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Solving Second Order Delay Differential Equations directly by a Four-step Multi-Hybrid Block Method

Familua A. B.[1], Areo E. A.[2], Olabode B. T.[3] and Owolabi K. M.[4]

Abstract

The aim of this paper is to compute the numerical solution of special second order delay differential equations directly by a four-step multi-hybrid block method. The methods were generated using collocation and interpolation approach by means of a combination of power series and exponential function at some selected grid and off-grid points. The developed schemes and its first derivatives was combined to form block methods to concurrently solve special second order delay differential equations directly without reducing it to the system of first order. The basic properties of the methods such as order, error constants, consistency and convergence were examined. The developed methods were applied to solve some second order delay differential equations, the methods also solve application problem in other to test for the efficiency and accuracy of the methods. The results are displays in the tables.

JEL classification numbers: 65L05; 65L06; 65L20.

Keywords: Multi-Hybrid Block Method; Special Second Order Delay Differential Equations; Convergence; Four-step Block Method; Order seventeen.

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1 Department of Mathematics and Computer Science, First Technical University, Ibadan, Oyo State, Nigeria.
2,3,4 Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State, Nigeria.

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1. Introduction

Differential equations in the company of a time delay are used to model a phenomenon which does not only depend on the current state of a system but also the earlier states. This category of equation is called delay differential equations (DDEs) [1]. It is differential equations in which the derivative at any time depends on the solution at earlier times. The special second order DDE can be written in the form:

$$y''(t) = f(t, y(t-\tau)),\ \ y(t_0) = \alpha,\ y'(t_0) = \beta,\ t \geq t_0,\ \ \tau > 0 \tag{1}$$

Where α is the initial function and τ is the delay term.

Most of the methods for solving special second order ODEs can be adopted for solving special second order delay differential equation. There are two different ways to calculate the delay term in the developed method. See [2-4].

DDEs have become significant tools to explore the complexities of the real-world problems concerning infectious diseases, biotic population, neuronal networks, and population dynamics.

Methods such as Runge-Kutta (RK), Runge-Kutta Nystrom (RKN), Hybrid techniques, and multistep are widely used for solving DDEs. Adegboyega [5] derived a class of Numerical Integrators of Adams Moulton type for solution of Delay Differential equations. Ismail et al. [6] proposed a RK method and Hermite interpolation to solve first-order DDEs. Machee et al. [7] proposed a Runge-kutta-Nystrom method for solving special second order delay differential equations. Taiwo and Odetunde [8], and Evans and Raslan [9] presented a decomposition method as an integrator for the solution of delay differential equations. Several authors also derived block linear multistep method (LMM) to solve DDEs; and such work can be seen in [10-12]. San et al. [10] developed a coupled block method for solving delay differential equations. Zanariah and Hoo [11] examined the numerical solution of DDEs by the block method. Hoo et al. [12] constructed Adams-Moulton Method for directly solving second-order DDEs. Mechee et al. [8] in their paper has adapted RKN for directly solving second-order DDEs. Suleiman and Ishak [13] investigated the numerical solution and the stability of a multistep method for solving DDEs. The work of Familua et. al [14], A class of numerical integrators of order 13 for solving special second order delay differential equations was presented, the analysis of the method was examined. It was found to be consistence, convergent and zero stable. The numerical results show that the method is more accurate than the method compared with in the literature.

Conversely, all the studies formerly mentioned method are having one limitation or the other, such as instability in nature, low order of accuracy and poor accuracy of the methods. Hence, we are motivated to derive a linear multistep with nine hybrid points through the combination of power series and exponential function as a basic function applied for solving DDEs. The method is implemented in block hybrid method because it is a faster numerical method to obtain the approximate solution at more than one point per step. Finally, we tested the new methods using DDEs test problems to indicate that it is better-quality and more efficient for solving special second order DDEs, the method was also applied to solve application problem form engineering namely, Mathieu equation.

