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The Convergence of Mann Iteration for Generalized $\Phi-$ hemi-contractive Maps

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Abstract

Charles[1] proved the convergence of Picard-type iterative for generalized $\Phi-$ accretive non-self maps in a real uniformly smooth Banach space. Based on the theorems of the zeros of strongly $\Phi-$ accretive and fixed points of strongly $\Phi-$ hemi-contractive we extend the results to Mann-type iterative and Mann iteration process with errors.

Keywords: strongly $\Phi-$ accretive, strongly $\Phi-$ hemi-contractive, Mann iteration process with errors, fixed point

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1. Introduction

In[2], we can see a Mann-type iteration method for a family of hemi-contractive mappings in Hilbert spaces; In[3], we can see a Halpern-Mann type Iteration for fixed point problems of a relatively nonexpansive mapping and a system of equilibrium problems; In[4], we can see that convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings; In[5], we can see that some Mann-type implicit iteration methods for triple hierarchical variational inequalities, systems variational inequalities and fixed point problems; In[6], we can see a Mann-type iteration method for solving the split common fixed point problem; In[7], we can see that Mann and Ishikawa-type iterative schemes for approximating fixed points of multi-valued non-self mappings.

In 2009, Charles[1] proved the convergence of Picard-type iterative for generalized $\Phi-$accretive non-self maps in a real uniformly smooth Banach space. In this paper, we will consider to extend the result of Charles[1] to Mann-type iterative and Mann iteration process with errors.

In 1995, Liu[8] introduced what he called the Mann iteration process with errors.

In 1998, Xu[9] introduced the following alternative definitions:

Let K be a nonempty convex subset of E and $T:K\to K$ be any map. For any given $x_0,u_0\in K$, the process defined by

$$x_{n+1}=a_nx_n+b_nTx_n+c_nu_n,\quad n\ge 0, \tag{1.1}$$

where $\{u_n\}$ is bounded sequences in K and the real sequences $\{a_n\},\{b_n\},\{c_n\}\subset[0,1]$ satisfy the conditions

$$a_n+b_n+c_n=1,\quad \forall n\ge 0.$$

It called the Mann iteration process with errors.

However, the most general Mann-type iterative scheme now studied is the following: $x_0\in K$,

$$x_{n+1}=(1-c_n)x_n+c_nTx_n,\ n=0,1,2,... \tag{1.2}$$

where $\{c_n\}_{n=1}^{\infty}\subset(0,1)$ is a real sequence satisfying appropriate conditions (see, e.g., Chidume[10], Edelstein and O'Brian[11], Ishikawa[12]). Under the following additional assumptions (i) $\lim c_n=0$; and (ii) $\sum_{n=0}^{\infty}c_n=\infty$, the sequence $\{x_n\}$

generated by (1.2) is generally referred to as the Mann sequence in the light of Mann[13] .

2. Preliminary Notes

Definition 2.1[1] *Let* (E, ρ) *be a metric space. A mapping* $T: E \to E$ *is called a contraction if there exists* $k \in [0,1)$ *such that* $\rho(Tx, Ty) \le k\rho(x, y)$ *for all* $x, y \in E$. *If* $k = 1$, *then* T *is called nonexpansive.*

Definition 2.2[1] *Given a gauge function* φ *, the mapping* $J_\varphi : E \to 2^{E^*}$ *defined by*

$$J_\varphi x := \left\{u^* \in E^* : \langle x, u^* \rangle = \|x\| \|u^*\|; \|u^*\| = \varphi(\|x\|)\right\} \tag{2.1}$$

is called the duality map with gauge function φ *where* E *is any normed space.*

In the particular case $\varphi(t) = t$, *the duality map* $J = J_\varphi$ *is called the normalized duality map.*

Proposition 2.3[14] *If a Banach space* E *has a uniformly Gateaux differentiable norm, then* $J : E \to E^*$ *is uniformly continuous on bounded subsets of* E *from the strong topology of* E *to the weak** *topology of* E^*.

