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The Weighted Average Cost of Capital as a Marginal Criterion

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Abstract

This paper addresses apparent definitional gaps and ambiguities in existing references to the marginal aspects of the weighted average cost of capital (WACC) and reviews its role as a benchmark in investment appraisal. A selection of typical statements, both from the finance and economics traditions, is presented to highlight limitations of the current state of the debate, and to reconcile modelling styles in the area. The commonly advanced justification for the use of WACC as a criterion on grounds of practicality only is criticised as subject to unduly restrictive conditions, especially unchanged gearing. Alternative Marginal Factor Cost (MFC) definitions for equity and debt are proposed, and their properties explored. The corresponding MFC curves cross over at the minimum WACC which can then serve as a marginal criterion under optimal gearing. Minimum WACC is also derived from a profit maximisation model and WACC is thus shown to be compatible with marginalism in so far as profit maximisation applies. A related aim of the analysis presented is to inform pedagogy in the area by crossing the boundary between economics and finance modelling.

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1 Introduction

The weighted average cost of capital (WACC), often advanced as a criterion in investment appraisal, has a somewhat uneasy coexistence with marginal analysis. There are numerous suggestions in the literature that this average concept can, in some circumstances, provide a measure acceptable as the unit cost of capital to be applied in investment appraisal without recourse to any corresponding marginal concept. The main justification provided is that there are practical, at least, difficulties in using such a concept. An unchanged debt/equity ratio (gearing) is almost universally stipulated as one of the conditions under which WACC may substitute for a marginal concept.

The condition of unchanged gearing seems however a curious one, when it is considered that gearing is changed by *any* application of more debt, or equity, at the margin. The stipulation of constant gearing is thus something of a contradiction. The gearing ratio can only be maintained if *every* input change, marginal or not, is followed by a rebalancing adjustment to maintain the previous ratio. There are indeed suggestions that such adjustments *may* be made to accommodate non marginal additions to debt or equity and also investments which would significantly alter the firm's current risk profile. But the standard treatment of the choice of capital structure stops short of assuming that an adjustment *would* be made.

It will be argued here that the stipulation of unchanged gearing as a condition for the use the WACC is misguided. An assumption of continuous rebalancing is *necessary* in order to align investment theory with the normal assumption of profit maximisation in the 'static' economic theory of the firm, or with the equivalent idea of present value maximisation in finance literature. Suggestions that rebalancing adjustments may not be continuous are inconsistent with profit maximisation and amount to a statement that the firm may not behave as a profit maximiser. Whereas such a view may not be devoid of merit, it is not addressed explicitly in the literature as a consequence of stipulations about gearing.

It will be argued, further, that the problem arises, in large measure, from a lack of a proper definition of marginal concepts relating to WACC. There are occasional references to such concepts but the definitions are not explicit. Formal tools are deployed here to define the marginal aspect of WACC in a way that encompasses necessary effects on gearing and brings out the optimisation of gearing as an integral part of the firm's profit maximisation problem. It is shown, furthermore, that profit maximisation requires *minimisation* of WACC. The proposed marginal definition then *coincides* with WACC. The latter therefore *can and should* be used as a proxy for the former in so far as profit maximisation applies.

Discussion of WACC in financial literature is on the whole conducted in a different format from models of the firm encountered in economics. The formal models provided here purport to contribute towards reconciliation of the two traditions and the formulae put forward contributes to that end. A related aim is to promote greater pedagogic clarity on the subject.

The paper is structured as follows. A sample of views, from both texts and research literature is provided in section 2, with our comment aimed at clarification and, in some instances, criticism. Section 3 introduces the proposed 'incremental WACC' concept and related marginal definitions and illustrates with examples the choice of optimal gearing in terms of achieving minimum WACC. Section 4 derives alternative but equivalent marginal definitions, not involving WACC, from a profit maximisation model to arrive at results consistent with section 3. The analysis of Sections 3 and 4 together brings out the essentially marginal character of the WACC concept. Section 5 attempts an overview of choices involving scale and risk level in the long run featuring use of WACC as a marginal criterion. Section 6 addresses definitional discrepancies in the treatment of costs in economics and finance models. The results and implications are summarised in the concluding section 7.

2 Indicative Perspectives from the Literature

References to the marginal aspect of WACC in finance texts, are generally without a precise definition, if they are not completely absent. A sample of treatments is provided here to represent the current state of the debate.

Banks (2016) suggests that "If we are considering each incremental dollar…we shouldn't be looking *only* (our emphasis) at the WACC. We must examine decisions in the light of the weighted *marginal* (our emphasis) cost of capital (WMCC) which is an upward sloping function." (p 76). This author appears to visualise, but without specifying, a marginal concept, corresponding to the average one of the WACC, with the usual upward sloping feature of the Marginal Cost (MC) function of economic theory. We will see later in what sense this is true.

Brealey, Myers and Allen (2016) on the other hand, without providing a definition of the marginal WACC, suggest that "the (WACC) formula assumes that the project or business to be valued is financed in the same debt/equity proportions as the company or industry as a whole. ….What if this is not true ?" (p 503). In answering this question the authors refer to rebalancing adjustments to restore a 'target' gearing ratio suggesting, however, that "Of course real companies will not rebalance capital structure in such a mechanical or compulsive way…For practical purposes it's sufficient to assume gradual and steady adjustment towards a long term target." (p 506). In what follows we question the nature of the target and of the adjustment.

Lumby and Jones (2015, pp 413-4) also argue that the current cost of the equity of an all equity company can properly be used as the discount rate, provided the projects under evaluation were marginal in size and would not cause any significant change in the company's overall level of risk. For a mixed finance company they draw on the 'pool of funds' concept to suggest that "It is neither practical nor especially sensible to try to identify a particular source of investment cash, physically, with a particular project. Once cash enters a company, it enters the general 'pool' of capital within that company and it is out of that pool that funds are drawn in order to be applied to particular investment projects" (p 416). We note, here too, the absence of a definition of the marginal aspect and the resulting reliance on the 'pool' approach.

Lumby and Jones (2015) also argue that "if a project is financed so as to cause a change in capital structure then an NPV/WACC analysis appears inappropriate. NPV…cannot adequately handle a simultaneous capital structure decision" (p. 423). In what follows we shall try to make some progress towards such simultaneity.

Watson and Head (2013), suggest that "strictly speaking the marginal cost of capital used to finance an investment project should be used rather than the average cost of capital" (p 280). But, due to difficulties of allocating particular funding to specific projects, and the economies of using one source of finance at a time, "it could be argued…that a *rolling average* marginal cost of capital is more appropriate than an *incremental* marginal cost of capital." (ibid). Two marginal concepts are launched here but without definition, or indication exactly how they would be applied in practice. Watson and Head (2013), like Lumby and Jones (2015) point out that use of an average concept is appropriate only under the restrictive assumptions that (i) the business risk of the investment project is the same as the risk of the company's current activities and (ii) incremental finance is raised in proportions which preserve the company's existing capital structure. Absent such conditions, they suggest that the cost of capital should be calculated on a marginal basis *and any effect on the existing average cost structure must also be reflected in the marginal cost of capital* (our emphasis). This appears to call for a formula which would involve both the existing average concept and the way in which the marginal concept impacts on it. Such a formula is indeed provided here in section 3.

A purportedly practical, approach is advanced by Pike, Neale and Linsley (2015). These authors do provide a calculation of the Marginal Capital Cost (MCC) of debt as the sum of the debt financing cost and the *additional* return required on the equity, without explicitly involving WACC. (p 540). A general formula corresponding to this is examined in Section 4 below.

Pike et al (2015) then argue however that "To all intents and purposes the capital structure is given……MCC has major operational limitations. In particular we are required to anticipate how the capital market is likely to react to the issue of additional debt. Given that we seem unable to define the WACC profile or pinpoint the optimal gearing at any one time this presents a problem. We could assume that the present gearing ratio is optimal but this prompts the question why different firms in the same industry have different gearing ratios. A solution commonly adopted in practice is to specify a target capital structure…The firm defines what it regards the optimal long term gearing range or ratio and then attempts to adhere to that ratio in financing future operations. The corollary then is to use the WACC as the cut off rate for new investment ….”(p 541).

