T \quad \text{and} \quad \Theta = \begin{bmatrix} 0 & -\mu \\ -\kappa & 0 \end{bmatrix}, \tag{3.2b}$$

where Θ is the *total attrition coefficient*, with $det\Theta = -\kappa\mu$. Based on equations (3.1) we have *Lanchester's square law*

$$\frac{K_0^2 - K^2(t)}{M_0^2 - M^2(t)} = \frac{\mu}{\kappa}, \tag{3.3}$$

for all $t \geq 0$, where $K(0) = K_0 > 0$ and $M(0) = M_0 > 0$ are the initial conditions. The term $\sqrt{\kappa\mu}$, representing, in particular, the positive eigenvalue of the total attrition coefficient Θ, defines *battle intensity*, since the terms $\sqrt{\mu/\kappa}$ and $\sqrt{\kappa/\mu}$ define *relative effectiveness*. Lanchester claimed that the system (3.1) describes real battles (see [27]). Despite the simplicity of Lanchester's model (3.1), the strategy of decision-making in a battlefield turns to be a complex procedure (see [31]). The origin $(0,0)$ is a saddle point. If $t = t_f$ is the time that combat ends and one (or both) side(s) is annihilated (see [9]), then the possible outcomes can be: (a) the army $K(t)$ becomes the winner if $K(t_f) > 0$ and $M(t_f) = 0$, (b) the army $M(t)$ is the winner of the battle if $M(t_f) > 0$ and $K(t_f) = 0$, (c) a draw with no survivors occurs if $M(t_f) = 0$ and $K(t_f) = 0$. In case (c), from (3.3) there is the *stalemate condition*

$$\Delta_1 = \kappa K_0^2 - \mu M_0^2 = 0. \tag{3.4}$$

Generally, a number of analysts have not been faithful to the compelling logic of the battlefield that Lanchester represented in mathematical form (see [20], [26]).

4 Deitchman's model

Several studies have proposed that Lanchester's models of human combat may describe conflicts among social animals, including vertebrates and insects (see [4], [15], [19], [20], [31]). Many social animals fight in groups, incurring substantial mortality (see [1], [6]). Supposing that two heterogeneous forces have a density of weapons units $M = M(t)$ and $K = K(t)$, respectively, as a function of time $t \geq 0$ and are engaging each other in combat. Army $M(t)$ uses area or direct fire, while target acquisition times are constant, acting independently of the force levels (a special case is when they are negligible). Both sides use a constant density defense. The army $K(t)$ uses direct fire and causes casualties in the opponent's military force, which is proportional to the number of remaining weapons units of the army $M(t)$. Army $M(t)$ is a tactical homogenous force, since army $K(t)$ is a guerilla force. Deitchman proposed the following model

$$\frac{dK(t)}{dt} = -\mu M(t)K(t), \qquad \frac{dM(t)}{dt} = -\kappa K(t), \tag{4.1}$$

with $K(0) = K_0$ and $M(0) = M_0$ (see [9]).

The attrition rates μ and κ of the armies $K(t)$ and $M(t)$, respectively, frame the efficiency of the firepower of the two armies and are especially assumed as constant and positive numbers. From equations (3.1):

$$M^2(t) - \frac{2\kappa}{\mu}K(t) = M_0^2 - \frac{2\kappa}{\mu}K_0, \tag{4.2}$$

for all $t \geq 0$.

5 General formulation

A generalized form of Lanchester’s equations is represented in the following schema

$$\frac{dK(t)}{dt} = -\mu M^{\mu_1}(t)K^{\kappa_1}(t), \qquad \frac{dM(t)}{dt} = -\kappa M^{\mu_2}(t)K^{\kappa_2}(t), \tag{5.1}$$

where κ, $\mu > 0$ are the attrition coefficients, $K(0) = K_0 > 0$ and $M(0) = M_0 > 0$ the initial conditions, since μ_1, μ_2, κ_1 and κ_2 are stoichiometric coefficients (see [4], [11], [12], [14]). Although Lanchester's models have been used in several physical procedures, the attempts made to fit them with historical data have resulted in a mixed success (see [21]).

