Characterization of Robust Solution Quadratically Constrained Quadratic Optimization Problem Subjected to Data Uncertainty

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Abstract

Robust Optimization (RO) arises in two stages of optimization, first level for maximizing over the uncertain data and second level for minimizing over the feasible set. It is the most suitable mathematical optimization procedure to solve real-life problem models. In the present work, we characterize robust solutions for both homogeneous and non-homogeneous quadratically constrained quadratic optimization problem where constraint function and cost function are uncertain. Moreover, we discuss about optimistic dual and strong robust duality of the considered uncertain quadratic optimization problem. Finally, we complete this work with an example to illustrate our solution method.

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1. Introduction

The present paper is devoted to the characterization of the solutions of optimization problems which are affected by data uncertainty. This means that all the data of the problem are neither known nor available when its optimal solution has to be computed. Such an optimization model assumes the usual one but it is clearly different. In fact, it allows to handle model error, to take into account imperfections, additive noises or external uncertainties. For examples: (a) An entrepreneur must estimate the time required to finish a job. Uncertainties due to the labor market, the supply of materials, bad weather conditions, etc. constitute unknown parameters to consider; (b) In robot design, even if the movement model is carefully planned to provide a stable locomotion, a humanoid robot is very likely to fall due to the uncertainties induced by the irregularities in the ground; (c) When designing a nuclear power plant, engineers have to consider the uncertainties about earthquakes, the strength of beams, etc. Moreover, one cannot always ignore uncertainties during modeling process (i.e. always considering a determinist model) because that could lead to security issues in real-world applications (e.g., designing nuclear power installations, autonomous car control system, aeronautical structures optimization). Then, it is certain that optimization model subject to data uncertainty is most adequate methodology for solving real-word problems: it is Robust Optimization (RO). In a nutshell, RO provides a solution which is immunized against the effect of parameters uncertainty: that solution is called robust solution [1,2,5]. Nevertheless, there is a similarity between RO and Stochastic Optimization (SO). The latter optimization model can be considered like a particular case of RO where the uncertainty set distribution is assumed to follow a probability law. Note that stochastic optimization is wholly a very important field of research and profound to be developed here. However, for more details, we recommend the works of Peter Kall and Stein W. Wallace [32], John R. Birge and François Louveaux [9], Alexander Shapiro, Andrzej Ruszczyński et al. [46,47], Kurt Marti [37], George B. Dantzig and Gerd Infanger [15,27].

Concerning the concept of robustness and uncertainty in mathematical programming models, it was shyly started with the pioneer works of S. K. Gupta, J. Rosenhead [22,44], G. B. Dantzig [15], F. J. Gould, and A. L. Soyster who first called it Inexact Linear Programming" [21,49]. Afterwards, the topic of RO was quickly expanded through mainly the contributions of J. M. Mulvey et al. [38], P. Kouvelis, and Gang Yu [34], L. El Ghaoui et al. [17,18], A. Ben-Tal and A. Nemirovski [2,3,4,5], D. Bertsimas and M. Sim [6,7,8,48], V. Jeyakumar et al. [30,31,35]. In these last years, RO has been a focus within many research communities starting with the field of control, convex optimization, mathematical programming, or even economics, and many fields of engineering science [2,26,33]. Basically, whenever an optimization problem is formulated, the question arises whether really all parameters and inputs are exactly known and what changes if they are not. In this sense, it is not surprising that many researchers were and are attracted by the challenges of robust optimization.

Finally in this work, our purpose is to analyze Quadratically Constrained Quadratic Optimization Problems (QCQOPs) subjected to uncertain data both in the constraint function and cost function. And we next investigate the characterization of their solutions. In fact, QCQOPs are very important in their own right but not enough studies with uncertainties. This choice is also motivated by the fact that QCPOPs are in some ways the general forms of almost all mathematical programming models. In addition, the literature about analysis of quadratic forms [13,16,19, 42], quadratic programming and QCQOP solution methods (basically indeterminist form) is well-supplied with well-known and sophisticated techniques.

For instance, they occur as subproblems in methods for numerical/algorithmic optimization

problems such as Newton-Lagrange methods (Sequential Quadratic Programming (SQP), Interior-Point Method (IMP), BFGS (Broyden, Fletcher,Goldfarb, and Shanno) method) in one hand. In other hand, any smooth optimization problem can be approximated with a quadratic model (thought of doing second order Taylor series approximation). Even if the considered problem is not enough smooth, it could be approximated quadratically. For this technique, we refer to the topic of Derivative Free Optimization (DFO) methods, see for example the introductory book of Andrew R. Conn, Katya Scheinberg and Luis N. Vicente [14]. We also recall that under some basic assumptions (at least symmetry and semi-dene positivity) on the data of a QCQOP, we can get convex or Second Order Conic Optimization Problem (SOCP), see Section 2. It is well-known that in the field of optimization an ideal and wanted framework is the convexity one [5,10,11,20,23,24,25,28,29,31,36,39,40,43,45] since that can make easiest to solve exactly optimization problems (i.e., the characterization of their global solution).

With the uncertainties, this advantage is in the most of time difficult to have in RO without well-thought-out assumptions.

The rest of the paper is organized as follows. Section 2 recall some preliminaries and describes the Quadratically Constrained Quadratic Optimization Problem Model which will be study throughout this paper. In Section 3, we give some fundamental results basically joint-range convexity conditions and S-Lemma.

The Section 4 is dedicated to the characterization of robust optimal solutions for an uncertain quadratic optimization problem with respect to homogeneous case and non-homogeneous one. In Section 5, we compare optimistic dual and robust strong dual of the uncertain optimization problem. We deal with an example in the homogeneous case in the Section 6 for an illustration. Section 7 concludes our work.

2. Quadratically Constrained Quadratic Optimization Problem Model

In this section, we start by recalling some basic definitions and fixing some notations. In what follow, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the linear space of n-dimension vector and $\mathbb{S}_n(\mathbb{R})$ denotes the space of n×n symmetric matrices with real entries. For all, A, B $\in \mathbb{S}_n(\mathbb{R})$, A $\succcurlyeq 0$ (respectively A $\succ 0$) means that the matrix A is positive semi-definite (respectively A is positive definite). By analogy, A $\succcurlyeq$ B (respectively A$\succ$B) is equivalent to A$-$B$\succcurlyeq$0 (respectively A$-$B$\succ 0$). For any twice differentiable function q: $\mathbb{R}^n \longrightarrow \mathbb{R}$, ∇q denotes the gradient (first derivative) of q and ∇^2q denotes the Hessian matrix (second derivative) of q. Given a set E, intE denotes the interior of E. If E_1 and E_2 are subsets of a set E, then the set $E_1 + E_2 = \{e = e_1 + e_2 \in E | (e_1, e_2) \in E_1 \times E_2\}$. The segment $[v_1, v_2]$ is defined by $[v_1, v_2] = \{v \mid v := tv_2 + (1-t)v_1, \ t \in [0,1]\}$. A set C is convex if the line segment between any two points in C lies in C i.e., if for any $c_1, c_2 \in C$ and any $\lambda \in [0,1]$, we have $\lambda c_1 + (1 - \lambda)c_2 \in C$. A function f: $\mathbb{R}^n \longrightarrow \mathbb{R}$ is convex if and only if its epigraph is a convex set.

A general formulation of a quadratic optimization problem under uncertainty can be formulated as follows:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A_u x + a_\omega^T x \qquad \text{(QCQP)}$$

$$\text{s.t. } \frac{1}{2} x^T B_v x + b_\varsigma^T x + \beta \leq 0$$

where: $\beta \in \mathbb{R}$, $a_\omega, b_\varsigma \in \mathbb{R}^n$, $A_u, B_v \in \mathbb{S}_n(\mathbb{R})$, and (u,v,$\omega$,$\varsigma$) belongs to the uncertainty set $\mathcal{U}$. In addition, we use an affine interpolation to define uncertainty data. Then, we set :

$A_u = A_0 + uA_1$, $B_v = B_0 + vB_1$, $a_\omega = a_0 + \omega a_1$, $b_\varsigma = b_0 + \varsigma b_1$,
with
$A_0, A_1, B_0, B_1 \in \mathbb{S}_n(\mathbb{R})$ and $a_0, a_1, b_0, b_1 \in \mathbb{R}^n$, $\mathcal{U}$ is an Euclidian product of sets and given by $\mathcal{U} = [u_1, u_2] \times [v_1, v_2] \times [\omega_1, \omega_2] \times [\varsigma_1, \varsigma_2]$.
We also put ;
$\mathcal{U}_1 = [u_1, u_2], \mathcal{U}_2 = [v_1, v_2], \mathcal{U}_3 = [\omega_1, \omega_2]$ and $\mathcal{U}_4 = [\varsigma_1, \varsigma_2]$.