Definition 2.4[15] *A mapping* $T: E \to E$ *is called strongly pseudo-contractive if for all* $x, y \in E$ *, the following inequality holds:*

$$\|x - y\| \le \|(1+r)(x-y) - rt(Tx - Ty)\| \tag{2.2}$$

for all $r > 0$ *and some* $t > 1$. *If* $t = 1$ *in inequality (2.2), then* T *is called pseudo-contractive. As we know that* T *is strongly pseudo-contractive if and only if*

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \ge k\|x-y\|^2 \tag{2.3}$$

holds for all $x, y \in E$ *and for some* $j(x-y) \in J(x-y)$ *, where* $k = \frac{1}{t}(t-1) \in (0,1)$.

Definition 2.5[1] *Recall that an operator* $T : D(T) \subseteq E \to E$ *is strongly accretive if there exists some* $k > 0$ *such that for each* $x, y \in D(T)$ *, there exists* $j(x-y) \in J(x-y)$ *such that*

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \qquad (2.4)$$

Proposition 2.6[1] *A mapping* $T: E \to E$ *is strongly pseudo-contractive if and only if* $(I-T)$ *is strongly accretive, and is strongly* $\varphi-$*pseudo-contractive if and only if* $(I-T)$*is strongly* $\varphi-$*accretive. Then* T *is generalized* $\Phi-$*pseudo-contractive if and only if* $(I-T)$*is generalized* $\Phi-$*accretive.*

Definition 2.7[1] *Let* E *be an arbitrary real normed linear space. A mapping* $T: D(T) \subseteq E \to E$ *is called strongly hemi-contractive if* $F(T) \neq \varnothing$*, and there exists* $t > 1$ *such that for all* $r > 0$*,*

$$\|x - x^*\| \leq \|(1+r)(x-x^*) - rt(Tx - x^*)\| \qquad (2.5)$$

holds for all $x \in D(T)$, $x^* \in F(T)$. *If* $t = 1$*, then* T *is called hemi-contractive. Finally,* T *is called generalized* $\Phi-$ *hemi-contractive, if for all* $x \in D(T)$, $x^* \in F(T)$*, there exists* $j(x-x^*) \in J(x-x^*)$ *such that*

$$\langle (I-T)x - (I-T)x^*, j(x-x^*)\rangle \geq \Phi(\|x-x^*\|). \qquad (2.6)$$

It follows from inequality (2.6) that T *is generalized* $\Phi-$*hemi-contractive if and only if*

$$\langle Tx - x^*, j(x-x^*)\rangle \leq \|x-x^*\|^2 - \Phi(\|x-x^*\|), \ \forall n \geq 0. \qquad (2.7)$$

Definition 2.8[1] *Let* $N(T) = \{x \in E : Tx = 0\} \neq \varnothing$. *The mapping* $T: D(T) \subseteq E \to E$ *is called generalized* $\Phi-$ *quasi-accretive if, for all* $x \in E$, $x^* \in N(T)$*, there exists* $j(x-x^*) \in J(x-x^*)$ *such that*

$$\langle Tx - Tx^*, j(x-x^*)\rangle \geq \Phi(\|x-x^*\|). \qquad (2.8)$$

Proposition 2.9[1] *If* $F(T) = \{x \in E : Tx = x\} \neq \varnothing$*, the mapping* $T: E \to E$ *is strongly hemi-contractive if and only if* $(I-T)$ *is strongly quasi-accretive; it is strongly* $\varphi-$ *hemi-contractive if and only if* $(I-T)$ *is strongly* $\varphi-$ *quasi-accretive; and* T *is generalized* $\Phi-$*hemi-contractive if and only if* $(I-T)$ *is generalized* $\Phi-$*quasi-accretive.*

Proposition 2.10[1] *Let* E *be a uniformly smooth real Banach space, and let*

$J:E\to 2^{E^*}$ *be a normalized duality mapping. Then*

$$\|x+y\|^2\le\|x\|^2+2\langle y,J(x+y)\rangle \tag{2.9}$$

for all $x,y\in E$.

Proposition 2.11[1] *Let* $\{\lambda_n\}$ *and* $\{\gamma_n\}$ *be sequences of nonnegative numbers and* $\{\alpha_n\}$ *be a sequence of positive numbers satisfying the conditions* $\sum_{n=1}^{\infty}\alpha_n=\infty$ *and* $\frac{\gamma_n}{\alpha_n}\to 0$ *, as* $n\to\infty$. *Let the recursive inequality*

$$\lambda_{n+1}\le\lambda_n-\alpha_n\psi(\lambda_n)+\gamma_n, n=1,2,... \tag{2.10}$$

be given where $\psi:[0,\infty)\to[0,\infty)$ *is strictly increasing continuous function such that it is positive on* $(0,\infty)$ *and* $\psi(0)=0$. *Then* $\lambda_n\to 0$, *as* $n\to\infty$.