For example, if we have the values $\mu_1 = \mu_2 = \kappa_1 = \kappa_2 = 1$, then we obtain Lanchester's model (2.1) with area fire.

If we have the values $\mu_1 = \kappa_2 = 1$ and $\mu_2 = \kappa_1 = 0$, then we obtain Lanchester's model (3.1) with aimed fire. If we have the values $\mu_1 = \kappa_1 = \kappa_2 = 1$ and $\mu_2 = 0$, then we obtain Deitchman's model (4.1). If

$$\lambda_\kappa = \kappa_2 - \kappa_1 \qquad \text{and} \qquad \lambda_\mu = \mu_1 - \mu_2, \tag{5.2}$$

where λ_κ and λ_μ are the net predation benefit parameters of the two armies, then

$$\frac{K^{1+\lambda_\kappa} - K_0^{1+\lambda_\kappa}}{M^{1+\lambda_\mu} - M_0^{1+\lambda_\mu}} = \frac{\mu}{\kappa}\left(\frac{1+\lambda_\kappa}{1+\lambda_\mu}\right). \tag{5.3}$$

If $\lambda_\kappa = \lambda_\mu = 0$, then we obtain the Lanchester's linear law (2.2), since for $\lambda_\kappa = \lambda_\mu = 1$ we obtain the Lanchester's square law (3.3). In case that the battle comes to a stalemate and there is no a winner, then $K(t) = M(t) = 0$ and we obtain the following stalemate condition

$$\Delta_2 = \kappa K_0^{1+\lambda_\kappa} - \left(\frac{1+\lambda_\kappa}{1+\lambda_\mu}\right)\mu M_0^{1+\lambda_\mu} = 0. \tag{5.4}$$

If $\Delta_2 > 0$, then $K(t)$ wins, since if $\Delta_2 < 0$, then $M(t)$ wins (see [12]). Hartley suggested that $\mu_1 = \kappa_2 = 0.75$ and $\mu_2 = \kappa_1 = 0.4$ (see [16]).

6 A model of warfare

Supposing that three tactical engagements of heterogeneous forces have a density of weapons units $A = A(t)$, $B = B(t)$ and $C = C(t)$, respectively, as a function of time $t \geq 0$ and are engaging each other in combat. These armies use aimed fire and target acquisition times are constant, acting independently of the force levels (a special case is when they are negligible). Both sides use a constant density defense. Therefore, we suppose that each army's rate of loss is proportional to the concentration of the other two armies. Then, the following autonomous system of differential equations is proposed

$$\frac{dA}{dt} = -bB - cC, \qquad (6.0.1a)$$

$$\frac{dB}{dt} = -aA - cC, \qquad (6.0.1b)$$

$$\frac{dC}{dt} = -aA - bB, \qquad (6.0.1c)$$

where $A = A(t)$, $B = B(t)$ and $C = C(t)$ represent the concentrations of the enemy troops and a, b and c correspond to the attrition rates of the armies A, B and C, respectively, assumed as constant and positive numbers. The initial conditions are: $A(0) = A_0 > 0$, $B(0) = B_0 > 0$, $C(0) = C_0 > 0$. We can write system (6.0.1) as follows

$$\frac{dX}{dt} = \Theta X, \qquad (6.0.2)$$

where $X = [A, B, C]^T$ is the concentration vector of the three weapons units and

$$\Theta = \Theta(a, b, c) = \begin{bmatrix} 0 & -b & -c \\ -a & 0 & -c \\ -a & -b & 0 \end{bmatrix},$$

is the *total attrition coefficient*. Also, it is: $det\Theta = -2abc < 0$ and $tr\Theta = 0$. The eigenvalues λ_i, $i = 1, 2, 3$, can be calculated by the characteristic equation

$$\lambda^3 - (ab + bc + ca)\lambda + 2abc = 0. \qquad (6.0.3)$$

Via Vieta's formula, there is for equation (6.0.3): $(\lambda_1 + \lambda_2 + \lambda_3) = 0$ and $\lambda_1\lambda_2\lambda_3 = -2abc = |\Theta|$. In phase portrait, the origin $(A, B, C) = (0,0,0)$ is the only equilibrium point of the system (6.0.1) and the Jacobian matrix $J(A, B, C)$ is

$$J(A, B, C) = \begin{bmatrix} 0 & -b & -c \\ -a & 0 & -c \\ -a & -b & 0 \end{bmatrix} = \Theta.$$

The Jacobian matrix J in origin is $J(0,0,0) = \Theta$, since J and Θ have the same eigenvalues.