3. Convex Analysis of Quadratic forms and S-procedures

The convex analysis and the study of quadratic forms are the essential ingredients to concoct a good recipe in the field of optimization. Indeed, in a convex world or in the presence of quadratic forms, the characterization of the solutions for an optimization problem can be carried out without major difficulty. This is why, R. T. Rockafellar stated that the great watershed in optimization is not between linearity and nonlinearity, but convexity and non-convexity (Rockafellar 1993, [11]). In this work, we use mainly joint-range convexity results for quadratic forms which were initially proposed by L. L. Dines [16] and improved later by other authors such that B.T. Polyak [42]. In addition, these convexity properties of quadratic are helpful to prove some results like Lemma 1, Theorem 2 and Corolary 1: we talk about S-procedure for generality. The S-Lemma is a useful tool especially in Control Theory and RO.

Theorem 1 (L. L. Dines (1941) [16])
Given, A, B $\in \mathbb{S}_n(\mathbb{R})$. Then the set $\{(x^T A x, x^T B x) \mid x \in \mathbb{R}^n\}$ is convex.

The generalization of convexity results for more than two homogeneous quadratic forms was established by B.T. Polyak [42] and given as follows:

Proposition 1 ([42])
Let $A_1, \dots, A_m \in \mathbb{S}_n(\mathbb{R})$. If the matrices $A_1, \dots, A_m$ commute, then the set
$$\{(x^T A_1 x, \dots, x^T A_m x) \mid x \in \mathbb{R}^n\}$$
is closed convex cone.

The following lemma is the homogeneous version of the famous S-lemma.

Lemma 1 (S-lemma [29,41])
Let $A_1, A_2 \in \mathbb{S}_n(\mathbb{R})$. Assume that there exists $x_0 \in \mathbb{R}^n$ such that $x_0^T A_1 x < 0$. Then, the following statements are equivalent:
(i) $\forall x \in \mathbb{R}^n$, $x_0^T A_2 x < 0 \implies x_0^T A_1 x < 0$
(ii) $\exists. \lambda \geq 0 \mid A_1 + \lambda A_2 \succcurlyeq 0$.

Theorem 2 Let $A_0, A_1, B_0, B_1 \in \mathbb{S}_n(\mathbb{R})$; $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\mathcal{U}_1 = [u_1, u_2], \mathcal{U}_2 = [v_1, v_2]$, and assume that

$$\Omega = \{(x^T A_{u_1} x, x^T A_{u_2} x, x^T B_{v_1} x, x^T B_{v_2} x) \mid x \in \mathbb{R}^n\} \text{ is convex.} \tag{1}$$

Then, for some $\alpha, \beta \in \mathbb{R}$, exactly one the following two statements holds:

(i) $\exists\, x \in \mathbb{R}^n |\ \frac{1}{2} x^T A_u x\ < \alpha \implies \frac{1}{2} x^T B_v x\ < 0,\ \forall (u, v) \in \mathcal{U}_1 \times \mathcal{U}_2$.

(ii) $\exists (\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}, \exists\, (\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that :

$$\lambda_1 \left(\frac{1}{2} x^T A_{\bar{u}} x - \alpha \right) + \lambda_2 \left(\frac{1}{2} x^T B_{\bar{v}} x + \beta \right) \geq 0\, ,\forall\, x \in \mathbb{R}^n.$$

Proof $[(ii) \implies \neg(i)]$. Assume that (ii) holds. If (i) is satisfied, then

$$\exists\, x \in \mathbb{R}^n |\ \frac{1}{2} x^T A_u x\ < \alpha \ \text{and}\ \frac{1}{2} x^T B_v x + \beta < 0,\ \forall (u, v) \in \mathcal{U}_1 \times \mathcal{U}_2.$$

This implies that for all $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$, we have :

$$\exists\, x \in \mathbb{R}^n |\ \lambda_1 \left(\frac{1}{2} x^T A_u x - \alpha \right) + \left(\frac{1}{2} x^T B_v x + \beta \right) < 0,\ \forall (u, v) \in \mathcal{U}_1 \times \mathcal{U}_2.$$

Which contradicts (ii).

Let us show that $[\neg(i) \implies (ii)]$.

Firstly, let us show that the set

$$\Gamma = \left\{ \left(\max_{u \in \mathcal{U}_1} x^T A_u x, \max_{v \in \mathcal{U}_2} x^T B_v x \right) \mid x \in\ \mathbb{R}^n \right\} + \mathbf{int} \mathbb{R}_+^2 \tag{2}$$

is convex.

Let $(q_1, r_1), (q_2, r_2) \in \Gamma$ and $\delta \in [0,1]$. *So, there exist*
$x_1, x_2\ \in \mathbb{R}^n$ such that:

$$\begin{cases} \max\limits_{u \in \mathcal{U}_1} x_1^T A_u x_1^T < q_1 \\[4pt] \max\limits_{u \in \mathcal{U}_1} x_1^T B_v x_1^T < r_1 \\[4pt] \max\limits_{u \in \mathcal{U}_1} x_2^T A_u x_2^T < q_2 \\[4pt] \max\limits_{u \in \mathcal{U}_1} x_2^T B_v x_2^T < r_2 \end{cases} \tag{3}$$

For all $x \in\ \mathbb{R}^n, u \longmapsto \frac{1}{2} x^T A_u x$ is an affine mapping with respect to (w.r.t) u. Consequently, it attains its maximum value on the compact set $\mathcal{U}_1$ more precisely, have:

$$\max_{u \in \mathcal{U}_1} x^T A_u x = \boldsymbol{max}\{x^T A_{u_1} x, x^T A_{u_2} x\}.$$

Reasoning in same manner that previously, one has**:**

$$\max_{u \in \mathcal{U}_2} x^T B_v x = \boldsymbol{max}\{x^T B_{v_1} x, x^T B_{v_2} x\}.$$

By reporting the maximum value equalities in inequality system (3), one finds:

$$\begin{cases} x^T A_{u_1} x < q_1, \\ x^T A_{u_2} x < q_1, \\ x^T B_{v_1} x < r_1, \\ x^T B_{v_2} x < r_1, \\ y^T A_{u_1} y < q_2, \\ y^T A_{u_2} y < q_2, \\ y^T B_{v_1} y < r_2, \\ y^T B_{v_2} y < r_2, \end{cases} \tag{4}$$

It follows from the system (4) above that:

$(q_1, q_1, r_1, r_1) \in \left\{ \left(x^T A_{u_1} x, x^T A_{u_2} x, x^T B_{v_1} x, x^T B_{v_2} x \right) \right\} + \mathbf{int}\mathbb{R}_+^4,$

$(q_2, q_2, r_2, r_2) \in \left\{ \left(y^T A_{u_1} y, y^T A_{u_2} y, y^T B_{v_1} y, y^T B_{v_2} y \right) \right\} + \mathbf{int}\mathbb{R}_+^4,$

which is equivalent to

$$\begin{cases} (q_1, q_1, r_1, r_1) \in \Omega + \mathbf{int}\mathbb{R}_+^4, \\ (q_2, q_2, r_2, r_2) \in \Omega + \mathbf{int}\mathbb{R}_+^4. \end{cases}$$

Since Ω *is co*nvex, the set $\Omega + \mathbf{int}\mathbb{R}_+^4$ *is also co*nvex. Then
$(\delta q_1 + (1 - \delta)q_2, \ \delta q_1 + (1 - \delta)q_2, \ \delta r_1 + (1 - \delta)r_2, \delta r_1 + (1 - \delta)r_2) \in \Omega + \mathbf{int}\mathbb{R}_+^4.$
Thus, there exists $z \in \mathbb{R}^n$ such that

$$\begin{cases} z^T A_{u_1} z < \delta q_1 + (1 - \delta)q_2, \\ z^T A_{u_2} z < \delta q_1 + (1 - \delta)q_2, \\ z^T B_{v_1} z < \delta r_1 + (1 - \delta)r_2, \\ z^T B_{v_2} z < \delta r_1 + (1 - \delta)r_2. \end{cases}$$

Hence,

$$\begin{cases} \max\limits_{u \in \mathcal{U}_1} z^T A_u z < \delta q_1 + (1 - \delta)q_2, \\ \max\limits_{u \in \mathcal{U}_1} z^T B_v z < \delta r_1 + (1 - \delta)r_2. \end{cases}$$

So, $\delta(q_1, r_1) + (1 - \delta)(q_2, r_2) \in \boldsymbol{\Gamma}$ and consequently $\boldsymbol{\Gamma}$ is **convex.**
Secondly since (i) fails, we can separate $\boldsymbol{\Gamma}$ and the set $\{(2\alpha, -2\alpha)\}$. Then, by hyperplane separation theorem (also known as the geometric form of Hahn-Banach Theorem, see H. Brezis (1983) [12]), there exists $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \backslash \{(0,0)\}$ such that

$$\lambda_1 \left(\max_{u \in \mathcal{U}_1} \frac{1}{2} x^T A_u x + \eta \right) + \lambda_2 \left(\max_{v \in \mathcal{U}_2} \frac{1}{2} x^T B_v x + \varepsilon \right) \geq$$

$$\alpha \lambda_1 - \beta \lambda_2, \forall \, x \in \mathbb{R}^n, \forall \, \eta, \varepsilon \in \mathbf{int}\mathbb{R}_+. \tag{5}$$

-Suppose that $\lambda_1 < 0$.