2. Derivation of Four-step Multi-Hybrid Method

This work considers an approximate solution that combines power series and exponential function of the form:

$$y(x)=\sum_{j=0}^{r+s-1} a_j x^j + a_{r+s}\sum_{j=0}^{r+s}\frac{\alpha^j x^j}{j!} \tag{2}$$

The second derivatives of (2) is given as,

$$y''(x)=\sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} + a_{r+s}\sum_{j=2}^{r+s}\frac{\alpha^j x^{j-2}}{(j-2)!} \tag{3}$$

Here, the interval of integration is taken by partitioning the Four-step length (x_n, x_{n+4}) into eight, that is, have twelve sub-steps. Collocating (3) at $x_{n+j},\ j=0(\tfrac{1}{4})k$ and interpolating (2) at $x_{n+j},\ j=0(1)k-1$ to yield a system of equations of the form:

$$AX=U \tag{4}$$

Where $A=\left[a_0,\ldots\ldots,a_{n+k-1},\ldots\ldots,a_{n+vi},\ldots\ldots,a_{4k+vi}\right]$

$U=\left[y_n,\ldots,y_{n+k-1},\ldots\ldots,f_n,\ldots\ldots,f_{n+vi},\ldots,f_{n+k}\right]$ and

$$X=\begin{bmatrix}
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \cdots & x_{n+1}^{16} & x_{n+1}^{17} & \cdots & \left(1+\alpha x_{n+1}+\frac{(\alpha^2x_{n+1}^2)}{2!}+\frac{(\alpha^3x_{n+1}^3)}{3!}+\frac{(\alpha^4x_{n+1}^4)}{4!}+\ldots\frac{(\alpha^{10}x_{n+1}^8)}{16!}\ldots\right)\\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & \cdots & x_{n+2}^{16} & x_{n+2}^{17} & \cdots & \left(1+\alpha x_{n+2}+\frac{(\alpha^2x_{n+2}^2)}{2!}+\frac{(\alpha^3x_{n+2}^3)}{3!}+\frac{(\alpha^4x_{n+2}^4)}{4!}+\ldots\frac{(\alpha^{10}x_{n+2}^8)}{10!}\ldots\right)\\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 47x_n^5 & 56x_n^6 & 72x_n^7 & \cdots & \left(\alpha^2+\alpha^2x_n+\frac{(\alpha^3x_n^2)}{2!}+\frac{(\alpha^4x_n^3)}{3!}+\frac{(\alpha^5x_n^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_n^8)}{8!}\ldots\right)\\
0 & 0 & 2 & 6x_{n+1/4} & 12x_{n+1/4}^2 & 20x_{n+1/4}^3 & 30x_{n+1/4}^4 & 47x_{n+1/4}^5 & 56x_{n+1/4}^6 & 72x_{n+1/4}^7 & \cdots & \left(\alpha^2+\alpha^2x_{n+1/4}+\frac{(\alpha^3x_{n+1/4}^2)}{2!}+\frac{(\alpha^4x_{n+1/4}^3)}{3!}+\frac{(\alpha^5x_{n+1/4}^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_{n+1/4}^8)}{8!}\ldots\right)\\
0 & 0 & 2 & 6x_{n+1/2} & 12x_{n+1/2}^2 & 20x_{n+1/2}^3 & 30x_{n+1/2}^4 & 47x_{n+1/2}^5 & 56x_{n+1/2}^6 & 72x_{n+1/2}^7 & \cdots & \left(\alpha^2+\alpha^3x_{n+1/2}\frac{(\alpha^4x_{n+1/2}^2)}{2!}+\frac{(\alpha^5x_{n+1/2}^3)}{3!}+\frac{(\alpha^6x_{n+1/2}^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_{n+1/2}^8)}{8!}\ldots\right)\\
0 & 0 & 2 & 6x_{n+3/4} & 12x_{n+3/4}^2 & 20x_{n+3/4}^3 & 30x_{n+3/4}^4 & 47x_{n+3/4}^5 & 56x_{n+3/4}^6 & 72x_{n+3/4}^7 & \cdots & \left(\alpha^2+\alpha^3x_{n+3/4}\frac{(\alpha^4x_{n+3/4}^2)}{2!}+\frac{(\alpha^5x_{n+3/4}^3)}{3!}+\frac{(\alpha^6x_{n+3/4}^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_{n+3/4}^8)}{8!}\ldots\right)\\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 47x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 & \cdots & \left(\alpha^2+\alpha^3x_{n+1}\frac{(\alpha^4x_{n+1}^2)}{2!}+\frac{(\alpha^5x_{n+1}^3)}{3!}+\frac{(\alpha^6x_{n+1}^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_{n+1}^8)}{8!}\ldots\right)\\
0 & 0 & 2 & 6x_{n+5/4} & 12x_{n+5/4}^2 & 20x_{n+5/4}^3 & 30x_{n+5/4}^4 & 47x_{n+5/4}^5 & 56x_{n+5/4}^6 & 72x_{n+5/4}^7 & \cdots & \left(\alpha^2+\alpha^3x_{n+5/4}\frac{(\alpha^4x_{n+5/4}^2)}{2!}+\frac{(\alpha^5x_{n+5/4}^3)}{3!}+\frac{(\alpha^6x_{n+5/4}^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_{n+5/4}^8)}{8!}\ldots\right)\\
0 & 0 & 2 & 6x_{n+3/2} & 12x_{n+3/2}^2 & 20x_{n+3/2}^3 & 30x_{n+3/2}^4 & 47x_{n+3/2}^5 & 56x_{n+3/2}^6 & 72x_{n+3/2}^7 & \cdots & \left(\alpha^2+\alpha^3x_{n+3/2}\frac{(\alpha^4x_{n+3/2}^2)}{2!}+\frac{(\alpha^5x_{n+3/2}^3)}{3!}+\frac{(\alpha^6x_{n+3/2}^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_{n+3/2}^8)}{8!}\ldots\right)\\
0 & 0 & 2 & . & . & . & . & . & . & . & & ..\\
0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 47x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 & \ldots. & \left(\alpha^2+\alpha^3x_{n+4}\frac{(\alpha^4x_{n+4}^2)}{2!}+\frac{(\alpha^5x_{n+4}^3)}{3!}+\frac{(\alpha^6x_{n+4}^4)}{4!}+\ldots.\frac{90(\alpha^{10}x_{n+4}^8)}{8!}\ldots\right)
\end{bmatrix}$$