3. Main Results

In this section, we will consider to extend the result of Charles[1] to Mann-type iterative and Mann iteration process with errors under the following assumptions.

First, we extend the result of Charles[1] to Mann-type iterative.

Theorem 3.1 *Suppose* K *is a closed convex subset of a real uniformly smooth Banach space* E. *Suppose* $T:K\to K$ *is a bounded generalized* $\Phi-$ *hemi-contractive map with strictly increasing continuous function* $\Phi:[0,\infty)\to[0,\infty)$ *such that* $\Phi(0)=0$ *and* $x^*\in F(T)\ne\varnothing$. *For arbitrary* $x_1\in K$, *define the sequence* $\{x_n\}$ *iteratively by*

$$x_{n+1}=(1-c_n)x_n+c_nTx_n, n=0,1,2,... \tag{3.1}$$

where $\{c_n\}\subseteq(0,1)$, $\lim c_n=0$ *and* $\sum c_n=\infty$. *Then, there exists a constant* $d_0>0$ *such that if* $0<c_n\le d_0$, $\{x_n\}$ *converges strongly to the unique fixed point* x^* *of* T.

Proof. Let r *be sufficiently large such that* $x_1\in B_r(x^*)$. *Define* $G:=\overline{B_r(x^*)}\cap K$. *Then, since* T *is bounded we have that* $(I-T)(G)$ *is bounded.*

Let $M=\sup\left\{\left\|\left(I-T\right)x_n\right\|:x_n\in G\right\}$. *As* j *is uniformly continuous on bounded subsets of* E, *for* $\varepsilon:=\dfrac{\Phi\left(\frac{r}{2}\right)}{2M}$, *there exists a* $\delta>0$ *such that* $x,y\in D(T),\|x-y\|<\delta$ *implies* $\left\|j(x)-j(y)\right\|<\varepsilon$. *Set* $d_0=\min\left\{1,\dfrac{r}{2M},\dfrac{\delta}{2M}\right\}$.

Claim1: $\{x_n\}$ *is bounded.*

Suffices to show that x_n *is in* G *for all* $n\geq 1$. *The proof is by induction. By our assumption,* $x_1\in G$. *Suppose* $x_n\in G$. *We prove that* $x_{n+1}\in G$. *Assume for contradiction that* $x_{n+1}\notin G$. *Then, since* $x_{n+1}\in K\ \forall n\geq 1$, *we have that* $\left\|x_{n+1}-x^*\right\|>r$.

Thus we have the following estimates:

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|&\leq\left\|x_n-c_n\left(I-T\right)x_n-x^*\right\|\\&\leq\left\|x_n-x^*\right\|+c_n\left\|\left(I-T\right)x_n\right\|\\&\leq r+d_0\cdot M\\&\leq 2r,\end{aligned}$$

$$\begin{aligned}\left\|x_n-x^*\right\|&\geq\left\|x_{n+1}-x^*\right\|-c_n\left\|\left(I-T\right)x_n\right\|\\&>r-c_n\cdot M\\&>\frac{r}{2},\end{aligned}$$

$$\begin{aligned}\left\|\left(x_{n+1}-x^*\right)-\left(x_n-x^*\right)\right\|&\leq c_n\left\|\left(I-T\right)x_n\right\|\\&\leq c_n\cdot M\\&\leq\frac{\delta}{2}<\delta,\end{aligned}$$