When (η, ε) tends to $(+\infty, 0)$ in inequality (5), we get at the limit: $-\infty \geq \alpha\lambda_1 - \beta\lambda_2$. Thus, $\lambda_1 \geq 0$.

-In the same way, we show that $\lambda_1 \geq 0$.

-And afterwards, one has $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \backslash \{(0,0)\}$.

We have just shown that there exists $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \backslash \{(0,0)\}$ such that:

$$\lambda_1 \, \boldsymbol{max} \left\{ \frac{1}{2} x^T A_{u_1} x, \frac{1}{2} x^T A_{u_2} x \right\} + \lambda_2 \, \boldsymbol{max} \left\{ \frac{1}{2} x^T B_{v_1} x, \frac{1}{2} x^T B_{v_2} x \right\} \geq \alpha\lambda_1 - \beta\lambda_2, \forall \, x \in \mathbb{R}^n.$$

This implies that the following inequalities system (6) below has no solution,

$$\left[\begin{array}{l} \frac{1}{2} x^T \left(\lambda_1 A_{u_1} + \lambda_2 B_{v_1} \right) \mathrm{x} < \alpha\lambda_1 - \beta\lambda_2, \\ \frac{1}{2} x^T \left(\lambda_1 A_{u_1} + \lambda_2 B_{v_2} \right) \mathrm{x} < \alpha\lambda_1 - \beta\lambda_2, \\ \frac{1}{2} x^T \left(\lambda_1 A_{u_2} + \lambda_2 B_{v_1} \right) \mathrm{x} < \alpha\lambda_1 - \beta\lambda_2, \\ \frac{1}{2} x^T \left(\lambda_1 A_{u_2} + \lambda_2 B_{v_2} \right) \mathrm{x} < \alpha\lambda_1 - \beta\lambda_2. \end{array} \right. \tag{6}$$

Let

$$\Lambda = \left\{ \left(\begin{array}{l} \frac{1}{2} x^T \left(\lambda_1 A_{u_1} + \lambda_2 B_{v_1} \right) x, \frac{1}{2} x^T \left(\lambda_1 A_{u_1} + \lambda_2 B_{v_2} \right) x, \\ \frac{1}{2} x^T \left(\lambda_1 A_{u_2} + \lambda_2 B_{v_1} \right) \mathrm{x}, \frac{1}{2} x^T \left(\lambda_1 A_{u_2} + \lambda_2 B_{v_2} \right) x \mid x \in \mathbb{R}^n \end{array} \right) \right\}.$$

Let us consider the following mapping:

$\Phi : \mathbb{R}^4 \longrightarrow \mathbb{R}^4, \ (t_1, t_2, t_3, t_4) \longmapsto (\lambda_1 t_1 + \lambda_2 t_3, \lambda_1 t_1 + \lambda_2 t_4, \lambda_1 t_2 + \lambda_2 t_3, \lambda_1 t_2 + \lambda_2 t_4)$.

Φ is a linear transformation and Ω is a convex set (assumption), it follows that $\Lambda = \Phi(\Omega)$ is a convex set, see Equation (1) in Theorem 2.

By remarking that
$((\alpha\lambda_1 - \beta\lambda_2), (\alpha\lambda_1 - \beta\lambda_2), (\alpha\lambda_1 - \beta\lambda_2), (\alpha\lambda_1 - \beta\lambda_2)) \notin \Lambda + \boldsymbol{int}\mathbb{R}_+^4$,
the hyperplane separation theorem shows that there are $(\gamma_i)_{1 \leq i \leq 4} \in \mathbb{R}_+^4 \backslash \{0_{\mathbb{R}^4}\}$ such that for all $x \in \mathbb{R}^n$, one has:

$$\gamma_1 [\frac{1}{2} x^T \left(\lambda_1 A_{u_1} + \lambda_2 B_{v_1} \right) x - (\alpha\lambda_1 - \beta\lambda_2)] +$$
$$\gamma_2 [\frac{1}{2} x^T \left(\lambda_1 A_{u_1} + \lambda_2 B_{v_2} \right) x - (\alpha\lambda_1 - \beta\lambda_2)] +$$
$$\gamma_3 [\frac{1}{2} x^T \left(\lambda_1 A_{u_2} + \lambda_2 B_{v_1} \right) x - (\alpha\lambda_1 - \beta\lambda_2)] +$$
$$\gamma_4 [\frac{1}{2} x^T \left(\lambda_1 A_{u_2} + \lambda_2 B_{v_2} \right) x - (\alpha\lambda_1 - \beta\lambda_2)] \geq 0. \tag{7}$$

Put $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. Then, we have the following equivalences:

$$(7) \Leftrightarrow \lambda_1[\frac{1}{2}x^T(\gamma_1 A_{u_1} + \gamma_2 A_{u_1} + \gamma_3 A_{u_2} + \gamma_4 A_{u_2})x - \alpha\gamma]$$
$$+ \lambda_2[\frac{1}{2}x^T(\gamma_1 B_{v_1} + \gamma_2 B_{v_2} + \gamma_3 B_{v_1} + \gamma_4 B_{v_2})x + \beta\gamma] \geq 0$$

$$\Leftrightarrow \lambda_1[\frac{1}{2}x^T((\gamma_1 + \gamma_2)A_{u_1} + (\gamma_3 + \gamma_4)A_{u_2})x - \alpha\gamma]$$
$$+ \lambda_2[\frac{1}{2}x^T((\gamma_1 + \gamma_3)B_{v_1} + (\gamma_2 + \gamma_4)B_{v_2})x + \beta\gamma] \geq 0,$$
$$\Leftrightarrow \lambda_1[\frac{1}{2}x^T((\gamma_1 + \gamma_2)(A_1 + u_1 A_2) +$$
$$(\gamma - \gamma_1 - \gamma_2)((A_1 + u_2 A_2))x - \alpha\gamma]$$
$$+ \lambda_2[\frac{1}{2}x^T((\gamma_1 + \gamma_3)(B_1 + v_1 B_2) +$$
$$(\gamma - \gamma_1 - \gamma_3)(B_1 + v_2 B_2))x + \beta\gamma] \geq 0,$$
$$\Leftrightarrow \lambda_1[\frac{1}{2}x^T((\gamma_1 + \gamma_2)u_1 A_2 + \gamma(A_1 + u_2 A_2)$$
$$-(\gamma_1 + \gamma_2)u_2 A_2)x - \alpha\gamma]$$
$$+ \lambda_2[\frac{1}{2}x^T((\gamma_1 + \gamma_3)v_1 B_2 + \gamma(B_1 + v_2 B_2)$$
$$-(\gamma_1 + \gamma_3)v_2 B_2)x + \beta\gamma] \geq 0,$$
$$\Leftrightarrow \lambda_1[\frac{1}{2}x^T(A_1 + u_2 A_2 + \frac{(\gamma_1 + \gamma_2)(u_1 - u_2)A_2}{\gamma})x - \alpha]$$
$$+ \lambda_2[\frac{1}{2}x^T(B_1 + v_2 B_2 + \frac{(\gamma_1 + \gamma_3)(v_1 - v_2)B_2}{\gamma})x + \beta] \geq 0,$$
$$\Leftrightarrow \lambda_1[\frac{1}{2}x^T(A_1 + \frac{\gamma u_2 + (\gamma_1 + \gamma_2)(u_1 - u_2)}{\gamma}A_2)x - \alpha]$$
$$+ \lambda_2[\frac{1}{2}x^T(B_1 + \frac{\gamma v_2 + (\gamma_1 + \gamma_3)(v_1 - v_2)}{\gamma}B_2)x + \beta] \geq 0,$$
$$\Leftrightarrow \lambda_1[\frac{1}{2}x^T(A_1 + \frac{(\gamma_1 + \gamma_2)u_1 + (\gamma_3 + \gamma_4)u_2}{\gamma}A_2)x - \alpha]$$
$$+ \lambda_1[\frac{1}{2}x^T(B_1 + \frac{(\gamma_1 + \gamma_3)v_1 + (\gamma_3 + \gamma_4)v_2}{\gamma}B_2)x + \beta] \geq 0.$$

Consequently, we obtain the existence of $(\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2$ et $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ such that:

$$\lambda_1\left(\frac{1}{2}x^T A_{\bar{u}}x - \alpha\right) + \lambda_2\left(\frac{1}{2}x^T B_{\bar{v}}x + \beta\right) \geq 0, \forall x \in \mathbb{R}^n$$

Where:
$$\bar{u} = \frac{(\gamma_1 + \gamma_2)}{\gamma}u_1 + \frac{(\gamma_3 + \gamma_4)}{\gamma}u_2 \text{ and } \bar{v} = \frac{(\gamma_1 + \gamma_3)}{\gamma}v_1 + \frac{(\gamma_2 + \gamma_4)}{\gamma}v_2.$$

Remark 1 Some particular cases of Theorem 2 are given as, if :
− $\mathcal{U}_1$ or $\mathcal{U}_2$ is a singleton, then Theorem 2 is reduced to Theorem 3.1 of [31].
− Both $\mathcal{U}_1$ and $\mathcal{U}_2$ are singletons, then Theorem 2 consists merely of Dines Theorem (see Theorem 1 or [16]).