Solving (4) with the Aid of Maple 18 Mathematical Software for the values of a_j's, $j=0(1/4)k$ to obtain values for the parameters: $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, .., a_{18}$ and substituting the values of the parameters into equation (2) and simplifying the result, to obtain a continuous scheme of the form:

$$y(t)=\sum_{j=0}^{k-2}\alpha_j(t)y_{n+j}(t)+\sum_{j=0}^{k}\beta_j f_{n+j}(t) \qquad (5)$$

Setting $t=\dfrac{x-x_{n+k-1}}{h}, k=4, x_n=0,\quad x_{n+1}=h,\ \ x_{n+2}=2h,\quad x_{n+3}=3h,\ \ x_{n+4}=4h$

The coefficients of $\alpha_j(t)$ and $\beta_j(t)$ are:

$\alpha_1(t)=(-t+2)y_{n+1}$, $\alpha_2(t)=(-1+t)y_{n+2}$

$$\beta_0(t)=\left(\begin{array}{l}\frac{34816}{76621545}t^{16}h^2+\frac{1476517439}{78586200}h^2t^8+\frac{993366559}{151351200}h^2t^4+\frac{2447904}{29469825}h^2t^{12}+\\ \frac{65536}{97692469875}h^2t^{18}-\frac{647718649}{29469825}h^2t^7+\frac{23613952}{638512875}h^2t^{14}+\frac{1300371871}{200930625}h^2t^{10}-\\ \frac{55192064}{273648375}h^2t^{13}-\frac{11726464}{4465125}h^2t^{11}+\frac{37319451569}{1891890000}h^2t^6-\frac{2436559}{1081080}h^2t^3-\\ \frac{16384}{638512875}h^2t^{17}-\frac{75588323677}{5675670000}h^2t^5+\frac{1}{2}h^2t^2-\frac{143219441}{11481750}h^2t^9-\frac{9469952}{1915538625}h^2t^{15}\\ +\frac{547902499}{173675502000}h^2-\frac{88668198397}{1420981380000}h^2t\end{array}\right)f_n$$

$$\beta_{\frac{1}{4}}(t)=\left(\begin{array}{l}-\frac{1957888}{273648375}t^{16}h^2+\frac{883334108}{3274425}h^2t^7+\frac{2498771668}{32564156625}h^2-\frac{2413491412}{9823275}h^2t^8-\\ \frac{1048576}{97692469875}h^2t^{18}-\frac{2636278528}{28704375}h^2t^{10}+\frac{338432}{110565}h^2t^{13}+\frac{16384}{212625}h^2t^{15}+\\ \frac{23229632}{606375}h^2t^{11}+\frac{31086766}{238875}h^2t^5-\frac{364514944}{29469825}h^2t^{12}+\frac{32}{3}h^2t^3+\frac{3114248}{18225}h^2t^9\\ +\frac{6863356}{135135}h^2t^4-\frac{1506345598}{2960377875}h^2t-\frac{27971584}{49116375}h^2t^{14}-\\ \frac{235395809336}{1064188125}h^2t^6+\frac{32768}{80405325}h^2t^{17}\end{array}\right)f_{n+\frac{1}{4}}$$