therefore,

$$\left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\|<\varepsilon.$$

Then from (3.1), the above estimates and Proposition 2.10 we have that

$$\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 &= \left\|x_n - c_n\left(I-T\right)x_n - x^*\right\|^2 \\
&\le \left\|x_n - x^*\right\|^2 - 2c_n\left\langle\left(I-T\right)x_n, j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\rangle \\
&\quad -2c_n\left\langle\left(I-T\right)x_n, j\left(x_n-x^*\right)\right\rangle \\
&\le \left\|x_n - x^*\right\|^2 + 2c_n\left\|\left(I-T\right)x_n\right\|\left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\| \\
&\quad -2c_n\Phi\left(\left\|x_n-x^*\right\|\right) \\
&\le \left\|x_n - x^*\right\|^2 - 2c_n\Phi\left(\frac{r}{2}\right) + 2c_n\cdot M\cdot\varepsilon \\
&\le r^2 + 2d_0\left[\frac{\Phi\left(\frac{r}{2}\right)}{2} - \Phi\left(\frac{r}{2}\right)\right] \\
&\le r^2 \qquad (3.2)
\end{aligned}$$

i.e., $\left\|x_{n+1}-x^*\right\| \le r$ *, a contradiction. Therefore* $x_{n+1}\in G$ *. Thus by induction* $\{x_n\}$ *is bounded. Then,* $\{Tx_n\}, \{\left(I-T\right)x_n\}$ *are also bounded.*

Claim2: $x_n \to x^*$.

Let $A_n = \left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\|$ *, Note that* $x_{n+1}-x_n \to 0$ *as* $n\to\infty$ *and hence by the uniform continuity of* j *on bounded subsets of* E *we have that*

$$A_n \to 0 \text{ as } n\to\infty.$$

We obtain that

$$\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 &\le \left\|x_n - x^*\right\|^2 + 2c_n\left\|\left(I-T\right)x_n\right\|\left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\| \\
&\quad -2c_n\Phi\left(\left\|x_n-x^*\right\|\right) \\
&\le \left\|x_n - x^*\right\|^2 + 2c_n\left[Z_n - \Phi\left(\left\|x_n-x^*\right\|\right)\right] \\
&\le \left\|x_n - x^*\right\|^2 - 2c_n\Phi\left(\left\|x_n-x^*\right\|\right) + 2c_nZ_n, \qquad (3.3)
\end{aligned}$$

where $Z_n = MA_n \to 0$ *as* $n \to \infty$.

Let $\lambda_n := \left\| x_n - x^* \right\|$ *and* $\gamma_n = 2c_n Z_n$ *, then from inequality (3.3) we obtain that*

$\lambda_{n+1} \le \lambda_n - 2c_n \Phi(\lambda_n) + \gamma_n$*, where* $\frac{\gamma_n}{c_n} \to 0$ *as* $n \to \infty$*. Therefore, the conclusion of the theorem follows from Proposition 2.11.*

We've done the proof of the theorem 3.1.

The following corollary follow trivially,since definition 2.5.

Corollary 3.2 *Suppose* E *is a real uniformly smooth Banach space, and* $T: E \to E$ *is a bounded generalized* $\Phi -$*accretive map with strictly increasing continuous function* $\Phi : [0,\infty) \to [0,\infty)$ *such that* $\Phi(0) = 0$ *and the solution* x^* *of the equation* $Tx = 0$ *exists. For arbitrary* $x_1 \in E$*, define the sequence* $\{x_n\}$ *iteratively by*

$$x_{n+1} = (1 - c_n) x_n - c_n T x_n, n = 0,1,2,...$$

where $\{c_n\} \subseteq (0,1)$, $\lim c_n = 0$ *and* $\sum c_n = \infty$*. Then, there exists a constant* $d_0 > 0$ *such that if* $0 < c_n \le d_0, \{x_n\}$ *converges strongly to the unique solution of* $Tx = 0$.

Now, we consider to generalize to a more general case, we extend the result of Charles[1] to Mann iteration process with errors as follows.

Theorem 3.3 *Suppose* K *is a closed convex subset of a real uniformly smooth Banach space* E*. Suppose* $T: K \to K$ *is a bounded generalized* $\Phi -$ *hemi-contractive map with strictly increasing continuous function* $\Phi : [0,\infty) \to [0,\infty)$ *such that* $\Phi(0) = 0$ *and* $x^* \in F(T) \ne \emptyset$*. For arbitrary* $x_1 \in K$*, define the sequence* $\{x_n\}$ *iteratively by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, n \ge 0, \tag{3.4}$$

where $\{u_n\}$ *is bounded sequence in* K *and* $\{a_n\},\{b_n\},\{c_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

$$(i) a_n + b_n + c_n = 1 \quad \forall n \geq 0; \ (ii) \sum_{n=0}^{\infty} b_n = \infty;$$

$$(iii) \lim b_n = 0; c_n = o(b_n); \ (iv) \sum_{n=0}^{\infty} c_n < \infty.$$

Then, there exists a constant $d_0 > 0$ *such that if* $0 < b_n, c_n \leq d_0$, $\{x_n\}$ *converges strongly to the unique fixed point* x^* *of* T.