Note that the following result is an uncertain version of the classical S-lemma given in Lemma 1.

Corollary 1 (Robust S-lemma)
Let $A_0, A_1, B_0, B_1 \in \mathbb{S}_n(\mathbb{R})$; $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Assume that:
$\Omega = \{ (x^T A_{u_1} x, x^T A_{u_2} x, x^T B_{v_1} x, x^T B_{v_2} x) \, | x \in \mathbb{R}^n \}$ is convex.
There exists $x_0 \in \mathbb{R}^n$ such that $x_0^T B_v x_0 < 0$ for all $v \in \mathcal{U}_2$.
Then, the following statements are equivalent:

(i) $x^T B_v x \leq 0, \forall v \in \mathcal{U}_2 \implies x^T A_u x \geq 0, \forall u \in \mathcal{U}_1$.
(ii) $\exists \lambda \geq 0, \exists (\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2 \mid A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0$.

Proof $[(ii) \implies (i)]$. This implication is always satisfied.
$[(i) \implies (ii)]$. Suppose that (i) holds. Hence, the inequalities system

$$\begin{cases} x^T A_u x < 0, \forall u \in \mathcal{U}_1, \\ x^T B_v x < 0, \forall v \in \mathcal{U}_2, \end{cases}$$

has no solution. From Theorem 2, there exist $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$,
$(\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that:

$$\lambda_1 x^T A_{\bar{u}} x + \lambda_2 x^T B_{\bar{v}} x \geq 0, \ \forall x \in \mathbb{R}^n.$$

If $\lambda_1 = 0$ then $\lambda_1 > 0$ and $x^T B_{\bar{v}} x \geq 0$ for all $\in \mathbb{R}^n$. Which is a contradiction since $x_0^T B_v x_0 < 0$ Hence $\lambda_1 > 0$. It follows that $A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0$, with: $\lambda = \frac{\lambda_2}{\lambda_1}$.

4. Characterization of Robust Solution of QCQOP under Data Uncertainty

4.1   Homogeneous Case
In this subsection, we give a characterization of robust optimal solution for homogeneous quadratic programming problem under data uncertainty.
For this purpose, we consider the following problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A_u x, \qquad \qquad \text{(H-QCQP)}$$
$$\text{s.t.} \ \frac{1}{2} x^T B_v x + \beta \leq 0.$$

The robust counterpart (worst case) of (H-QCQP) is given by

$$\min_{x \in \mathbb{R}^n} \max_{u \in \mathcal{U}_1} \frac{1}{2} x^T A_u x, \qquad \qquad \text{(RH-QCQP)}$$
$$\text{s.t.} \ \max_{v \in \mathcal{U}_2} \frac{1}{2} x^T B_v x + \beta \leq 0.$$

Theorem 3 (Characterization of robust solution)
Let $A_0, A_1, B_0, B_1 \in \mathbb{S}_n(\mathbb{R})$; $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.
Assume that:
(H_1) There exists $x_0 \in \mathbb{R}^n$ such that $x_0^T B_v x_0 + \beta < 0$ for all $v \in \mathcal{U}_2$.
(H_2) The set $\Omega = \{ (x^T A_{u_1} x, x^T A_{u_2} x, x^T B_{v_1} x, x^T B_{v_2} x) \, | x \in \mathbb{R}^n \}$ is convex.

Then, the following statements are equivalent:
(i) $\bar{x}$ is a robust optimal solution of (H-QCQP);
(ii) there exist $\lambda \geq 0$ and $(\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that:

$$(A_{\bar{u}} + \lambda B_{\bar{v}})\bar{x} = 0, \qquad \text{(First-order Condition)}.$$

$$\left(\frac{1}{2}\bar{x}^T B_{\bar{v}}\bar{x} + \beta\right) = 0, \qquad \text{(Complementary Slackness)} \tag{9}$$

$$A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0, \qquad \text{(Second-order Condition)}.$$

Proof $[(ii) \Longrightarrow (i)]$. Assume that $\bar{x}$ is a robust optimal solution of (H-QCQP).

Then, for all $x \in \mathbb{R}^n$:

$$\frac{1}{2}x^T B_v x + \beta \leq 0, \forall v \in \mathcal{U}_2 \Longrightarrow \max_{u \in \mathcal{U}_1} \frac{1}{2}x^T A_u x \geq \alpha := \max_{u \in \mathcal{U}_1} \frac{1}{2}\bar{x}^T A_u \bar{x}. \tag{10}$$

This implies that

$$\max_{u \in \mathcal{U}_1} \frac{1}{2}x^T A_u x < \alpha,$$

$$\frac{1}{2}x^T B_v x + \beta \leq 0, \forall v \in \mathcal{U}_2$$

has no solution. This means that the following inequalities system

$$\begin{cases} \dfrac{1}{2}x^T A_u x < \alpha, \forall u \in \mathcal{U}_1, \\[2mm] \dfrac{1}{2}x^T B_v x + \beta \leq 0, \forall v \in \mathcal{U}_2, \end{cases}$$

has no solution. Thanks to hypothesis (H2) by applying Theorem 2, there exist
$(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ and $(\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that:

$$\lambda_1 \left(\frac{1}{2}x^T A_{\bar{u}} x - \alpha\right) + \lambda_2 \left(\frac{1}{2}x^T B_{\bar{v}} x + \beta\right) \geq 0, \forall x \in \mathbb{R}^n.$$

If $\lambda_1 = 0$, then: $\lambda_2 \left(\frac{1}{2}x^T B_{\bar{v}} x + \beta\right) \geq 0$ $for\ all\ x \in \mathbb{R}^n$.
For $x = x_0$, with (H_1), we see that $\lambda_1 \leq 0$ and $\lambda_2 = 0$, which contradicts the fact that $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$. Hence, $\lambda_1 < 0$. This means that

$$\left(\frac{1}{2}x^T A_{\bar{u}} x - \alpha\right) + \lambda \left(\frac{1}{2}x^T B_{\bar{v}} x + \beta\right) \geq 0, \forall x \in \mathbb{R}^n \tag{11}$$

Where $\lambda = \frac{\lambda_2}{\lambda_1}$. Inequality (11) implies that

$$\left(\max_{u \in \mathcal{U}_1} \frac{1}{2}x^T A_u x - \alpha\right) + \lambda \left(\frac{1}{2}x^T B_{\bar{v}} x + \beta\right) \geq 0, \forall x \in \mathbb{R}^n$$

It is clear that for $x = \bar{x}$, one has :

$$\left(\frac{1}{2}\bar{x}^T B_{\bar{v}}\bar{x} + \beta\right) = 0 \text{ (by using the definition of in Inequality (10))}.$$

Then, the complementary slackness conditions hold. Let us consider the function defined by:
$\varphi: \mathbb{R}^n \to \mathbb{R}, x \mapsto \varphi(x) = \left(\frac{1}{2}x^T A_{\bar{u}} x - \alpha\right) + \lambda \left(\frac{1}{2}x^T B_{\bar{v}} x + \beta\right)$. The Inequality (11) implies that
$\varphi(x) \geq 0$ and $\varphi(\bar{x}) \geq 0$. $\bar{x}$ minimize φ on $\mathbb{R}^n$ and φ is twice differentiable. Then, $\bar{x}$ solve the
Euler equation $\nabla\varphi(x) = 0$ and inequalities system $\nabla^2\varphi(x) \succcurlyeq 0$. Thus, the first and second order
necessary conditions of optimality in the point $\bar{x}$ lead to:

$$\begin{cases} (A_{\bar{u}} + \lambda B_{\bar{v}})\bar{x} = 0, & \text{(First-order Condition)}. \\ A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0, & \text{(Second-order Condition)}. \end{cases}$$

[(ii) =) (i)]. Now, suppose that there exist $\lambda \geq 0$ and $(\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that :

$$\begin{cases} (A_{\bar{u}} + \lambda B_{\bar{v}})\bar{x} = 0, & \text{(First-order Condition)}. \\ \lambda(\frac{1}{2}\bar{x}^T B_{\bar{v}}\bar{x} + \beta) = 0, & \text{(Complementary Slackness)}. \\ A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0, & \text{(Second-order Condition)}. \end{cases}$$

Let x be a robust feasible solution of (H-QCQP). By using complementary slackness condition, one has:

$$\frac{1}{2}x^T B_{\bar{v}}x + \beta \leq \lambda\left(\frac{1}{2}\bar{x}^T B_{\bar{v}}\bar{x} + \beta\right).$$

Let us consider $\mathbb{R}^n \longrightarrow \mathbb{R}$, $x \longmapsto \varphi(x) := \frac{1}{2}x^T A_{\bar{u}}x + \lambda\left(\frac{1}{2}x^T B_{\bar{v}}x + \beta\right)$.