Pike et al (2015) further argue, in apparent contradiction with the foregoing, that ‘The somewhat pragmatic solution proposed assumes that the new project will have no appreciable effect on gearing. In other words that the company already operates at or close to the optimal gearing ratio and does not significantly deviate from it. *Obviously* (our emphasis) the WACC and MCC will coincide in this case” (p 541). This discussion seems to allow for the possibility that the targeted ratio may be other than the one that minimises WACC and seems ambivalent as to whether the use of WACC is restricted to small projects, with negligible effects on gearing, or else may apply to any project, as gearing can always be adjusted to maintain the targeted ratio.

In a rather neglected pedagogic note, however, Rose (1987) refers to a MCC “graphed on a stair-stepped or curvilinear basis against total new capital raised by the firm. The stair steps are sometimes termed ‘break points’ at which the marginal cost of capital jumps due to the exhaustion of a cheaper source of funds” (p 16). But Rose argues that “students… often have difficulty relating the WACC and MCC concepts as presented in this diagrammatic approach. In particular, students frequently fail to see that the stair-steps (or upward curvature) of the MCC schedule simply reflects the path of the minimum point on the WACC curve along the total-new-capital axis. That is, as the firm encounters higher costs associated with raising larger and larger amounts of new funds during a given time period, *the WACC curve rises* (our emphasis). The path followed by the minimum point of the WACC curve along the total-new-capital axis defines the MCC schedule.” (p 16). This is the clearest, perhaps, attempt to define a marginal concept in relation to WACC, although not necessarily in line with what Banks (2016) has in mind. The definition is considered further in section 5.

Economics literature, in contrast, features rather sporadic, and not very recent, direct references to WACC or to its links with marginal concepts. E.g. a textbook reference by Moschandreas (2000) simply defines the cost of capital in an essentially *ex post* way as “the rate at which the firm’s expected returns are capitalised” (p 328).

A rather forgotten piece of research, however, (Bronfenbrenner, 1960) is surprising, perhaps, in its advocacy of the use of WACC as a criterion rate, advanced with no apparent qualification. “The competitive entrepreneur should be looked upon as adjusting the use of productive services so as to equate the values of the marginal products not to their contractual market prices but to *weighted averages* of contractual and entrepreneurial prices, where these diverge and where entrepreneurial inputs are used” (p 307). An ‘entrepreneurial’ input is defined in the present context as a *residually remunerated* one (e.g. the equity, where the input is financial capital), while a ‘contractual’ input is one remunerated on fixed terms (debt generally, such as bonds or bank loans). We will confirm in section 4 that this insight is correct.

Paulo (1992), defending ‘ex ante marginalism’ proposes that “the marginal cost curve for finance should take the form of a *sequential* (our emphasis) marginal cost curve..…Projects are then screened sequentially qualifying for approval when the marginal return exceeds the marginal cost of the finance component used” (p 181). In reply Wang (1994) suggests that “the WACC is actually the weighted average of the marginal cost of each new marginal dollar of capital raised. It is not the average cost of capital the firm has raised in the past or will raise in the future. The principle of marginalism is fully applied..”(p 188). Here we do have another definition, which varies however, from the one to be proposed in what follows.

The framework to be adopted in the remainder of this paper will draw on standard economic theory. The resulting analysis will be related mainly to Bronfenbrenner (1960), but also Rose (1987). Other of the above contributions will be revisited as necessary.

3 Marginal Definitions and Minimisation of WACC

The uncertainty surrounding the marginal aspect of WACC is in sharp contrast with the definitions of the marginal cost of output (MC) or the Marginal Factor Cost (MFC) of standard microeconomic theory. These measure the change in total cost consequent upon a unit change of output (TC) or factor input (TFC) respectively. The marginal concepts are also expressed, equivalently, in terms of the respective averages AC or AFC. That is, where a factor input is concerned, formula $MFC = AFC + F[(dAFC)/dF]$ produces the same result as formula $MFC = d(TFC)/dF$.

To compute MFC here, however, we need the effect on total cost not only of a unit change in one component of cost but, *additionally*, the effect of that unit change on the unit cost of another. Equivalently, to evaluate MFC with reference to AFC, we need to examine the effect of a unit change in one cost component on the *weighted* average of the cost of two (or, by extension, possibly more) components of cost, where *the weights necessarily vary* with any change in one of the components. That is precisely the consequence of a change in *gearing* occasioned by a unit change in one component. As will be seen presently, the MFC definition provided here in the form of a *sum* of two terms, of which WACC itself is the first term.

Economics-style modelling might represent the input decision problem as follows. Assuming other inputs fixed, the firm is faced with the choice of an appropriate level of capital utilisation overall and, simultaneously with that, between finance to be obtained contractually (K_c), and finance to be supplied entrepreneurially (K_e), both components measured in physical units.[2] While both variables K_c and K_e represent physically undifferentiated capital, with identical productivity characteristics, they are different 'factors' of production, in as much as they have different unit costs. Let K denote the total number of physical units of capital to be utilised, where $K = K_c + K_e$. Denote the unit cost of K_c by i, the unit cost of K_e by r and the *gearing* ratio by g, thus defining gearing as

$$g = \frac{K_c}{K_c + K_e} \qquad (1)$$

We can also define, correspondingly, the equity or 'own resources' ratio as

$1- g = K_e/(K_c. +K_e)$ weights within WACC will then be g and $1- g$ and WACC can be written as

$$WACC = gi + (1 - g)r \qquad (2)$$

where i and r are the respective unit costs.

It is true that the TFC of K can be found form a known WACC through multiplication by K. It is also the case that division of the total cost of K by the unweighted K will still yield the *WACC*, i.e. the AFC per *composite* unit of K, when the weights have been allowed for.[3] But the derivative of TFC necessary to yield the MFC, can only be taken for a unit increase in *one* of the components of the average at a time. The MFC is the MC of *one or the other* component. It cannot be defined until it is specified *which* component would be varied and the

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[2] The paradigm in finance literature is couched in terms of the *values*, rather than the physical quantities of production factors, and their costs as rates of return. In economics, on the other hand, inputs are measured in terms of *physical units* and absolute rental costs per unit. This problem may be overcome for our present purposes, if K is interpreted as denoting physical units of capital, priced at *one* money unit each, Rates of return can then be interpreted also as absolute rental costs. Further justification for this procedure is provided in section 6.

[3] If $WACC = (K_c/K)i + (K_e/K)r$ then *TFC* would be *WACC* multiplied by K. But the conventional *MFC* cannot then be obtained unambiguously as $d(TFC)/dK$, only as as $d(TFC)/dK_c$, or $d(TFC)/dK_e$.

implications of that for the weighting. *Any* change in the use of either or both components of the input would *also* change the weighting of the mix, unless compensating changes in the inputs are made to restore gearing to the original, or other desired level. Stipulations of a constant gearing ratio are thus *not compatible* with changes in the amounts of either or both components. Only after the effect on gearing has been allowed for, can there be a calculation of the impact of an input change on the total or the (weighted) average factor cost.

The effect of a unit change in either K_c or K_e on g and thus on the weights of WACC can be obtained from the definition of g in (1). We obtain [4]

$$\frac{\partial g}{\partial K_c} = \frac{K_e}{(K_c + K_e)^2} = \frac{K_e}{K^2} > 0 \qquad (1.1)$$

and

$$\frac{\partial g}{\partial K_e} = -\frac{K_c}{(K_c + K_e)^2} = -\frac{K_c}{K^2} < 0 \qquad (1.2)$$

It may be noted that a simultaneous increase of both components at the margin would also not leave g unchanged.[5]

We can now set out formally the definitions proposed here for the two marginal factor costs (MFCs), terms of WACC

$$MFC_c = WACC + \frac{\partial (WACC)}{\partial K_c} K \qquad (3.1)$$

and

$$MFC_e = WACC + \frac{\partial (WACC)}{\partial K_e} K \qquad (3.2)$$

It can be seen now that the MFC in each case consists of two parts, the first of which is the (starting) WACC itself. The second term will measure the effect on WACC of a unit change in one of the capital 'factors', taking each separately and incorporating the effect of the consequent change in the g ratio. The second terms of (3.1) and (3.2) will be referred to as the *incremental* WACCs (labelled

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[4] As a numerical illustration, 40 units of K_c and 60 of K_e will give a starting g of 40/100. Adding one unit of K_c changes g to 41/101, an increase of 0.006. Adding a unit of K_e on the other hand changes g to 40/101, a decrease of 0.004. Both changes are closely approximated by (1.1) and (1.2).