Proof. Let r *be sufficiently large such that* $x_1 \in B_r(x^*)$. *Define* $G := \overline{B_r(x^*)} \cap K$. *Then, since* T *is bounded we have that* $(I-T)(G)$ *is bounded.*

Let $M = \max\{\sup\|(I-T)x_n\|, \sup\|x_n - u_n\| : x_n \in G\}$. *As* j *is uniformly continuous on bounded subsets of* E, *for* $\varepsilon := \dfrac{\Phi\left(\frac{r}{2}\right)}{4M}$, *there exists a* $\delta > 0$ *such that* $x, y \in D(T)$, $\|x-y\| < \delta$ *implies* $\|j(x) - j(y)\| < \varepsilon$.

Set $d_0 = \min\left\{1, \dfrac{r}{4M}, \dfrac{\delta}{4M}, \dfrac{\Phi\left(\frac{r}{2}\right)}{8Mr}\right\}$.

Claim1: $\{x_n\}$ *is bounded.*

Suffices to show that x_n *is in* G *for all* $n \geq 1$. *The proof is by induction. By our assumption,* $x_1 \in G$. *Suppose* $x_n \in G$. *We prove that* $x_{n+1} \in G$. *Assume for contradiction that* $x_{n+1} \notin G$. *Then, since* $x_{n+1} \in K \ \forall n \geq 1$, *we have that* $\|x_{n+1} - x^*\| > r$. *Equation (3.4) becomes*

$$x_{n+1} = x_n - b_n(I-T)x_n - c_n(x_n - u_n). \tag{3.5}$$

Thus we have the following estimates:

$$
\begin{aligned}
\left\|x_{n+1}-x^*\right\| &\le \left\|x_n-x^*\right\| + b_n\left\|(I-T)x_n\right\| + c_n\left\|x_n-u_n\right\| \\
&\le r + d_0\left(M+M\right) \\
&\le 2r\ ,
\end{aligned}
$$

$$
\begin{aligned}
\left\|x_n-x^*\right\| &\ge \left\|x_{n+1}-x^*\right\| - b_n\left\|(I-T)x_n\right\| - c_n\left\|x_n-u_n\right\| \\
&> r - d_0\left(M+M\right) \\
&> \frac{r}{2}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\left\|\left(x_{n+1}-x^*\right)-\left(x_n-x^*\right)\right\| &\le b_n\left\|(I-T)x_n\right\| + c_n\left\|x_n-u_n\right\| \\
&\le 2d_0M \\
&\le \frac{\delta}{2} < \delta\ ,
\end{aligned}
$$

therefore,

$$
\left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\| < \varepsilon\ .
$$

Then from (3.5), the above estimates and Proposition 2.10 we have that

$$
\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 &= \left\|x_n - b_n\left(I-T\right)x_n - c_n\left(x_n-u_n\right) - x^*\right\|^2 \\
&\le \left\|x_n-x^*\right\|^2 - 2b_n\left\langle\left(I-T\right)x_n,\ j\left(x_{n+1}-x^*\right)\right\rangle - 2c_n\left\langle x_n-u_n,\ j\left(x_{n+1}-x^*\right)\right\rangle \\
&\le \left\|x_n-x^*\right\|^2 + 2b_n\left\langle\left(I-T\right)x_n,\ j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\rangle \\
&\quad -2b_n\left\langle\left(I-T\right)x_n,\ j\left(x_n-x^*\right)\right\rangle + 2c_n\left\langle x_n-u_n,\ j\left(x_{n+1}-x^*\right)\right\rangle \\
&\le \left\|x_n-x^*\right\|^2 + 2b_n\left\|\left(I-T\right)x_n\right\|\left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\| \\
&\quad -2b_n\Phi\left(\left\|x_n-x^*\right\|\right) + 2c_n\left\|x_n-u_n\right\|\left\|x_{n+1}-x^*\right\| \\
&\le r^2 + 2d_0\left[M\cdot\varepsilon - \Phi\left(\frac{r}{2}\right) + 2Mr\right] \\
&\le r^2 + 2d_0\left[\frac{\Phi\left(\frac{r}{2}\right)}{2} - \Phi\left(\frac{r}{2}\right)\right] \\
&\le r^2 \qquad (3.6)
\end{aligned}
$$