The first-order condition and second-order condition (necessary condition of optimality) mean that $\nabla\varphi(\bar{x}) = 0$ and φ is convex function over $\mathbb{R}^n$. Thus,

$$\varphi(x) \geq \varphi(\bar{x}) \implies \frac{1}{2}x^T A_{\bar{u}}x \geq \frac{1}{2}\bar{x}^T A_{\bar{u}}\bar{x} + \lambda\left(\frac{1}{2}\bar{x}^T B_{\bar{v}}\bar{x} + \beta\right) -$$
$$\lambda\left(\frac{1}{2}x^T B_{\bar{v}}x + \beta\right)$$
$$\implies \frac{1}{2}x^T A_{\bar{u}}x \geq \frac{1}{2}\bar{x}^T A_{\bar{u}}\bar{x} - \lambda\left(\frac{1}{2}x^T B_{\bar{v}}x + \beta\right)$$
$$\implies \frac{1}{2}x^T A_{\bar{u}}x \geq \frac{1}{2}\bar{x}^T A_{\bar{u}}\bar{x}.$$

Hence, $\bar{x}$ is robust optimal solution of (H-QCQP).

Remark 2 Note that the robust optimal solution characterization of the uncertain homogeneous quadratic optimization problem given in Theorem 3 was established in [31] by assuming that $\mathcal{U}_1$ is a singleton and where the corresponding additive scalar is assumed to be negative (regarding to our problem formulation).

4.2 Non-homogeneous Case

In this subsection, at first we will derive necessary and sufficient conditions of robust optimal solution for quadratic programming under bounded interval uncertainty. Secondly, we will give a characterization of optimal solutions of the optimistic dual.
Let us consider the following uncertain quadratic problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}x^T A_u x + a_\omega^T x \qquad \text{(NH-QCQP)}$$
$$\text{s.t. } \frac{1}{2}x^T B_v x + b_\varsigma^T x + \beta$$

where $A_u, B_v \in \mathbb{S}_n(\mathbb{R}), a_\omega, b_\varsigma \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ and $(u, v, \omega, \varsigma)$ belongs to the uncertainty set $\mathcal{U} = \prod_{i=1}^4 \mathcal{U}_i = [u_1, u_2] \times [v_1, v_2] \times [\omega_1, \omega_2] \times [\varsigma_1, \varsigma_2]$.

The robust counterpart (worst case) of (NH-QCQP) is given by

$$\min_{x \in \mathbb{R}^n} \max_{u \in \mathcal{U}_1} \frac{1}{2} x^T A_u x + \max_{\omega \in \mathcal{U}_3} a_\omega^T x, \qquad \text{(RHN-QCQP)}$$

$$s.t. \ \max_{v \in \mathcal{U}_2} \frac{1}{2} x^T B_v x + \max_{\varsigma \in \mathcal{U}_4} b_\varsigma^T x + \beta \le 0.$$

Given a $\bar{x}$ in $\mathbb{R}^n$, we set $\alpha = - \max_{(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3} \left\{ \frac{1}{2} \bar{x}^T A_u \bar{x} + a_\omega^T \bar{x} \right\}$ and

$$\Xi_{\bar{x}} = \left\{ \binom{x}{t}^T H_1 \binom{x}{t}, \binom{x}{t}^T H_2 \binom{x}{t}, \binom{x}{t}^T H_3 \binom{x}{t}, \binom{x}{t}^T H_4 \binom{x}{t} \mid (x,t) \in \mathbb{R}^n \times \mathbb{R} \right\},$$

Where:

$$H_1 = \begin{pmatrix} A_{u_1} & a_{\omega_1} \\ a_{\omega_1}^T & 2\alpha \end{pmatrix}, H_2 = \begin{pmatrix} A_{u_2} & a_{\omega_2} \\ a_{\omega_2}^T & 2\alpha \end{pmatrix}, H_3 = \begin{pmatrix} B_{v_1} & b_{\varsigma_1} \\ b_{\varsigma_1}^T & 2\beta \end{pmatrix} \text{ and } H_4 = \begin{pmatrix} B_{v_2} & b_{\varsigma_2} \\ b_{\varsigma_2}^T & 2\beta \end{pmatrix}.$$

We give the following definition with regards to convexity of QCQOPs at a given point in $\mathbb{R}^n$. This definition is inspired by the works of Dines (1941), Polyak (1998) and Jeyakumar (2012) about analysis of convexity properties of quadratic forms, and named after them.

Definition 1 (Convexity in the sense of Dines-Polyak-Jeyakumar) The problem (NH-QCQP) is said to be DPJ-convex (or regular) w.r.t. $\bar{x}$ if the set $\Xi_{\bar{x}}$ is convex.

Theorem 3 (Characterization of robust solution)
Let $\bar{x}$ be a robust feasible solution of (NH-QCQP) such that;
(H_1) (NH-QCQP) is DPJ-convex at $\bar{x}$;
(H_2) there exists $x_0 \in \mathbb{R}^n$ such that $x_0^T B_v x_0 + b_\varsigma^T x_0 + \beta < 0$ for all $(v, \varsigma) \in \mathcal{U}_2 \times \mathcal{U}_4$.
Then, the following statements are equivalent:
(i) $\bar{x}$ is a robust optimal solution of (NH-QCQP);
(ii) there exist $\lambda \ge 0$ and $(\bar{u}, \bar{v}, \overline{\omega}, \bar{\varsigma}) \in \mathcal{U}$ such that:

$$\begin{cases} (A_{\bar{u}} + \lambda B_{\bar{v}}) \bar{x} + a_{\overline{\omega}} + \lambda b_{\bar{\varsigma}} = 0, & \text{(First-order Condition).} \\ \lambda (\frac{1}{2} \bar{x}^T B_{\bar{v}} \bar{x} + b_{\bar{\varsigma}} + \beta) = 0, & \text{(Complementary Slackness).} \\ A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0. & \text{(Second-order Condition).} \end{cases} \qquad (12)$$

Proof $[(ii) \Longrightarrow (i)]$. Let $\bar{x}$ be a robust optimal solution of
(NH-QCQP) and $\alpha = - \max_{(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3} \left\{ \frac{1}{2} \bar{x}^T A_u \bar{x} + a_\omega^T \bar{x} \right\}$.

Then for all $x \in \mathbb{R}^n$, one has:

$$\max_{(v,\varsigma) \in \mathcal{U}_2 \times \mathcal{U}_4} \left\{ \frac{1}{2} x^T B_v x + b_\varsigma^T x \right\} + \beta \le 0$$

$$\Longrightarrow \max_{(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3} \left\{ \frac{1}{2} x A_u x + a_\omega^T x \right\} \ge -\alpha.$$

Consequently, the system

$$\left[\begin{array}{l} \max\limits_{(u,\omega)\in\mathcal{U}_1\times\mathcal{U}_3} \left\{\frac{1}{2}xA_ux + a_\omega^T x\right\}.< -\alpha, \\ \max\limits_{(v,\varsigma)\in\mathcal{U}_2\times\mathcal{U}_4} \left\{\frac{1}{2}x^T B_v x + b_\varsigma^T x\right\} + \beta < 0, \end{array} \right.$$

has no solution. Thus, the system

$$\frac{1}{2}x^T A_u x + a_\omega^T x + \alpha < 0, \forall(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3, \qquad (S_1)$$
$$\frac{1}{2}x^T B_v x + b_\varsigma^T x + \beta < 0, \ \forall(v,\varsigma) \in \mathcal{U}_2 \times \mathcal{U}_4,$$

has no solution.