[5] The ratio would now move to 41/102 = 0.402, an overall increase. An increase of 1.5 in K_e would be required to restore g to 0.4.

IWACC), to distinguish them from the more conventional use of 'marginal'. The term 'marginal' is reserved to describe the change in *total* factor cost following a unit change in one or other 'factor'. As we are referring, however, to unit changes in one 'factor' at a time we shall distinguish between IWACC_c and IWACC_e for the incremental WACC of entrepreneurial and contractual capital respectively.

As a change in one 'factor' will in turn influence the cost of the other, the effect on TFC must be evaluated over the *whole K*. That is, the *interdependenc*e of the costs of K_c and K_e means that $\partial(\mathrm{WACC})/\partial K_c$ or $\partial(\mathrm{WACC})/\partial K_c$ must be multiplied by the *entire K*. The IWACC terms thus have no analogue in standard microeconomic theory. But it bears repeating that neither the WACC term alone, nor the IWACC term alone, represent the MFC. Only the complete expressions of (3.1) and (3.2) do. These are the measures which purport to meet the requirement of Watson and Head (2013), cited earlier, for changes in the average cost structure to be fully allowed for in the MFC definition.

Now it may be argued that a definition of the marginal aspect of WACC such as the above is strictly not necessary, if the impact on total factor cost is all that is desired. Or, that the calculation of incremental WACC term is but an 'intermediate' step in this process, arguably only necessary to the extent that it may be desired to bring out explicitly the average/marginal relationship. But such explicitness is indeed necessary to dispel the notion, implicit in some of the earlier citations, that there exists a marginal WACC concept which is independent of gearing. The essential variability of the gearing, *ceteris paribus*, as a result of *any* change in capital usage needs to be recognised.

To proceed to more specific definitions of (3.1) and (3.2) some functional characteristics may be attributed to the unit costs of each of the two capital 'factors' i and r. Both can be thought of as increasing functions of g, with $di/dg > 0$ and also $dr/dg > 0$. To avoid a possibly excessive level of generality both functions will henceforth be treated as consisting of a constant and a variable part. The contractual component would have a constant at i_0, possibly at the economy's risk free rate, but probably higher as lenders may not regard the firm as entirely risk free. The variable part would rise with gearing g. The equity would have a constant of r_0, the market rate of return for the firm's risk class, higher than i_0, and applicable to an *all-equity* firm.[6] The variable part would again rise with g, as the equity investors would need further compensation for accepting more risk through the gearing.

Treating the firm's risk class as constant, WACC can be more fully written as

[6] A g of 0 would make WACC equal to r_0.

$$WACC = gi(g) + (1-g)r(g) \tag{2.1}$$

with the constants i_0 and r_0 subsumed under *i(g)* and *r(g)*.

The expressions of (3.1) and (3.2) may then be spelled out more by substituting in the expression of (2.1) and its partial derivatives. These will involve also the expressions of (1.1) and (1.2) above

$$\begin{aligned}\frac{\partial(TFC)}{\partial K_c} &= WACC + \frac{\partial(WACC)}{\partial K_c}K = \\ &= gi(g) + (1-g)r(g) + \frac{\partial g}{\partial K_c}[i(g) - r(g) + g\frac{di}{dg} - g\frac{dr}{dg} + \frac{dr}{dg}]K\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\frac{\partial(TFC)}{\partial K_e} &= WACC + \frac{\partial(WACC)}{\partial K_e}K = \\ &= gi(g) + (1-g)r(g) + \frac{\partial g}{\partial K_e}[i(g) - r(g) + g\frac{di}{dg} - g\frac{dr}{dg} + \frac{dr}{dg}]K\end{aligned} \tag{4.2}$$

Numerical illustrations are provided below to help develop a 'feel' for the proposed measures. These can be interpreted as exploring gearing changes 'in the small' i.e. in the vicinity of a historically given capital level of $K = 100$. Table 1.1 shows the effects of changes in K_c and K_e. Linear functions are selected initially for *i(g)* and *r(g)*. WACC figures are shown (and verified) as calculated by alternative formulas in the bold columns headed WACC and AFC (columns 9 and 11). The bold columns headed MFC = ΔTFC (columns 12/13) measure the overall effect on cost, of changes in K_c or K_e, calculated, conventionally, as the difference between TFCs before and after the unit change in the input concerned. These figures can then be compared with columns 20/21 (also in bold) which now calculate the MFCs in the manner of equations (4.1) and (4.2) above. Columns 22/23 perform the calculation in the alternative manner of equations (7.3) and (7.4) which are discussed in the next section. As can be seen, the figures in the three pairs of columns match each other very closely, confirming the validity of the definitions offered. It is worth noting that the approximation is very close also for non-marginal input changes of five units, rather than one unit, illustrated for the sake of brevity only around the $g = 0.5$ position.

The (constant) values *di/dg = 0.01* and *dr/dg = 0.02*, representing the derivatives of the linear functions, mean that WACC falls indefinitely as *g* increases up to a limit of *g =1*. AFC_c and MFC_c rise monotonically. It will be noted however that, although AFC_e rises from an initial value of *0.10* to a final one of *0.11*, MFC_e, as defined here, *falls* over the same range from *0.10* to *0.09*. Contrary, perhaps, to what the usual AC/MC relationships of basic economic theory would have led us to expect, this effect is due to the interactions of the factor costs through the gearing, alluded to already.

Table 1.1: Factor costs and gearing. Linear AFC functions

Column1	Column2	Column3	Column4	Column5	Column6	Column7	Column8	Column9	Column10	Column11	Column12	Column13	Column14	Column15	Column16	Column17	Column18	Column19	Column20	Column21	Column22	Column23
Effect of gearing in vicinity of K = 100 with linear i and r functions								*i = 0.06 + 0.02g* *r = 0.10 + 0.01g*														
		Kc	*Ke*	*K*	*g = Kc/K*	*AFCc = i(g)*	*AFCe = r(g)*	*WACC*	*TFC*	*AFC = TFC/K*	*MFCc = =ΔTFCc*	*MFCe = =ΔTFCe*	*dg/dKc*	*dg/dKe*	*di/dg*	*dr/dg*	*IWACCc*	*IWACCe*	*MFCc = = WACC + + (IWACCc)K*	*MFCe = = WACC + + (IWACCe)K*	*MFCc = = i(g) + g(1-g)di/dg + + (1-g)²dr/dg*	*MFCe = = r(g) - g2di/dg - - g(1-g)dr/dg*
No gearing		0	100	100	**0.0000**	0.0600	0.1000	**0.1000**	10.0000	**0.1000**												
Unit Change ΔKc = 1		1	100	101	**0.0099**	0.0602	0.1001	**0.0997**	10.0701	**0.0997**	**0.0701**		0.010		0.02	0.01	-0.00030		**0.0700**		**0.0700**	
Unit Change ΔKe = 1		0	101	101	**0.0000**	0.0600	0.1000	**0.1000**	10.1000	**0.1000**		**0.1000**		0.000	0.02	0.01		0.00000		**0.1000**		**0.1000**
Low Gearing		20	80	100	**0.2000**	0.0640	0.1020	**0.0944**	9.4400	**0.0944**												
Unit Change ΔKc = 1		21	80	101	**0.2079**	0.0642	0.1021	**0.0942**	9.5137	**0.0942**	**0.0737**		0.008		0.02	0.01	-0.00021		**0.0736**		**0.0736**	
Unit Change ΔKe = 1		20	81	101	**0.1980**	0.0640	0.1020	**0.0945**	9.5396	**0.0945**		**0.0996**		-0.002	0.02	0.01		0.00005		**0.0996**		**0.0996**
50% Gearing		50	50	100	**0.5000**	0.0700	0.1050	**0.0875**	8.7500	**0.0875**												
Unit Change ΔKc = 1		51	50	101	**0.5050**	0.0701	0.1050	**0.0874**	8.8275	**0.0874**	**0.0775**		0.005		0.02	0.01	-0.00010		**0.0775**		**0.0775**	
Unit Change ΔKe = 1		50	51	101	**0.4950**	0.0699	0.1050	**0.0876**	8.8475	**0.0876**		**0.0975**		-0.005	0.02	0.01		0.00010		**0.0975**		**0.0975**
50% Gearing		50	50	100	**0.5000**	0.0700	0.1050	**0.0875**	8.7500	**0.0875**												
Change ΔKc = 5		55	50	105	**0.5238**	0.0705	0.1052	**0.0870**	9.1381	**0.0870**	**0.0776**		0.005		0.02	0.01	-0.00010		**0.0775**		**0.0775**	
Change ΔKe = 5		50	55	105	**0.4762**	0.0695	0.1048	**0.0880**	9.2381	**0.0880**		**0.0976**		-0.005	0.02	0.01		0.00010		**0.0975**		**0.0975**
High Gearing		80	20	100	**0.8000**	0.0760	0.1080	**0.0824**	8.2400	**0.0824**												
Unit Change ΔKc = 1		81	20	101	**0.8020**	0.0760	0.1080	**0.0824**	8.3196	**0.0824**	**0.0796**		0.002		0.02	0.01	-0.00003		**0.0796**		**0.0796**	
Unit Change ΔKe = 1		80	21	101	**0.7921**	0.0758	0.1079	**0.0825**	8.3337	**0.0825**		**0.0937**		-0.008	0.02	0.01		0.00011		**0.0936**		**0.0936**
Maximal Gearing		100	0	100	**1.0000**	0.0800	0.1100	**0.0800**	8.0000	**0.0800**												
Unit Change ΔKc = 1		101	0	101	**1.0000**	0.0800	0.1100	**0.0800**	8.0800	**0.0800**	**0.0800**		0.000		0.02	0.01	0.00000		**0.0800**		**0.0800**	
Unit Change ΔKe = 1		100	1	101	**0.9901**	0.0798	0.1099	**0.0801**	8.0901	**0.0801**		**0.0901**		-0.010	0.02	0.01		0.00010		**0.0900**		**0.0900**