$$\beta_{\frac{1}{2}}(t)=\left(\begin{array}{l}\frac{2076199}{9009}h^2t^4+\frac{1008382489\ \ 3}{6548850}h^2t^8-\frac{1758324563}{1091475}h^2t^7\\ -\frac{425577227}{382725}h^2t^9-\frac{4308289792}{16372125}h^2t^{11}-\frac{12053504}{552825}h^2t^{13}\\ -\frac{702698699}{1051050}h^2t^5-40h^2t^3+\frac{851354368}{9823275}h^2t^{12}-\frac{2195456}{723647925}h^2t^{17}\\ +\frac{392801432}{637875}h^2t^{10}+\frac{229376}{4343625}t^{16}h^2-\frac{27656192}{49116375}h^2t^{15}\\ +\frac{524288}{6512831325}h^2t^{18}+\frac{428756404}{6512831325}h^2+\frac{8793376444\ \ 4}{70945875}h^2t^6\\ +\frac{2036676969\ \ 4}{3256415662\ \ 5}h^2t+\frac{35016704}{8513505}h^2t^{14}\end{array}\right)f_{n+\frac{1}{2}}$$

$$\beta_{\frac{3}{4}}(t)=\left(\begin{array}{l}
-\frac{66363392}{273648375}t^{16}h^{2}+\frac{481440256}{4975425}h^{2}t^{13}-\frac{59356594408}{13030875}h^{2}t^{6}\\
-\frac{1594752128}{4209975}h^{2}t^{12}+\frac{1120}{9}h^{2}t^{3}+\frac{5233929976}{1148175}h^{2}t^{9}\\
+\frac{182193548156}{29469825}h^{2}t^{7}+\frac{622592}{44304975}h^{2}t^{17}-\frac{278572577126}{97692469875}h^{2}t\\
+\frac{377307136}{147349125}h^{2}t^{15}+\frac{1101971788}{2960377875}h^{2}-\frac{2972115968}{1148175}h^{2}t^{10}\\
-\frac{2361782272}{127702575}h^{2}t^{14}+\frac{7927160128}{7016625}h^{2}t^{11}-\frac{8785278}{11583}h^{2}t^{4}\\
-\frac{1048576}{2791213425}h^{2}t^{18}+\frac{4730260714}{2027025}h^{2}t^{5}-\frac{60179298428}{9823275}h^{2}t^{8}
\end{array}\right)f_{n+\frac{3}{4}}$$

$$\beta_{1}(t)=\left(\begin{array}{l}
\frac{16326656}{21049875}t^{16}h^{2}+\frac{43814603596}{5740875}h^{2}t^{10}+\frac{2256379}{1188}h^{2}t^{4}\\
+\frac{14746475327}{2481078600}h^{2}t-\frac{910}{3}h^{2}t^{3}-\frac{398393344}{49116375}h^{2}t^{15}\\
-\frac{5052098138}{297675}h^{2}t^{7}-\frac{32768}{722925}h^{2}t^{17}-\frac{55184384}{184275}h^{2}t^{13}\\
-\frac{7934911232}{2338875}h^{2}t^{11}-\frac{5037429121}{382725}h^{2}t^{9}+\frac{262144}{214708725}h^{2}t^{18}\\
-\frac{278473343}{46200}h^{2}t^{5}+\frac{7400298496}{127702575}h^{2}t^{14}+\frac{4866476672}{4209975}h^{2}t^{12}\\
-\frac{1284218459}{6202696500}h^{2}+\frac{339345112759}{19646550}h^{2}t^{8}+\frac{113582746049}{9355500}h^{2}t^{6}
\end{array}\right)f_{n+1}$$