6.1 Equal attrition rates: $a = b = c$

Suppose that the three armies A, B and C have equal attrition rates, that presupposes that they are tactical engagements of homogeneous forces. For the case $a = b = c$, system (6.0.1) is attributed the form (6.0.2), with $\Theta = \Theta(a, a, a)$, since the characteristic equation of Θ is $(\lambda + 2a)(\lambda - a)^2 = 0$. The matrix Θ has the following eigenvalues: $\lambda_1 = -2a < 0$, with algebraic multiplicity 1, and $\lambda_2 = a > 0$ with algebraic multiplicity 2. If u_1 is the corresponding eigenvector of the eigenvalue $\lambda_1 = -2a$, then $u_1 = [1,1,1]^T$. Similarly, for the eigenvalue $\lambda_2 = a$ we have the following eigenvectors: $u_{21} = [-1,1,0]^T$ and $u_{22} = [-1,0,1]^T$. The general solution of the system is

$$X(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_{21} + c_3 e^{\lambda_2 t} u_{22}, \qquad (6.1.1)$$

with c_1, c_2 and c_3 arbitrary coefficients, which can be calculated from the initial conditions $A(0) = A_0$, $B(0) = B_0$ and $C(0) = C_0$. Then, we have

$$\begin{cases} c_1 = \frac{1}{3}(A_0 + B_0 + C_0) \\ c_2 = \frac{1}{3}(2B_0 - A_0 - C_0). \\ c_3 = \frac{1}{3}(2C_0 - A_0 - B_0) \end{cases} \tag{6.1.2}$$

If we consider

$$z(t) = A(t) + B(t) + C(t) = 3c_1 e^{-2at},$$

where $z = z(t)$ is the total concentration of the three armies, then:

$$\lim_{t \to +\infty} z(t) = 0,$$

which means that noone will survive at the end of the battle. We can get the following system

$$\frac{dA}{dC} = \frac{B+C}{A+B}, \qquad \frac{dB}{dC} = \frac{A+C}{A+B}, \tag{6.1.3}$$

and if we add the two equations of the system (6.1.3)

$$\frac{d}{dC}(A + B) = 1 + \frac{2C}{(A+B)}. \tag{6.1.4a}$$

A cyclic rotation of the symbols A, B and C gives

$$\frac{d}{dC}(B + C) = 1 + \frac{2A}{(B+C)}, \qquad \frac{d}{dC}(C + A) = 1 + \frac{2B}{(C+A)}. \tag{6.1.4b}$$

If we solve the differential equations (6.1.4), we get

$$\begin{cases} \frac{A_0+B_0+C_0}{A+B+C} = \left(\frac{A+B-2C}{A_0+B_0-2C_0}\right)^2 \\ \frac{A_0+B_0+C_0}{A+B+C} = \left(\frac{B+C-2A}{B_0+C_0-2A_0}\right)^2. \\ \frac{A_0+B_0+C_0}{A+B+C} = \left(\frac{C+A-2B}{C_0+A_0-2B_0}\right)^2 \end{cases} \tag{6.1.5}$$

If we asume: $A = B = C = 0$, and from (6.1.5) arises that

$$A_0 + B_0 = 2C_0, \qquad B_0 + C_0 = 2A_0, \qquad C_0 + A_0 = 2B_0.$$

Then, we have the following three stalemate conditions

$$\begin{cases} \Delta_1 = A_0 + B_0 - 2C_0 \\ \Delta_2 = B_0 + C_0 - 2A_0. \\ \Delta_3 = C_0 + A_0 - 2B_0 \end{cases} \quad (6.1.6)$$