The findings of Table 1.1 are depicted in Figure 1.1 The figure is a variant of Figure 19.3 in Lumby and Jones (2015, p 464), among other similar treatments, which is a typical representation of the 'traditional'[7] view of WACC. But as well as the WACC and the AFCs *i* and *r* of debt and equity, it features also the respective *marginal* costs (in bold). Assuming, as we have, linear functions for *i* and *r*, and reflecting the fact that *r* starts at a higher level than *i*, it shows the

[7] As opposed to the Modigliani and Miller (1958) view

effects of a unit increase of K_c or K_e at various *g* levels. The calculations illustrate the gradual reduction in WACC from *0.10* to *0.8* as gearing moves from *0* to *1*.

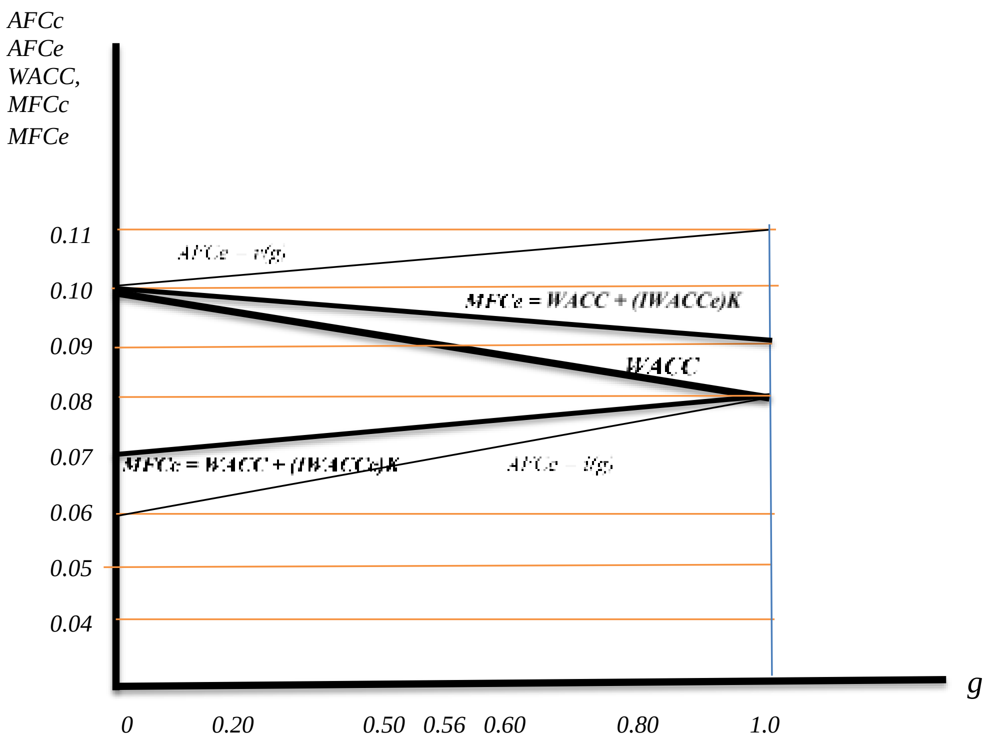

Figure 1.1: Factor costs and gearing. Linear AFC functions

Figure 1.1 depicts the five aspects of unit cost, all linear functions. The AFC curves of the usual textbook treatments are however drawn fainter than the MFC ones, the latter being the relevant ones for optimisation. Given any current gearing level the firm will clearly seek to tap the *less expensive* of the contractual or entrepreneurial capital to finance any expansion. Conversely, if contraction were required, the firm would reduce the *more expensive* component at the margin. It will be noted that the two lines indicating marginal quantities have no point of contact, while WACC has a minimum at $g = 1$. Given our present assumptions, the firm would in any case be using only contractual capital, entrepreneurial capital having a higher MFC_e throughout.

Table 1.2 (which follows the layout of Table 1.1) and Figure 1.2 illustrate the more plausible case of a nonlinear *i(g)* function, although still, in the interests of simplicity, with a linear *r(g).* This enables us to describe a typical gearing optimisation exercise, in the usual sense of seeking the minimum WACC. From the starting value of *0.1,* WACC now passes through a minimum at just under *0.09*, with $g = 0.56$, before returning to *0.1* at $g = 1$. Comparing the ranges of the other variables with Fig. 1.1, only AFC_e remains unchanged under our assumptions, all the others having bigger ranges. The MFC curves *intersect the WACC at its minimum* at just under *0.09.* To the right of that point however we note the continuing rise of MFC_c to a terminal value of *0.10,* whereas MFC continues its *fall* to a new low of *0.05.*

In this more general case, the equality of the MFCs, as defined here, with the minimum WACC conforms in this regard to what might have been expected from conventional economics[8]. Other characteristics of the proposed MFCs are however rather harder to reconcile with conventional MC curves. They generally *move in opposite directions*, crossing over momentarily at the minimum WACC, and diverging from each other to the left and the right of that. WACC can be rising while a marginal measure (in this case the MFC_e) runs below it, and falling while MFC_e is above it. Still, the cross over point is the most significant as the place where the firm switches financing mode from contractual to entrepreneurial, or the reverse.

A property of the proposed IWACC concept in particular is also worth noting. As is clear from Table 1.2, the IWACCs *change signs* at the minimum WACC point. The change is from negative to positive in the case of $IWACC_c$ and the reverse, from positive to negative, in the case of $IWACC_e$. The algebraic sum of these IWACCS and the (positive) value of the starting WACC, determines whether the MFCs of K_c or K_e *would* be higher or lower than the starting WACC at each gearing level.

A final noteworthy feature is that the expressions inside the square brackets of (4.1) and (4.2) are identical in their general form, demonstrating the essential interdependence of the marginal costs of entrepreneurial and contractual capital. The size of the derivatives *di/dg* and *dr/dg* would generally reflect the strength of the prospective loan and equity investors' requirements for increased returns in response to potential rises in gearing. The derivatives will not, however, take the same numerical values throughout, as such values will vary at different points of the respective curves.

[8] Confirming also the intuitive insight of Pike et al (2013) cited earlier.

Table 1.2: Factor costs and gearing. Nonlinear AFC_c function(s).