$$\beta_{\frac{5}{4}}(t)=\left(\begin{array}{l}
-\frac{96029567896}{3898125}h^{2}t^{6}-\frac{3664437376}{1403325}h^{2}t^{12}-\frac{4284416}{2338875}t^{16}h^{2}\\
-\frac{119540167676}{3274425}h^{2}t^{8}-\frac{53348582656}{3189375}h^{2}t^{10}+\frac{1894302208}{2764125}h^{2}t^{13}\\
-\frac{1048576}{357847875}h^{2}t^{18}+\frac{932102144}{49116375}h^{2}t^{15}+\frac{4292608}{39760875}h^{2}t^{17}\\
+\frac{17669056192}{2338875}h^{2}t^{11}-\frac{1833932}{495}h^{2}t^{4}+\frac{54249456808}{1913625}h^{2}t^{9}\\
+\frac{1729982638}{144375}h^{2}t^{5}+\frac{2912}{5}h^{2}t^{3}+\frac{38375279396}{1091475}h^{2}t^{7}\\
-\frac{1936381517594}{162820783125}h^{2}t-\frac{28556275712}{212837625}h^{2}t^{14}+\frac{33787997092}{32564156625}h^{2}
\end{array}\right)f_{n+\frac{5}{4}}$$

$$\beta_{\frac{3}{2}}(t)=\begin{pmatrix} \frac{6324224}{1913625}t^{16}h^{2}+\frac{1901557580\ 8}{4209975}h^{2}t^{12}+\frac{1246186651\ 84}{3189375}h^{2}t^{6} \\ +\frac{1378275737\ 6}{58046625}h^{2}t^{14}+\frac{1066205346\ 91}{1786050}h^{2}t^{8}-\frac{3797426233\ 4}{3256415662\ 5}h^{2} \\ +\frac{2316439}{405}h^{2}t^{4}-\frac{425984}{2168775}h^{2}t^{17}-\frac{454426624}{13395375}h^{2}t^{15} \\ -\frac{1516775723\ 99}{2679075}h^{2}t^{7}-\frac{5968550912}{4975425}h^{2}t^{13}+\frac{1717412488\ 381}{9769246987\ 5}h^{2}t \\ -\frac{8008}{9}h^{2}t^{3}-\frac{5403521306\ 9}{1148175}h^{2}t^{9}-\frac{9051909555\ 2}{7016625}h^{2}t^{11} \\ -\frac{2655798217}{141750}h^{2}t^{5}+\frac{8087322037\ 84}{28704375}h^{2}t^{10}+\frac{524288}{97594875}h^{2}t^{18} \end{pmatrix} f_{n+\frac{3}{2}}$$

$$\beta_{\frac{7}{4}}(t)=\begin{pmatrix} \frac{211042304}{4465125}h^{2}t^{15}-\frac{62234624}{13395375}t^{16}h^{2}-\frac{760745984}{2321865}h^{2}t^{14} \\ -\frac{7316183376\ 8}{1488375}h^{2}t^{6}+\frac{3354780472}{54675}h^{2}t^{9}+\frac{28572146}{1225}h^{2}t^{5} \\ +\frac{22880}{21}h^{2}t^{3}+\frac{905742848}{55825}h^{2}t^{13}-\frac{9334396}{1322}h^{2}t^{4} \\ +\frac{2144597798\ 8}{297675}h^{2}t^{7}+\frac{1409024}{5050475}h^{2}t^{17}-\frac{6865908472\ 4}{893025}h^{2}t^{8} \\ +\frac{1066161427\ 6}{6512831325}h^{2}+\frac{2820892250\ 24}{16372125}h^{2}t^{11}-\frac{1798258727\ 68}{29469825}h^{2}t^{12} \\ -\frac{1048576}{136632825}h^{2}t^{18}-\frac{7029656921\ 38}{3256415662\ 5}h^{2}t-\frac{1494282409\ 472}{40186125}h^{2}t^{10} \end{pmatrix} f_{n+\frac{7}{4}}$$

$$\beta_{2}(t)=\begin{pmatrix} \frac{459559936}{1289925}h^{2}t^{14}-\frac{77201408}{1488375}h^{2}t^{15}+\frac{1651688018}{42525}h^{2}t^{10} \\ +\frac{2130443052\ 8}{3274425}h^{2}t^{12}+\frac{3123187411\ 1}{396900}h^{2}t^{8}+\frac{2346469}{336}h^{2}t^{4} \\ -\frac{2145}{2}h^{2}t^{3}+\frac{6533687205\ 7}{1323000}h^{2}t^{6}-\frac{9923955660\ 8}{5457375}h^{2}t^{11} \\ -\frac{56952509}{2450}h^{2}t^{5}-\frac{2415519572}{33075}h^{2}t^{7}+\frac{3659245731\ 353}{1736755020\ 00}h^{2}t \\ +\frac{364544}{70875}t^{16}h^{2}-\frac{8092719928}{127575}h^{2}t^{9}+\frac{131072}{15181425}h^{2}t^{18} \\ -\frac{524288}{1686825}h^{2}t^{17}-\frac{325144576}{184275}h^{2}t^{13}-\frac{1337068653\ 61}{8683775100\ 0}h^{2} \end{pmatrix} f_{n+2}$$