i.e., $\left\|x_{n+1}-x^*\right\| \le r$ *, a contradiction. Therefore* $x_{n+1} \in G$ *. Thus by induction* $\{x_n\}$ *is bounded. Then,* $\{Tx_n\}, \{(I-T)x_n\}$ *are also bounded.*

Claim2: $x_n \to x^*$.

Let $A_n = \left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\|$ *, Note that* $x_{n+1}-x_n \to 0$ *as* $n \to \infty$ *and hence by the uniform continuity of* j *on bounded subsets of* E *we have that*

$$A_n \to 0 \ \text{as} \ n \to \infty.$$

We obtain that

$$\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 &\le \left\|x_n-x^*\right\|^2 + 2b_n\left\|(I-T)x_n\right\|\left\|j\left(x_{n+1}-x^*\right)-j\left(x_n-x^*\right)\right\| \\
&\quad -2b_n\Phi\left(\left\|x_n-x^*\right\|\right)+2c_n\left\|x_n-u_n\right\|\left\|x_{n+1}-x^*\right\| \\
&\le \left\|x_n-x^*\right\|^2 + 2b_nM\cdot A_n - 2b_n\Phi\left(\left\|x_n-x^*\right\|\right)+2c_nM\cdot 2r \\
&\le \left\|x_n-x^*\right\|^2 + 2b_n\left[M\cdot A_n + 2\frac{c_n}{b_n}Mr - \Phi\left(\left\|x_n-x^*\right\|\right)\right] \\
&\le \left\|x_n-x^*\right\|^2 + 2b_n\left[Z_n - \Phi\left(\left\|x_n-x^*\right\|\right)\right] \\
&\le \left\|x_n-x^*\right\|^2 - 2b_n\Phi\left(\left\|x_n-x^*\right\|\right) + 2b_nZ_n \ ,
\end{aligned} \tag{3.7}$$

where $Z_n = MA_n + 2\frac{c_n}{b_n}Mr \to 0$ *as* $n \to \infty$.

Let $\lambda_n := \left\|x_n-x^*\right\|$ *and* $\gamma_n = 2b_nZ_n$ *, then from inequality (3.7) we obtain that*

$\lambda_{n+1} \le \lambda_n - 2b_n\Phi(\lambda_n)+\gamma_n$*, where* $\frac{\gamma_n}{b_n} \to 0$ *as* $n \to \infty$*. Therefore, the conclusion of the theorem follows from Proposition 2.11.*

We've done the proof of the theorem 3.3.

The following corollary follow trivially,since definition 2.5.

Corollary 3.4 *Suppose* E *is a real uniformly smooth Banach space, and* $T: E \to E$ *is a bounded generalized* $\Phi-$*accretive map with strictly increasing* *continuous function* $\Phi:[0,\infty)\to[0,\infty)$ *such that* $\Phi(0)=0$ *and the solution* x^* *of the equation* $Tx=0$ *exists. For arbitrary* $x_1\in E$ *, define the sequence* $\{x_n\}$ *iteratively by*

$$x_{n+1}=a_nx_n-b_nTx_n+c_nu_n, n\geq 0,$$

where $\{u_n\}$ *is bounded sequence in* K *and* $\{a_n\},\{b_n\},\{c_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

$$(i)a_n+b_n+c_n=1 \quad \forall n\geq 0;\ (ii)\sum_{n=0}^{\infty}b_n=\infty;$$

$$(iii)\sum_{n=0}^{\infty}b_n^2<\infty;\ c_n=o(b_n);\ (iv)\sum_{n=0}^{\infty}c_n<\infty.$$

Then, there exists a constant $d_0>0$ *such that if* $0<b_n, c_n\leq d_0$, $\{x_n\}$ *converges strongly to the unique solution of* $Tx=0$.

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