Column1	Column2	Column3	Column4	Column5	Column6	Column7	Column8	Column9	Column10	Column11	Column12	Column13	Column14	Column15	Column16	Column17	Column18	Column19	Column20	Column21	Column22	Column23
Effect of gearing in vicinity of K = 100 with quadratic i function								*i = 0.06 + 0.02g + 0.02g² r = 0.10 + 0.01g*														
		Kc	*Ke*	*K*	*g = Kc/K*	*AFCc = i(g)*	*AFCe = r(g)*	*WACC*	*TFC*	*AFC = TFC/K*	*MFCc = =ΔTFCc*	*MFCe = =ΔTFCe*	*dg/dKc*	*dg/dKe*	*di/dg*	*dr/dg*	*IWACCc*	*IWACCe*	*MFCc = = WACC + + (IWACCc)K*	*MFCe = = WACC + + (IWACCe)K*	*MFCc = = i(g) + g(1-g)di/dg + + (1-g)² dr/dg*	*MFCe = = r(g) - g2di/dg - - g(1-g)dr/dg*
No gearing		0	100	100	**0.0000**	0.0600	0.1000	**0.1000**	10.0000	**0.1000**												
Unit Change ΔKc = 1		1	100	101	**0.0099**	0.0602	0.1001	**0.0997**	10.0701	**0.0997**	**0.0701**		0.010		0.020	0.01	-0.000300		**0.0700**		**0.0700**	
Unit Change ΔKe = 1		0	101	101	**0.0000**	0.0600	0.1000	**0.1000**	10.1000	**0.1000**		**0.1000**		0.000	0.020	0.01		0.000000		**0.1000**		**0.1000**
Low Gearing		20	80	100	**0.2000**	0.0648	0.1020	**0.0946**	9.4560	**0.0946**												
Unit Change ΔKc = 1		21	80	101	**0.2079**	0.0650	0.1021	**0.0944**	9.5318	**0.0944**	**0.0758**		0.008		0.028	0.01	-0.000189		**0.0757**		**0.0757**	
Unit Change ΔKe = 1		20	81	101	**0.1980**	0.0647	0.1020	**0.0946**	9.5553	**0.0946**		**0.0993**		-0.002	0.028	0.01		0.000047		**0.0993**		**0.0993**
50% Gearing		50	50	100	**0.5000**	0.0750	0.1050	**0.0900**	9.0000	**0.0900**												
Unit Change ΔKc = 1		51	50	101	**0.5050**	0.0752	0.1050	**0.0900**	9.0876	**0.0900**	**0.0876**		0.005		0.040	0.01	-0.000025		**0.0875**		**0.0875**	
Unit Change ΔKe = 1		50	51	101	**0.4950**	0.0748	0.1050	**0.0900**	9.0926	**0.0900**		**0.0926**		-0.005	0.040	0.01		0.000025		**0.0925**		**0.0925**
Min WACC Gearing		56	44	100	**0.5600**	0.0775	0.1056	**0.0898**	8.9848	**0.0898**												
Unit Change ΔKc = 1		57	44	101	**0.5644**	0.0777	0.1056	**0.0898**	9.0748	**0.0898**	**0.0899**		0.004		0.042	0.01	0.000000		**0.0899**		**0.0899**	
Unit Change ΔKe = 1		56	45	101	**0.5545**	0.0772	0.1055	**0.0898**	9.0748	**0.0898**		**0.0900**		-0.006	0.043	0.01		-0.000001		**0.0898**		**0.0898**
High Gearing		80	20	100	**0.8000**	0.0888	0.1080	**0.0926**	9.2640	**0.0926**												
Unit Change ΔKc = 1		81	20	101	**0.8020**	0.0889	0.1080	**0.0927**	9.3615	**0.0927**	**0.0975**		0.0020		0.052	0.01	0.000049		**0.0975**		**0.0975**	
Unit Change ΔKe = 1		80	21	101	**0.7921**	0.0884	0.1079	**0.0925**	9.3375	**0.0925**		**0.0735**		-0.0080	0.052	0.01		-0.000196		**0.0731**		**0.0731**
Maximal Gearing		100	0	100	**1.0000**	0.1000	0.1100	**0.1000**	10.0000	**0.1000**												
Unit Change ΔKc = 1		101	0	101	**1.0000**	0.1000	0.1100	**0.1000**	10.1000	**0.1000**	**0.1000**		0.0000		0.060	0.01	0.000000		**0.1000**		**0.1000**	
Unit Change ΔKe = 1		100	1	101	**0.9901**	0.0994	0.1099	**0.0995**	10.0507	**0.0995**		**0.0507**		-0.0100	0.060	0.01		-0.000500		**0.0500**		**0.0500**

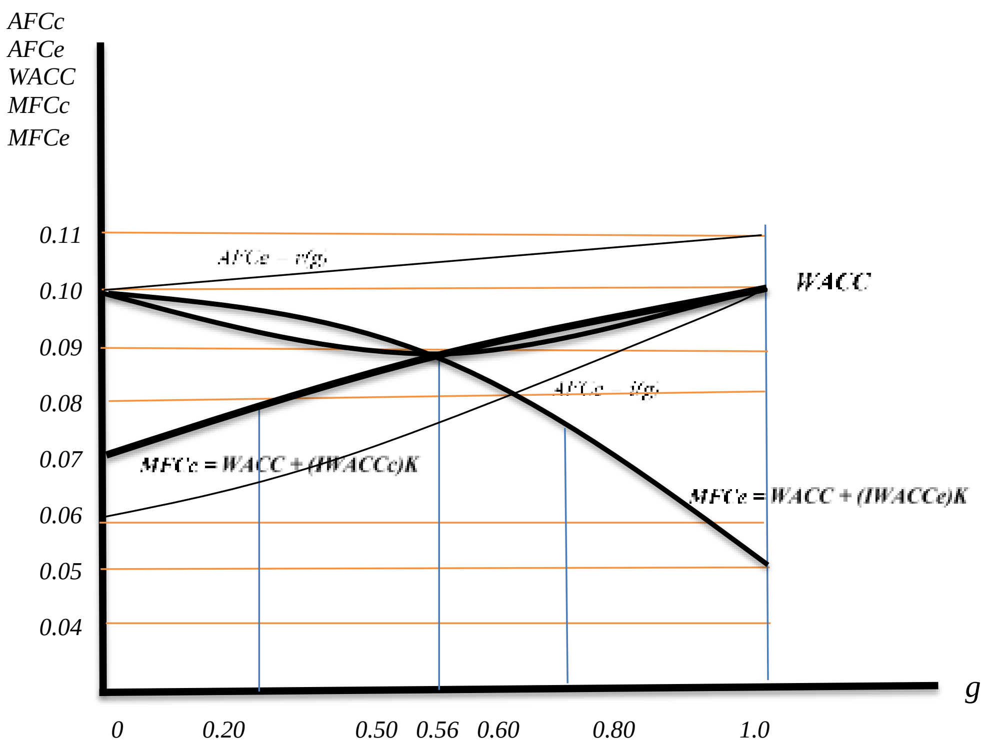

Figure 1.2: Factor costs and gearing. Non-linear AFC function(s)

There are two aspects to the exercise described here. The first (a) is the choice of capital 'factor' to increase in expansion (or reduce in contraction), always in the vicinity of the current level of K and the current g. The second (b) is the move to an optimal g.

The former choice (a) is made easily by comparing MFCs at the current gearing level. Suppose that the firm is operating at a historically given g of *0.25* and that some expansion is now desired. MFC_c being lower than MFC_e, clearly calls for an injection of contractual capital and hence higher gearing. Contraction on the other hand calls for withdrawal of some entrepreneurial capital, again increasing g. If however the starting g were at *0.75*, to the right of the minimum WACC, expansion would require application at the margin of more of the now cheaper K_e, leading to a lower value of g (to the left of $g = 0.75$). Contraction in this region would call for the withdrawal of some K_c, again reducing gearing.