Moreover, we can show that for all t ∈ ℝ, the following system

$$\left[\begin{array}{l} \frac{1}{2}x^T A_u x + t a_\omega^T x + t^2\alpha < 0, \forall(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3, \qquad (S_2) \\ \frac{1}{2}x^T B_v x + t b_\varsigma^T x + t^2\beta < 0, \ \forall(v,\varsigma) \in \mathcal{U}_2 \times \mathcal{U}_4, \end{array} \right.$$

has no solution in $\mathbb{R}^n \times \mathbb{R}$. Indeed if $(x^*, t^*) \in \mathbb{R}^n \times \mathbb{R}$ is a solution of (S_2), then

$$\left[\begin{array}{l} \frac{1}{2}x^{*^T} A_u x^* + t^* a_\omega^T x^* + t^{*^2}\alpha < 0, \forall(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3, \\ \frac{1}{2}x^{*^T} B_v x^* + t^* b_\varsigma^T x^* + t^{*^2}\beta < 0, \ \forall(v,\varsigma) \in \mathcal{U}_2 \times \mathcal{U}_4. \end{array} \right.$$

So, for $t^* \neq 0$, one has:

$$\left[\begin{array}{l} \frac{1}{2}\left(\frac{x^*}{t^*}\right)^T A_u \left(\frac{x^*}{t^*}\right) + a_\omega^T \left(\frac{x^*}{t^*}\right) + \alpha < 0, \forall(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3, \\ \frac{1}{2}\left(\frac{x^*}{t^*}\right)^T B_v \left(\frac{x^*}{t^*}\right) + b_\varsigma^T \left(\frac{x^*}{t^*}\right) + \beta < 0, \ \forall(v,\varsigma) \in \mathcal{U}_2 \times \mathcal{U}_4. \end{array} \right.$$

Which contradicts the fact that (S_1) has no solution.

If $t^* = 0$, it follows that x^* is a solution of the system

$$\left[\begin{array}{l} \frac{1}{2}x^T A_u x < 0, \forall u \in \mathcal{U}_1, \\ \frac{1}{2}x^T B_v x < 0, \forall v \in \mathcal{U}_2. \end{array} \right.$$

In this case $x^* = 0$ and for a fairly large number $n \in \mathbb{N}, x_n = nx^*$ is a solution of the system (S_1). Which is a contradiction. We conclude that the system (S_2) has no solution.

Let us show that there exist $(\bar{u}, \bar{v}, \overline{\omega}, \bar{\varsigma}) \in \mathcal{U}$ and $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ such that

$$\lambda_1\left(\frac{1}{2}x^T A_{\bar{u}} x + a_{\overline{\omega}}^T x + \alpha\right) + \lambda_2\left(\frac{1}{2}x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta\right) \geq 0, \forall x \in \mathbb{R}^n. \qquad (13)$$

One proceeds by distinguishing four cases as follows.

First case : $u_1 < u_2$ and $v_1 < v_2$.

Let

$$M_1 = \begin{pmatrix} A_1 & a_1 + \dfrac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} a_2 \\ a_1^T + \dfrac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} a_2^T & 2\alpha \end{pmatrix},$$

$$M_2 = \begin{pmatrix} A_2 & \dfrac{\omega_2 - \omega_1}{u_2 - u_1} a_2 \\ \dfrac{\omega_2 - \omega_1}{u_2 - u_1} a_2^T & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} B_1 & b_1 + \dfrac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} b_1 \\ b_1^T + \dfrac{\varsigma_1 v_2 - \varsigma_2 v_1}{v_2 - v_1} b_1^T & 2\beta \end{pmatrix},$$

and $$M_4 = \begin{pmatrix} B_2 & \dfrac{\varsigma_2 - \varsigma_1}{v_2 - v_1} b_2 \\ \dfrac{\varsigma_2 - \varsigma_1}{v_2 - v_1} b_2^T & 0 \end{pmatrix}.$$

Then, we have the following linear matrix equalities :

$$M_1 + u_1 M_2 = H_1 \; ; \quad M_1 + u_2 M_2 = H_2 \; ;$$
$$M_3 + v_1 M_4 = H_3 \; and \; M_3 + v_2 M_4 = H_4.$$

By applying the fact that (S_1) has no solution, we deduce that the system below

$$\left[\begin{array}{l} \dfrac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + uM_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0, \forall u \in \mathcal{U}_1, \\[2mm] \dfrac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + vM_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0, \forall v \in \mathcal{U}_2, \end{array} \right.$$

also has no solution.

Thanks to the convexity of the set $\Xi_{\bar{x}}$ and from Theorem 2 there exist
$(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ $(\bar{u}, \bar{v}) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\lambda_1 \left[\dfrac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + \bar{u}M_2) \begin{pmatrix} x \\ t \end{pmatrix} \right] +$$
$$\lambda_2 \left[\dfrac{1}{2}\begin{pmatrix} x \\ t \end{pmatrix}^T (M_3 + \bar{v}M_4) \begin{pmatrix} x \\ t \end{pmatrix} \right] \geq 0$$

For all $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

By setting t=1, one has :

$$\lambda_1 \left[\frac{1}{2} x^T A_{\bar{u}} x + \left(a_1 + \left(\frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} \right) + \bar{u} \frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} a_2 \right) x + \alpha \right] +$$

$$\lambda_2 \left[\frac{1}{2} x^T B_{\bar{v}} x + \left(\left(b_1 + \frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} + \bar{v} \frac{\varsigma_2 - \varsigma_1}{v_2 - v_1} \right) b_2 \right) x + \beta \right] \geq 0,$$

$$\forall \, x \in \mathbb{R}^n.$$

Therefore, there exist $(\bar{u}, \bar{v}, \overline{\omega}, \bar{\varsigma}) \in \mathcal{U}$ and $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ such that relation (13) holds with:

$$\overline{\omega} = \frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} + \bar{u} \frac{\omega_2 - \omega_1}{u_2 - u_1} \quad \text{and} \quad \bar{\varsigma} = \frac{\varsigma_1 v_2 - \varsigma_2 v_1}{v_2 - v_1} + \bar{v} \frac{\varsigma_2 - \varsigma_1}{v_2 - v_1}$$

Second case: $u_1 < u_2$ and $v_1 = v_2 = \bar{v}$.
Let

$$M_1 = \begin{pmatrix} A_1 & a_1 + \frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} a_2 \\ a_1^T + \frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} a_2^T & 2\alpha \end{pmatrix},$$

$$M_2 = \begin{pmatrix} A_2 & \frac{\omega_2 - \omega_1}{u_2 - u_1} a_2 \\ \frac{\omega_2 - \omega_1}{u_2 - u_1} a_2^T & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} B_{\bar{v}} & b_1 \\ b_1^T & 2\beta \end{pmatrix} \text{ and}$$

$$M_4 = \begin{pmatrix} 0_{\mathbb{S}_n(\mathbb{R})} & b_2 \\ b_2^T & 0 \end{pmatrix}.$$

Then, we have the following linear matrix equalities :

$$M_1 + u_1 M_2 = H_1 \; ; \quad M_1 + u_2 M_2 = H_2 \, ;$$
$$M_3 + \varsigma_1 M_4 = H_3 \text{ and } M_3 + \varsigma_2 M_4 = H_4.$$

From system (S_2), we obtain that the system

$$\begin{cases} \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + uM_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0, \forall u \in \mathcal{U}_1, \\ \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + \varsigma M_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0, \forall \varsigma \in \mathcal{U}_2, \end{cases}$$

has no solution.
Since the set $\Xi_{\bar{x}}$ *is* convex, we can apply Theorem 2, thus we can find (λ_1, λ_2) *in* $\mathbb{R}_+^2 \setminus \{(0,0)\}$ and $(\bar{u}, \bar{\varsigma}) \in \mathcal{U}_1 \times \mathcal{U}_4$ such that

$$\lambda_1 \left[\frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + \bar{u}M_2) \begin{pmatrix} x \\ t \end{pmatrix} \right] +$$

$$\lambda_2 \left[\frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^T (M_3 + \bar{\varsigma}M_4) \begin{pmatrix} x \\ t \end{pmatrix} \right] \geq 0, \forall \, (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Letting t=1, one has:
$$\lambda_1 \left[\frac{1}{2} x^T A_{\bar{u}} x + \left(a_1 + \left(\frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} \right) + \bar{u} \frac{\omega_2 - \omega_1}{u_2 - u_1} a_2 \right) x + \alpha \right] +$$
$$\lambda_2 \left[\frac{1}{2} x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right] \geq 0, \ \forall \, x \in \mathbb{R}^n.$$

Hence, there exist $(\bar{u}, \bar{v}, \overline{\omega}, \bar{\varsigma}) \in \mathcal{U}$ and $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ such that relation (13) holds with :
$$\overline{\omega} = \frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} + \bar{u} \frac{\omega_2 - \omega_1}{u_2 - u_1}.$$