Based on (6.1.6), if a battle is held between three forces, with equal attrition rates, then the winner is determined only by the initial concentrations. According to (6.1.5), if we are generally aware of the array between the three initial concentrations (for example, if $A_0 > B_0 > C_0$), then the same array of the concentrations replies for $t \geq 0$ ($A > B > C$, respectively). (6.1.6) arises the following properties:

(i) If two of the coefficients Δ_i ($i = 1, 2, 3$) are equivalent to zero, then the third coefficient is also zero. That means: $A = B = C = 0$ for $t \to \infty$.

(ii) If two of the coefficients Δ_i ($i = 1, 2, 3$) are nonpositive numbers, then the third coefficient is a positive.

(iii) If two of the coefficients Δ_i ($i = 1, 2, 3$) are positive numbers, then the third coefficient is a nonpositive.

(iv) If one of the coefficients Δ_i ($i = 1, 2, 3$) is zero and the second coefficient is a positive (or nonposotive) number, then the third coefficient is a nonpositive (or positive, respectively) number.

6.2 Two equal attrition rates: $a \neq b = c$

Supposing that the armies B and C have the same attrition rate (they are tactical engagements of homogeneous forces), the linear system (6.0.1) possesses the form (6.0.2), with $\Theta = \Theta(a, b, b)$ and $det\Theta = -2ab^2$, since the characteritic equation (6.0.3) is

$$\lambda^3 - b(b + 2a)\lambda + 2ab^2 = 0. \quad (6.2.1)$$

The matrix Θ has the three distinguished real eigenvalues: $\lambda_1 = b > 0$, $\lambda_2 = -\frac{\sqrt{b}}{2}\left(\sqrt{8a+b}+\sqrt{b}\right) < 0$ and $\lambda_3 = \frac{\sqrt{b}}{2}\left(\sqrt{8a+b}-\sqrt{b}\right) > 0$. Conclusively, the general solution is

$$X(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 + c_3 e^{\lambda_3 t} u_3, \qquad (6.2.2)$$

where

$$u_1 = [0,-1,1]^T, \qquad u_2 = \left[\frac{\lambda_3}{a}, 1,1\right]^T, \qquad u_3 = \left[\frac{\lambda_2}{a}, 1,1\right]^T,$$

and c_1, c_2 and c_3 arbitrary constants, which can be determined from the initial conditions $A(0) = A_0$, $B(0) = B_0$ and $C(0) = C_0$. So, we get

$$\begin{cases} c_1 = \frac{1}{2}(C_0 - B_0) \\ c_2 = \frac{1}{(\lambda_2-\lambda_3)}\left[\frac{\lambda_2}{2}(B_0 + C_0) - aA_0\right]. \\ c_3 = \frac{1}{(\lambda_2-\lambda_3)}\left[aA_0 - \frac{\lambda_3}{2}(B_0 + C_0)\right] \end{cases} \qquad (6.2.3)$$

Based on the equations of (6.0.2) we have

$$\frac{dB}{dA} = \frac{aA+bC}{b(B+C)}, \qquad \frac{dC}{dA} = \frac{aA+bB}{b(B+C)} \qquad (6.2.4)$$

and if we decide to add the two equations of the system (6.2.3) we get

$$\frac{d}{dA}(B+C) = 1 + \frac{2a}{b}\frac{A}{(B+C)}. \qquad (6.2.5)$$

From (6.2.5) we have the following *stalemate condition*

$$\left(\frac{B+C-z_1 A}{B_0+C_0-z_1 A_0}\right)^{z_1} = \left(\frac{B_0+C_0+z_2 A_0}{B+C+z_2 A}\right)^{z_2}, \qquad (6.2.6)$$

where

$$z_1 = \frac{1}{2}\left(\sqrt{1+\frac{8a}{b}}+1\right) > 0 \quad \text{and} \quad z_2 = \frac{1}{2}\left(\sqrt{1+\frac{8a}{b}}-1\right) > 0.$$

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