The latter choice (b) of the optimal g, i.e. of the minimum WACC, can be read from Table 1.2, or figure 1.2. To determine the same position formally, we need the total differential d(WACC), defined as the sum of the differentials of K_c and K_e , set equal to zero. From the definition of (2.1) we obtain

$$\begin{aligned}&\frac{\partial g}{\partial K_c}[i(g)+g\frac{di}{dg}+\frac{dr}{dg}-r(g)-g\frac{dr}{dg}]dK_c+\\&+\frac{\partial g}{\partial K_e}[i(g)+g\frac{di}{dg}+\frac{dr}{dg}-r(g)-g\frac{dr}{dg}]dK_e=0\end{aligned} \tag{5}$$

The (identical) expressions in the square brackets have already been encountered in the IWACC terms in (4.1) and (4.2). Here we require equality of the two terms of (5) in their entirety, i.e. including the gearing derivatives outside the brackets. Recalling (1.2) the second term is negative but transferring it to the RHS changes its sign to a plus. It can now be seen that d(WACC) would generally be zero, if the expression inside the brackets took the value of zero, i.e. if

$$i(g)+g\frac{di}{dg}=r(g)+g\frac{dr}{dg}-\frac{dr}{dg} \tag{5.1}$$

Equalisation of the two sides of (5.1) is made possible by the fact that *r(g)* starts from a value higher than *i(g)*, each starting position corresponding to minimum gearing. The general case however requires at least one of the *i(g)* and *r(g)* functions to be non-linear, with the terms eventually taking mutually offsetting values.

Substituting into (5.1) the numerical values from Table 1.2 we have the quadratic

$$0.06 + 0.02g + 0.02g^2 + (0.02+0.04g)g + 0.01 - 0.1 - 0.01g - 0.01g =$$

$$= 0.06g^2 + 0.02g - 0.03 = 0 \qquad (6)$$

Solving (6) we find the root $g = 0.56$, which confirms the results of Table 1.2
The relevance of choice (a) or (b) depends on how far the firm is already committed to significant costs in its historically given position. A desired move from that will be dictated by updated perceptions of profitability and/or changes in the cost functions of $i(g)$ and $r(g)$, but in any case subject to possible rigidities of previous commitments. The underlying presumption is that the functions of $i(g)$ and $r(g)$, and their derivatives, are amenable to estimation. Practical difficulties about such estimation in alluded to by various authors in this regard are somewhat exaggerated.

4 The Link with Profit Maximisation

How does the above exercise relate to the firm's profit maximisation problem? This can now be examined in its general form and in the absence of any rigidity constraining the choice of the overall level of K. For the sake of focusing on capital choices, however, all other inputs (notably labour) are assumed fixed. A revenue function of the form $R(K_c + K_e)$ is assumed which incorporates a production function and a product demand function. The profit π to be maximised can be then represented in the economist's style, albeit somewhat unconventionally, as

$$\pi = R(K_c + K_e) - i(g)K_c - r(g)K_e \qquad (7)$$

The maximum is defined by the two first order conditions (FOCs), corresponding to the two decision variables K_c and K_e

$$\frac{\partial\pi}{\partial K_c} = \frac{\partial R}{\partial K_c} - i(g) - K_c\frac{di}{dg}\frac{\partial g}{\partial K_c} - K_e\frac{dr}{dg}\frac{\partial g}{\partial K_c} = 0 \qquad (7.1)$$

$$\frac{\partial\pi}{\partial K_e} = \frac{\partial R}{\partial K_e} - r(g) - K_c\frac{di}{dg}\frac{\partial g}{\partial K_e} - K_e\frac{dr}{dg}\frac{\partial g}{\partial K_e} = 0 \qquad (7.2)$$

Recalling once again the derivatives of (1.1) and (1.2) the above FOCs imply that

$$\frac{\partial R}{\partial K_c} = i(g) + K_c \frac{di}{dg}\frac{K_e}{K^2} + K_e \frac{dr}{dg}\frac{K_e}{K^2} \qquad (7.3)$$

$$\frac{\partial R}{\partial K_e} = r(g) - K_c \frac{di}{dg}\frac{K_c}{K^2} - K_e \frac{dr}{dg}\frac{K_c}{K^2} \qquad (7.4)$$

The RHSs of (7.3) and (7.4) will be recognised as the separate marginal costs of capital in each mode of utilisation. The MFC definitions here do not directly involve WACC. Each consists of the sum of the AFC (of the varying factor), the marginal effects on the own AFC cost and the marginal effect on the cost of the other factor. This is the formal equivalent of the Pike et al (2015) definition encountered earlier. These marginal effects on the average are positive in the case of K_c, raising MFC_c above AFC_c. The reverse applies to K_e, the marginal effects of which are negative, making MFC_e of entrepreneurial capital lower than AFC cost. As is confirmed from the last pair of columns in tables 1.1 and 1.2 these definitions are equivalent to those introduced in section 3.

The FOCs are then seen to require equalisation of the marginal revenue product of capital (MRPK = $\partial R/\partial K$), undifferentiated, to repeat, by the mode of employment as regards its physical productivity, with each of the separate MFCs of capital. The latter must therefore also equal one another.

We can pursue explicitly the equalisation of the two MFCs required by (7.3) and (7.4). Expressing in terms of g and $(1 - g)$, rather than the K values, we obtain

$$i(g) + g(1-g)\frac{di}{dg} + (1-g)^2\frac{dr}{dg} = r(g) - g^2\frac{di}{dg} - g(1-g)\frac{dr}{dg}$$

or

$$i(g) + g\frac{di}{dg} - g^2\frac{di}{dg} + \frac{dr}{dg} - 2g\frac{dr}{dg} + g^2\frac{dr}{dg} = r(g) - g^2\frac{di}{dg} - g\frac{dr}{dg} + g^2\frac{dr}{dg} \qquad (8)$$

Cancelling terms and rearranging we obtain

$$i(g) + g\frac{di}{dg} = r(g) + g\frac{dr}{dg} - \frac{dr}{dg} \qquad (8.1)$$

That is none other but our previous expression (5.1) which defined the minimum WACC in the previous section. The same condition has now been derived from the necessary equality, at the profit maximum, between the marginal costs of the two types of capital.

The move towards the optimum can be thought of as a stepwise process of applying more units of capital, financed by the the mode of utilisation offering the *lower* of the two MFCs as long as MRPK exceeded that MFC. The process continues in principle, in the absence of any constraints, until an overall *K* level is reached where MRPK and the two MFCs are brought to equality. The implied process is therefore again one of simultaneous determination of overall *K*, and of the amounts of each form of capital necessary to bring their MFCs to equality with MRPK and with one another.

We should note here that the diagram of Figure 1.2, with gearing on the horizontal axis, does not allow us to show directly the actual quantities of K_c and K_e which would represent the firm's preferred overall *K* and would, at the same time, make up a preferred *g* ratio. Representational limitations should not however conceal the simultaneous determination of overall *K* and optimal *g* which is implied by the analysis. This is in contrast with Lumby and Jones's (2015) earlier suggestion that such simultaneity cannot be handled analytically.[9]

Having established that profit maximisation also requires the minimisation of WACC, it may be asked why the profit (or present value) maximising firm would not *always* be, at its WACC minimising *g* as well at its preferred *K* level? We have already seen references to 'target' levels of gearing which are not necessarily optimal in terms of minimising WACC. For such minimisation should be regarded as a corollary of the profit (or present value) maximisation hypothesis itself, at least in the 'long run', when possible constraints would by definition not apply. Unlike in Pike et al (2015), the firm would *never* 'target' any other *g* level. Differences in the WACC minimising level of *g* across firms would reflect characteristics specific to these.

The implication of this for WACC as a criterion rate is rather startling. At the minimum WACC, where we have found the IWACC terms of (4.1 - 4.2) to become zero, the MFC measure of *either* K_c or K_e reduces to equality with

[9] Figure 1.2 is still put forward as an advance on Fig 19.3 of Lumby and Jones (2015), and similar versions of it in finance literature. For these plot WACC against the AFC_c and AFC_e curves only which, unlike the MFC ones, in principle play no part in determining a preferred position.

WACC, as that consists simply of the remaining non zero term. The MFC curves, whether upward or downward sloping, would go through the minimum of the WACC curve and, at that point, there would be no question of having to choose one or the other. *WACC itself would proxy either*. So WACC *is* the requisite marginal concept, after all! We can interpret the earlier quotations form Bronfenbrenner (1960), Rose (1987), along with the somewhat intuitive statement of Pike et al (2015) as envisaging exactly this.