Third case : $u_1 = u_2 = \bar{u}$ and $v_1 < u_2$
Let
$$M_1 = \begin{pmatrix} A_{\bar{u}} & a_1 \\ a_1^T & 2\alpha \end{pmatrix} ; \ M_2 = \begin{pmatrix} 0_{\mathbb{S}_n(\mathbb{R})} & a_2 \\ a_2^T & 0 \end{pmatrix} ;$$
$$M_3 = \begin{pmatrix} B_1 & b_1 + \frac{\omega_1 u_2 - \omega_2 u_1}{u_2 - u_1} b_1 \\ b_1^T + \frac{\varsigma_1 v_2 - \varsigma_2 v_1}{v_2 - v_1} b_1^T & 2\beta \end{pmatrix} \quad \text{and}$$
$$M_4 = \begin{pmatrix} B_2 & \frac{\varsigma_2 - \varsigma_1}{v_2 - v_1} b_2 \\ \frac{\varsigma_2 - \varsigma_1}{v_2 - v_1} b_2^T & 0 \end{pmatrix}.$$

We proceed like in the second case by changing the role of u and v with
$$\bar{\varsigma} = \frac{\varsigma_1 v_2 - \varsigma_2 v_1}{v_2 - v_1} + \bar{v} \frac{\varsigma_2 - \varsigma_1}{v_2 - v_1}.$$

Fourth case: $u_1 = u_2 = \bar{u}$ and $v_1 = v_2 = \bar{v}$.
Putting :
$$M_1 = \begin{pmatrix} A_{\bar{u}} & a_1 \\ a_1^T & 2\alpha \end{pmatrix} ; \ M_2 = \begin{pmatrix} 0_{\mathbb{S}_n(\mathbb{R})} & a_2 \\ a_2^T & 0 \end{pmatrix} ; \ M_3 = \begin{pmatrix} B_{\bar{v}} & b_1 \\ b_1^T & 2\beta \end{pmatrix} \text{ and } \ M_4 = \begin{pmatrix} 0_{\mathbb{S}_n(\mathbb{R})} & b_2 \\ b_2^T & 0 \end{pmatrix}.$$

In this case, we remark that :
$$M_1 + \omega_1 M_2 = H_1 \ ; \ \ M_1 + \omega_2 M_2 = H_2 \ ;$$
$$M_3 + \varsigma_1 M_4 = H_3 \ and \ M_3 + \varsigma_2 M_4 = H_4.$$

The system
$$\begin{bmatrix} \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + \omega M_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0, \forall \omega \in \mathcal{U}_3, \\ \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^T (M_1 + \varsigma M_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0, \forall \varsigma \in \mathcal{U}_2, \end{bmatrix}$$

has no solution because it is equivalent to another one like (S_2).

Again from the convexity of $\Xi_{\bar{x}}$ and Theorem 2, there exist
$(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ and $(\bar{\omega}, \bar{\varsigma}) \in \mathcal{U}_3 \times \mathcal{U}_4$ such that

$$\lambda_1 \left[\frac{1}{2} \binom{x}{t}^T (M_1 + \bar{\omega} M_2) \binom{x}{t} \right] +$$

$$\lambda_2 \left[\frac{1}{2} \binom{x}{t}^T (M_3 + \bar{\varsigma} M_4) \binom{x}{t} \right] \geq 0, \forall \; (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

Letting t=1, one has:
$$\lambda_1 \left[\frac{1}{2} x^T A_{\bar{u}} x + a_{\bar{\omega}}^T x + \alpha \right] +$$
$$\lambda_2 \left[\frac{1}{2} x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right] \geq 0, \; \forall \; x \in \mathbb{R}^n.$$

Finally, in all cases, there exist $(\bar{u}, \bar{v}, \bar{\omega}, \bar{\varsigma}) \in \mathcal{U}$ and $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ such that (13) holds.

From Slater condition, one obtains that: $\lambda_1 > 0$ and
$$\left(\frac{1}{2} x^T A_{\bar{u}} x + a_{\bar{\omega}}^T x + \alpha \right) + \lambda \left(\frac{1}{2} x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right) \geq 0, \forall \; x \in \mathbb{R}^n, (14)$$
with: $\lambda = \frac{\lambda_2}{\lambda_1}$.

From this inequality (14), we get:
$$\max_{(u,\omega) \in \mathcal{U}_1 \times \mathcal{U}_3} \left(\frac{1}{2} x^T A_u x + a_\omega^T x \right) + \alpha + \lambda \left(\frac{1}{2} x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right) \geq 0, \qquad \forall \; x \in \mathbb{R}^n.$$

For $x = \bar{x}$, since $\bar{x}$ is a feasible solution of (NH-QCQP), we obtain:
$$\left(\frac{1}{2} \bar{x}^T B_{\bar{v}} \bar{x} + b_{\bar{\varsigma}}^T \bar{x} + \beta \right) = 0. \tag{15}$$

Let us consider:
$$\mathbb{R}^n \longrightarrow \mathbb{R}, x \longmapsto \phi(x) := \frac{1}{2} x^T A_{\bar{u}} x + a_{\bar{\omega}}^T x + \alpha + \lambda \left(\frac{1}{2} x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right).$$
By using the definition of α, (14) and (15), we get:
$\phi(x) \geq 0$ and $\phi(\bar{x}) \leq 0$. It is follows that $\phi(x) \geq \phi(\bar{x})$. $\bar{x}$ minimize ϕ on $\mathbb{R}^n$ and φ is twice differentiable. Then, $\nabla \phi(\bar{x}) = 0$ and $\nabla^2 \phi(\bar{x}) \succcurlyeq 0$.
Therefore,
$$\begin{cases} (A_{\bar{u}} + \lambda B_{\bar{v}}) \bar{x} + a_{\bar{\omega}}^T + b_{\bar{\varsigma}}^T = 0 \\ A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0. \end{cases}$$

[(ii) =) (i)]. Suppose there exist $\lambda \geq 0$ and $(\bar{u}, \bar{v}, \bar{\omega}, \bar{\varsigma}) \in \mathcal{U}$

$$\begin{cases} (A_{\bar{u}} + \lambda B_{\bar{v}}) \bar{x} + a_{\bar{\omega}} + \lambda b_{\bar{\varsigma}} = 0, \text{ (First-order Condition)} \\ \lambda (\frac{1}{2} \bar{x}^T B_{\bar{v}} \bar{x} + b_{\bar{\varsigma}}^T \bar{x} + \beta) = 0, \quad \text{(Complementary Slackness)} \\ A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0. \qquad\qquad\quad \text{(Second-order Condition)} \end{cases}$$

Let us consider the map,

$\psi : \mathbb{R}^n \to \mathbb{R}, \ x \longmapsto \ \psi(x) := \frac{1}{2}x^T A_{\bar{u}} x + a_{\bar{\omega}}^T x + \lambda \left(\frac{1}{2}x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right)$

and assume that x be a robust feasible solution of (NH-QCQP).

First-order condition and second-order condition gives us that
$\nabla \psi(\bar{x}) = 0$ and ψ is convex. Then, $\bar{x}$ is a global minimizer of ψ
and one has

$$\psi(x) \geq \ \psi(\bar{x}) \Longrightarrow \frac{1}{2}x^T A_{\bar{u}} x + a_{\bar{\omega}}^T x \geq \frac{1}{2}\bar{x}^T A_{\bar{u}} \bar{x} + a_{\bar{\omega}}^T \bar{x}$$

$$+\lambda \left(\frac{1}{2}\bar{x}^T B_{\bar{v}} \bar{x} + b_{\bar{\varsigma}}^T \bar{x} + \ \beta \right) - \lambda \left(\frac{1}{2}x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right)$$

$$\Longrightarrow \ \frac{1}{2}x^T A_{\bar{u}} x + a_{\bar{\omega}}^T x \geq \frac{1}{2}\bar{x}^T A_{\bar{u}} \bar{x} + a_{\bar{\omega}}^T \bar{x}$$

$$-\lambda \left(\frac{1}{2}x^T B_{\bar{v}} x + b_{\bar{\varsigma}}^T x + \beta \right)$$

$$\Longrightarrow \ \frac{1}{2}x^T A_{\bar{u}} x + a_{\bar{\omega}}^T x \geq \frac{1}{2}\bar{x}^T A_{\bar{u}} \bar{x} + a_{\bar{\omega}}^T \bar{x}.$$

Therefore, $\bar{x}$ is robust optimal solution of (NH-QCQP).