The MFC definitions of (3.1 - 3.2) and (4.1 - 4.2) offered here do not however align very well with the other views among those cited from the literature. Possibly, in as much as the changes in the marginal and average quantities are linked, the definitions are still in the spirit of Watson and Head's (2013) idea of 'rolling average marginal cost', as far as that can be interpreted. But our definitions do not agree with Wang's (1988) idea of the 'weighted average of the marginal costs', as that has no clear justification as an optimising tool and seems to come *ex post* the choice of gearing.

In any event the foregoing has hopefully highlighted the central place occupied by changes in gearing (automatic or intended) in the firm's overall optimisation problem. To that extent it has been argued, contrary to some views cited earlier, that stipulations of constant gearing are unhelpful.

5 Overview of Choices of Scale and Risk Level

The exercise of Table 1.2 has focused on input adjustments and gearing optimisation 'locally' i.e. around a scale characterised by a given $K = 100$ and subject to the assumption of no change in the firm's risk profile. That would correspond to a situation where the $i(g)$ and $r(g)$ functions, and in particular the constants i_0 (in eq. 3.1) and r_0 (in eq. 3.2) indicating the general levels of contractual and entrepreneurial equity costs, were unchanged. Any changes in riskiness, and hence in factor costs, are so far due entirely to the changes in gearing.

As suggested earlier, the relevance of the exercise is governed by whether the firm is significantly committed to its current scale of operation or is at a major planning stage (say before significant reconstruction). In the latter case we would be looking for a 'globally' optimal scale, with total K now serving as a proxy for the *scale of output.*

In the absence of specific changes however the preceding analysis should not apply also to a different scale of operation. Table 1.2 could be repeated at, say, $K = 200$, *mutatis mutandis,* and Fig 1.2 reproduced without essential change. On the other hand the constants i_0 and r_0 might change with higher scale, say falling to *0.05* and *0.09* respectively. Such lower capital costs might be the result of the firm acquiring market power as it got larger. The table would then change to show minimum WACC at just under *0.08* and terminal MFC values of *0.09* and *0.04*. But again that would simply be a uniform downward move by *0.01* throughout and Figure 1.2 would show all the curves lowered by that amount. The WACC and other curves would move uniformly, downward or upward, as in Rose (1987, p.17). The table and figures thus revised, are not presented here, in the interests of brevity.

A more general tool is presented instead to allow for changing gearing, as well as a changing WACC, as the firm changes scale and also, possibly, risk profile. This is presented most conveniently through Figures 2 and 3.

Figure 2 is a variant of a similar figure in Bronfenbrenner (1960, p. 306). It depicts

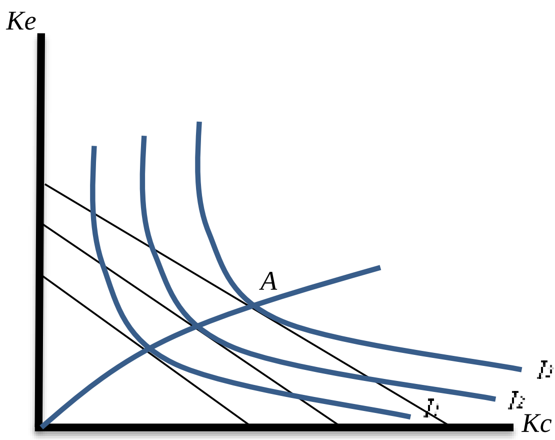

Figure 2: Choice of Gearing as Firm expands

the choice between finance to be obtained contractually (K_c) or supplied entrepreneurially (K_e), as the firm expands its total use of capital and assuming, as before, other inputs fixed. The straight lines represent different budget levels and are drawn with a slope of less than 45^0 to the horizontal axis, to indicate a higher cost per unit of equity than per unit of debt. The *I's* are production isoquants reflecting, as before, the *interchangeability* of the two types of capital in production, as their mode of employment does not affect their physical productivity. The *I's* are also meant however to incorporate *attitudes* as between equity and debt finance, such as fear of loss of control or fear of excessive debt burden etc. The *I's* have thus now acquired something of the character of 'indifference curves'. They are not drawn completely symmetric to the axes, as their curvature may well vary at different overall levels of *K*. The budget lines also display different slopes illustrating here the possibility of falling K_c costs, relative to K_e as the utilisation of *K* grows. The resulting expansion path is not linear but indicates equilibrium positions involving increasing *g* ratios.

Position A in Figure 2 represents the optimum within the limitation represented by the budget line furthest out to the right, although such a limitation may be overcome at a later decision point. The sum of K_c and K_e in the vicinity of A defines a new table, which would not now be a mere replica of Table 1.2 but might well reflect changes in the *i(g)* and *r(g)* functions, e.g. making the latter non-linear as well. Again we omit presenting a new table. On the production side the move to a significantly different *K* level might have a more (or less) than proportionate effect on output, while revenue might also reflect a downward sloping demand curve.

The picture of Figure 3, purports to encompass the range of possibilities shown in Figure 2 and extend the analysis to define a profit maximum. Relaxing also the assumption of other inputs constant, we can still use the new level of total *K* (i.e. the sum of K_c and K_e), to represent the scale of output on the horizontal axis. *K* is obtained by selecting, sequentially at each point, the amounts of K_c and $K_{e,}$ (and hence also the *g* ratio) that are best combined with other inputs to define optimal scale of operation. The MPRK can be plotted against *K* as a downward sloping curve and helps to identify the profit maximising position shown as point A. That is however the point where MRPK intersects with *WACC*, not with any conventionally shaped MFC curve. The gearing decision is allowed for as an integral part of the overall optimisation sequence, whereby the firm seeks the minimum WACC achievable given its profit maximising overall capital requirement.

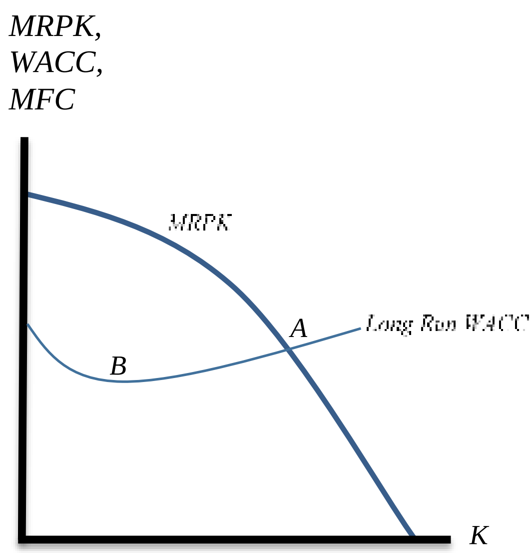

Figure 3: Simultaneous Choice of Scale, Gearing and, possibly, Risk Profile

The above position A is *not the same* as seeking the scale at which WACC reaches a 'global' minimum (point B). That would be inconsistent with profit maximisation. The level of capital costs to the firm may, as already suggested, vary with scale.[10] The WACC curve is drawn such as to show a higher overall cost of capital at different output scales, i.e. generally upward sloping, save for an initial phase where the firm transitions from a start up to a more viable scale. After reaching a minimum at B, the WACC curve begins to rise, reflecting increasing capital costs, probably due to difficulties in accessing additional capital resources.[11]

The process may be again seen as a twofold one, of determining scale first and optimal gearing thereafter. But essentially we have *simultaneity* of the two decisions. Neither K nor g is to be taken as given. The position of A is the result of the firm's adopting, as it expands, the (optimal) levels of K_c and $K_{e,}$ and hence WACC and g.

That indeed seems to be the procedure described by Rose (1987). But unlike his labelling of the curve as a 'MCC' one, we shall describe it here as a *WACC* curve. This is now defined as a curve each of the points of which represent a WACC minimising choice of g at each possible output scale, and involving equality of WACC with the MFCs of K_c and K_e at each point. Hence, labelling this curve as a MFC one is misleading. Rather, we have here a WACC curve *in a different sense* from that of Figure 1.2. The curve of Figure 3 will be labelled the *Long Run WACC curve*, to reflect conditions of essentially unrestricted choice of scale as part of the profit maximisation exercise.

Nomenclature apart, it should be clear that the present analysis only calls for *a single curve* rather than the conventional configuration of two, an AFC and a MFC. And in locating the equilibrium A at the intersection of MRPK with WACC we have the apparent resolution of the puzzle of the use of the WACC as a marginal criterion, consistently with Bronfenbrenner (1960).

Figure 3 is meant to represent, in addition, the possibility of a firm involved in *riskier projects*, as it expands. Again we can hypothesise an initially high starting level of risk, falling in the firm's initial growth phase, but then rising again as the firm seeks to expand. The firm can be visualised as having ordered projects in ascending order of risk, and it is logical to expect it to go for the less risky first. The 'marginal' project would then be by definition riskier than the earlier ones and the horizontal axis can represent *ascending overall risk* as well as increasing

[10] Indicatively, assume that a g of *0.4* at or near start up (K =100) might offer moderately risk averse investors returns of $i = 0.08$ and $r = 0.11$, thus an acceptable WACC of 0.098. But at a higher scale of K = 300, and with both i and r higher, a higher g of, say *0.5*, would be called for giving debt holders *0.10*, equity holders *0.14* and a thus WACC of *0.12*.

[11] Analytically this resembles the position of a monopsonistic firm in the market for a factor.

scale. The shape of the Long Run WACC curve illustrated above however, although perhaps the most likely, is not the only possible one. The curve may be flatter or even become downward sloping e.g. if the firm acquires monopoly power as it expands, thereby becoming less risky.

The diagrammatic and analytical representation of the optimum in this section has clearly been less detailed than that of sections 3 and 4. The five sections together however contain the core of our attempt to apply the usual tools of economics to integrate financing (gearing) choices with scale and risk aspects of the firm's behaviour.[12]

A final defence of WACC as a criterion rate which appears in Bronfenbrenner (1960, pp 306-7), calls for comment. He defends use of WACC for another reason, namely the need to preserve the possibility of some 'profit' (positive or negative) in the face of the 'adding up theorem'. Under postulated conditions of long run competitive equilibrium and production at minimum average cost output is distributed in its entirety, once all their factors are paid returns equal to their marginal products. Clearly that would not apply if, in our terms,[13] K_c and K_e, with a common marginal product, were paid different rates r and i. Compatibility with the adding up theorem would be restored if, instead, the *composite* factor K were paid the WACC and its use adjusted accordingly.[14]

This line of reasoning can also be accommodated conceptually within our present scheme. The 'profit' maximised here is embodied in the return on the equity as a surplus over and above the equity's opportunity cost. That in turn we have defined as a minimum of r_0, on account of the risk class, and a further amount, on account of gearing. In this general framework, and without stipulating competitive equilibria and minimum costs, the 'adding up' aspect should not be an issue. As one 'factor', the equity capital, is remunerated residually, the sum of all factor payments will *ex post* come to exactly the amount of output in any event. *Ex ante* the factor payments are *expectations* shaped by opportunity costs, even though payments on borrowed capital are, in principle, secured contractually. If the two capital 'factors' are offered *expected* remunerations in line with the MFCs of (3.1) and (3.2), the *ex post* return to the equity will include a profit or loss depending on overall performance above or below expectation. 'Profit' will 'normally' *not* be zero, but hopefully, from the standpoint of equity holders, positive.

[12] For a statement with a focus on utility, rather than profit maximisation see Zafiris (2016).

[13] The adding up debate is typically also conducted in terms of physical product rather than monetary values.

[14] For then, with one more input L (labour), we would be back to the required
$K(\partial Q/\partial K) + L(\partial Q/\partial L) = Q$

6 Capital Quantities vs Values

One of the aims of this paper has been to address disparities in the definitions of capital costs in economics and finance literature. One (perhaps the main) difference between the two approaches, already alluded to, is that in financial models capital costs are expressed in terms of percentage *rates of return* applied to the *values* of financial instruments, such as equity shares and bonds. Such values can be either book or, more usually, market ones (which may be very different) and sometimes combinations of the two.[15] In economics modelling, on the other hand, inputs are measured in terms of *physical units* and *absolute* rental costs per unit. Such indeed is the case with our models of sections 3 and 4.[16] The economist's approach thus separates clearly quantities from prices (or costs), values being derived simply as the product of the multiplication of quantities by prices. In contrast the finance approach to valuation conflates quantities and prices (costs) in a rather more ambiguous concept of 'value'.

This problem has been addressed in the foregoing, by interpreting physical units of an input, as priced at *one* per unit in money terms, as at a particular point in time. Percentage rates of return have thus been interpreted also as absolute rental costs. To differentiate more clearly between quantity and price in the treatment of capital we need to distinguish between two aspects of cost. The first is the price of physically identical units of capital ('machines' for simplicity), to be denoted here by P_k. Then K_c or K_e would measure, as before, the number of machines financed by either debt or equity. The second aspect arises from the fact that the firm's ability to acquire machines depends not only on the machine price P_k. but also on the market prices of firm's own debt and equity financial instruments, ('bonds' and 'stocks', respectively). Let us denote these by P_c and P_e and again assume initial values of *one*. All three prices P_c, P_e and P_k are subject to volatility which would make them diverge from these initial levels.[17] The firm might then be faced with higher costs due to a rise in P_k. But even with unchanged machine prices, the firm would face higher *effective costs* for its machines, if P_c, and/or P_e *fell*, reducing the amounts of finance that could be raised through the corresponding instrument, and vice versa in the event of rises in P_c, and/or P_e .

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[15] Capital' values are not usually imputed to labour resources, although 'human capital' is increasingly reckoned with, alongside other 'intangibles'

[16] The original version of the 'adding up theorem' discussed in the last section is also couched in terms of physical units of input and output.

[17] Listed companies will have instruments with readily ascertainable market values at most times. This applies rather less to incorporated businesses, although the principle is the same. The 'prices' P_c, P_e may be thought of as indices taking values above or below 1.

Consider now the status of these prices in the optimisation exercise. Unlike a possibly variable product price P_c, which eq. (7) has been subsumed in a downward sloping revenue function, the machine price P_k is best thought of as determined exogenously. The same may be said however of P_c and $P_{e,}$ which, although specific to the firm's financial instruments, should still be regarded as determined exogenously, and thus outside the firm's control. Movements in these prices will represent the joint effect of a) market forces affecting bond and/or equity returns generally and b) perceptions of the firm's prospects in particular. But these are precisely the determinants of the general levels of the returns on debt and equity encountered in the previous sections, when the influence of gearing is removed. It is only a short further step to argue that P_c and P_e would mirror such market determined levels of return, standing in a generalised *inverse* relationship with these. It would seem then that the effective machine price can then be subsumed by our earlier.

To make the transition from physical capital to financial capital the profit function of (7) may be rewritten as

$$\pi = R(K_c + K_e) - i(g)P_k K_c - r(g)P_k K_e \qquad (9)$$

with the effects on P_c and P_e embedded in i and r. The restrictive assumption of $P_k = 1$ may now be removed. But treating P_k as a constant means that this reformulation would add little of substance to our earlier results.

7 Conclusion

This paper has surveyed some literature advancing the use of WACC as a possible benchmark rate of return in investment, under unduly restrictive conditions, especially unchanged gearing. Some of the suggestions made have been found to derive largely from incomplete conceptualisation and definition of the marginal aspects of WACC. Also from failure to address directly the likely impact of gearing changes on WACC as the firm varies its use of contractual and/or entrepreneurial capital.

Having made explicit such changes in gearing, we have identified the requisite marginal concept as the marginal cost of one *or* the other of contractual and entrepreneurial capital, after allowing for the change in gearing. A MFC definition for either has been proposed which consists of the sum of WACC and a term labelled 'IWACC', for 'incremental WACC' in each case. The properties of the resulting measures have been explored and compared with those of conventional cost curves. The implied decision process is one of selecting iteratively the lower of the marginal costs of debt or equity, except at the minimum WACC where the two cross over. We have thus shown the relevance of WACC as a marginal criterion has been where gearing is optimal in the sense of minimising WACC.

A similar conclusion has been obtained from our attempt to trace the implications of the firm's profit maximisation problem. The formal exercise has been shown to encompass the optimisation of the gearing at the minimum WACC at any level of capital utilisation. Minimum WACC has thus been shown to be the necessary consequence of the profit maximisation hypothesis and negation of its relevance for marginalism is tantamount to a questioning of the hypothesis itself.

The formal framework presented here is felt to offer some advantages in terms of integrating the decision processes involved and bridging an apparent gap between relevant models in economics and finance. These may be pursued in further research. The illustrations provided have hopefully also been of some pedagogic value.

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