5. Optimistic Dual and Strong Robust Duality of QCQOP

In this section, we characterize the solutions of dual optimistic. We also show that the robust strong
duality property holds.
Let us consider

$\mathbb{R}^n \times \mathcal{U} \times \mathbb{R}^+ \to \mathbb{R}, \ (x, \xi, \lambda) \longmapsto \ \mathcal{L}(x, \xi, \lambda) := q_0(x, \xi) + \lambda q_1(x, \xi)$

where $\xi = (u, v, \omega, \varsigma), \quad q_0(x, \xi) := \frac{1}{2}x^T A_u x + a_{\omega}^T x$

and $q_1(x, \xi) := \frac{1}{2}x^T B_v x + b_{\varsigma}^T x + \beta.$

The problem

$$\max_{} \ \inf_{x \in \mathbb{R}^n} \ \mathcal{L}(x, \xi, \lambda) \qquad\qquad \text{(OD-QCQP)}$$
$$\text{s.t. } (\xi, \lambda) \ \in \mathcal{U} \times \mathbb{R}^+$$

is called the optimistic dual of (NH-QCQP).

Theorem 5 Assume that assumptions of Theorem 4 hold. If $\bar{x}$ is a robust solution of (NH-QCQP),
then there exist $(\bar{\xi}, \bar{\lambda}) \in \mathcal{U} \times \mathbb{R}^+$ such that $(\bar{\xi}, \bar{\lambda})$ is an optimal solution of (OD-QCQP).

Proof. From Theorem 4, there exists $\left(\bar{u}, \bar{v}, \bar{\omega}, \bar{\varsigma}, \bar{\lambda} \right) \in \mathcal{U} \times \mathbb{R}^+$ such that

$$\begin{cases} (A_{\bar{u}} + \lambda B_{\bar{v}})\bar{x} + a_{\bar{\omega}} + \lambda b_{\bar{\varsigma}} = 0, \ \text{(First-order Condition)}. \\ \lambda(\frac{1}{2}\bar{x}^T B_{\bar{v}} \bar{x} + b_{\bar{\varsigma}} + \beta) = 0, \qquad \text{(Complementary Slackness)} \\ A_{\bar{u}} + \lambda B_{\bar{v}} \succcurlyeq 0. \qquad\qquad\qquad \text{(Second-order Condition)} \end{cases}$$

We set $\bar{\bar{\xi}} = (\bar{u}, \bar{v}, \overline{\omega}, \bar{\varsigma})$. For all feasible solution
$(u, v, \omega, \varsigma, \lambda) \in \mathcal{U} \times \mathbb{R}^+$ of (OD-QCQP), one has :

$$\begin{aligned}
\mathcal{L}(\bar{x}, \bar{\bar{\xi}}, \bar{\lambda}) &= q_0(\bar{x}, \bar{\bar{\xi}}) + \bar{\lambda} q_1(\bar{x}, \bar{\bar{\xi}}) \\
&= q_0(\bar{x}, \bar{\bar{\xi}}) \\
&= \max_{\xi \in \mathcal{U}} q_0(\bar{x}, \xi) \\
&\geq q_0(\bar{x}, \xi), \forall \, \xi \in \mathcal{U} \\
&\geq q_0(\bar{x}, \xi) + \lambda q_1(\bar{x}, \xi), \forall \, (\xi, \lambda) \in \mathcal{U} \times \mathbb{R}^+ \\
&= \mathcal{L}(\bar{x}, \xi, \lambda), \forall \, (\xi, \lambda) \in \mathcal{U} \times \mathbb{R}^+.
\end{aligned}$$

Hence, $(\bar{\bar{\xi}}, \bar{\lambda})$ is an optimal solution of (OD-QCQP).

Corollary 2 Suppose that assumptions of Theorem 5 hold.
If (RNH-QCQP) has an optimal solution, then the strong duality property is fulfilled.

Proof. Let $\bar{x}$ be an optimal solution of (RNH-QCQP). So,

$$\begin{aligned}
\min \text{ (RNH-QCQP)} &= \frac{1}{2} \max_{u \in \mathcal{U}_1} \bar{x}^T A_u \bar{x} + \max_{\omega \in \mathcal{U}_3} a_\omega^T \bar{x} \\
&= \frac{1}{2} \bar{x}^T A_u \bar{x} + a_\omega^T \bar{x}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\max \text{ (OD-QCQP)} &= \max_{(\xi, \lambda) \in \mathcal{U} \times \mathbb{R}^+} \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \xi, \lambda) \\
&= \mathcal{L}\left(\bar{x}, \bar{\bar{\xi}}, \bar{\lambda}\right) \\
&= q_0(\bar{x}, \bar{\bar{\xi}}) + \bar{\lambda} q_1(\bar{x}, \bar{\bar{\xi}}) \\
&= \frac{1}{2} \bar{x}^T A_u \bar{x} + a_\omega^T \bar{x}.
\end{aligned}$$

Thus, min (RNH-QCQP) = max (OD-QCQP).

6. Example

We deal with the following 2-dimensions QCQOP

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A_u x, \qquad\qquad (\mathcal{P})$$
$$\text{s.t. } \frac{1}{2} x^T B_v x + \beta \leq 0,$$

where:

$$A_0 = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_0 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -5 & -10 \\ -10 & -5 \end{pmatrix}, \beta = -1, \, |u| \leq 10^{-1} \text{ and } |v| \leq 5.\,10^{-1}.$$

With the interval interpolation approach regarding to :

− matrices A_0 and A_1, $A_u = A_0 + uA_1 = \begin{pmatrix} \frac{2u+1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{2u+11}{2} \end{pmatrix}$,

− matrices B_0 and B_1, $B_v = B_0 + vB_1 = \begin{pmatrix} 1 - 5v & \frac{1-20v}{2} \\ \frac{1-20v}{2} & 1 - 5v \end{pmatrix}$.

Let $x_0 = \left(\frac{1}{2}, \frac{1}{2}\right)^T$. Then we verify that the strictly feasibility condition holds with this point x_0.
The matrices A_{u_1}, A_{u_2}, B_{v_1} and B_{v_2} commute. From Proposition 1, the set
$\Xi = \{(x^T A_{u_1} x, x^T A_{u_2} x, x^T B_{v_1} x, x^T B_{v_2} x) \mid x \in \mathbb{R}^n\}$
is convex. Where:

$$A_{u_1} = \begin{pmatrix} \frac{2}{5} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{2}{5} \end{pmatrix}, A_{u_2} = \begin{pmatrix} \frac{3}{5} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{5} \end{pmatrix}, B_{v_1} = \begin{pmatrix} \frac{5}{4} & 1 \\ 1 & \frac{5}{4} \end{pmatrix} \text{ and } B_{v_2} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$$

The problem $(\mathcal{P})$ can be rewritten under the analytically form still denoted $(\mathcal{P})$ and given by:
$$\min_{x \in \mathbb{R}^n} \frac{1}{4}(2u + 1)x_1^2 - \frac{3}{2}x_1 x_2 + \frac{1}{4}(2u + 1)x_2^2, \qquad (\mathcal{P})$$
$$\text{s.t. } \frac{1}{2}(1 - 5v)x_1^2 + \frac{1}{2}(1 - 20v)x_1 x_2 + \frac{1}{2}(1 - 5v)x_2^2 \leq 0.$$

From Theorem 3, we deduce that the robust solution of $(\mathcal{P})$ satisfies

$$\begin{cases} [\, 2u + 1 + 2\lambda(1 - 5v)]x_1 + [-3 + \lambda(1 - 20v)]x_2 = 0, \\ [-3 + \lambda(1 - 20v)]x_1 + [2u + 1 + 2\lambda(1 - 5v)]x_2 = 0, \\ \lambda\left[\frac{1}{2}(1 - 5v)x_1^2 + \frac{1}{2}(1 - 20v)x_1 x_2 + \frac{1}{2}(1 - 5v)x_2^2 - 1\right] = 0, \\ \begin{pmatrix} \frac{2u+1}{2} + \lambda(1 - 5v) & \frac{1}{2}(-3 + \lambda(1 - 20v)) \\ \frac{1}{2}(-3 + \lambda(1 - 20v)) & \frac{2u+1}{2} + \lambda(1 - 5v) \end{pmatrix} \succcurlyeq 0. \end{cases}$$

Letting $\bar{u} = 3.10^{-2}$, $\bar{v} = 4.10^{-2}$ and $\bar{\lambda} = \frac{97}{90}$. The resolution of this system gives us the following robust solutions:

$$\bar{x} = \pm \left(\frac{\sqrt{10}}{3}, \frac{\sqrt{10}}{3}\right)^T.$$

7. Conclusion

We have dealt with a non-convex quadratic optimization problem subjected to data uncertainties both in constraint and in criterion. Such an optimization model is extremely difficult to solve thanks its two levels of optimization (in short "mini-max"). We propose a new definition of convexity for a RO problem and an adequate S-procedure in order to characterize the global robust optimal solution of these quadratic uncertain problem in homogeneous case in one hand. We have also dealt with the non-homogeneous case in other hand. The example given in Section 6 allows to validate this paradigm in the reality. In this work, we bring some necessary and sufficient conditions in Theorem 5 and Corollary 2 to make strong duality property holds. A next step could be to address some numerical optimization algorithms for solving a large scale of these kind of QCQOPs.

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