$$\beta_{\frac{9}{4}}(t)=\left(\begin{array}{l}
-\frac{1848639326\ 72}{5740875}h^2t^{10}+\frac{4406821148}{3618239625}h^2+\frac{7479666176}{4975425}h^2t^{13}\\
+\frac{7472014720\ 64}{49116375}h^2t^{11}-\frac{1618792794\ 88}{29469825}h^2t^{12}+\frac{5985707837\ 6}{1148175}h^2t^{9}\\
-\frac{3239966607\ 86}{1953849239\ 75}h^2t+\frac{1840107758}{99225}h^2t^{5}-\frac{349108}{63}h^2t^{4}\\
-\frac{1048576}{136632825}h^2t^{18}+\frac{602619904}{13395375}h^2t^{15}-\frac{3555807232}{11609325}h^2t^{14}\\
+\frac{4161536}{15181425}h^2t^{17}-\frac{8609792}{1913625}t^{16}h^2+\frac{1582078284\ 44}{2679075}h^2t^{7}\\
+\frac{22880}{27}h^2t^{3}-\frac{5901136434\ 4}{1499375}h^2t^{6}-\frac{5726884072\ 4}{893025}h^2t^{8}
\end{array}\right)f_{n+\frac{9}{4}}$$

$$\beta_{\frac{5}{2}}(t)=\left(\begin{array}{l}
-\frac{45744128}{1488375}h^2t^{15}+\frac{3842507164}{151875}h^2t^{6}+\frac{2364487}{657}h^2t^{4}\\
+\frac{1204474265\ 6}{58046625}h^2t^{14}+\frac{7399807789\ 9}{1786050}h^2t^{8}-\frac{4989017864}{6512831325}h^2\\
-\frac{2791543808}{2764125}h^2t^{13}-\frac{7837060864}{779625}h^2t^{11}-\frac{1128319855\ 3}{297675}h^2t^{7}\\
+\frac{1695388134\ 44}{1628078312\ 5}h^2t+\frac{5931008}{1913625}t^{16}h^2-\frac{8008}{15}h^2t^{3}\\
-\frac{2163645511\ 1}{637875}h^2t^{9}+\frac{524288}{97594875}h^2t^{18}+\frac{6059560732\ 24}{28704375}h^2t^{10}\\
+\frac{3081291008}{841995}h^2t^{12}-\frac{32768}{172125}h^2t^{17}-\frac{309053641}{26250}h^2t^{5}
\end{array}\right)f_{n+\frac{5}{2}}$$

$$\beta_{\frac{11}{4}}(t)=\left(\begin{array}{l}
-\frac{155648}{93555}t^{16}h^2+\frac{169638698}{28875}h^2t^{5}-\frac{1048576}{357847875}h^2t^{18}\\
-\frac{9713024}{5103}h^2t^{12}+\frac{32768}{318087}h^2t^{17}+\frac{803815424}{49116375}h^2t^{15}\\
-\frac{2332945203\ 2}{212837625}h^2t^{14}+\frac{2083298543\ 6}{1091475}h^2t^{7}-\frac{4942151437\ 6}{3898125}h^2t^{6}\\
-\frac{3452111257\ 6}{3189375}h^2t^{10}+\frac{3218776576}{6081075}h^2t^{13}-\frac{1373784516\ 4}{654885}h^2t^{8}\\
-\frac{862196}{495}h^2t^{4}+\frac{2912}{11}h^2t^{3}-\frac{4790797346}{930404475}h^2t\\
+\frac{1103563712}{212625}h^2t^{11}+\frac{6612791672}{382725}h^2t^{9}+\frac{1764299092}{4652022375}h^2
\end{array}\right)f_{n+\frac{11}{4}}$$

$$\beta_3(t)=\left(\begin{array}{l}\frac{1602031039\ 99}{19646550}h^2t^8+\frac{14360576}{21049875}t^{16}h^2+\frac{1130412032}{25540515}h^2t^{14}\\ +\frac{2376499}{3564}h^2t^4+\frac{1371079236\ 19}{28066500}h^2t^6-\frac{1015808}{23856525}h^2t^{17}\\ +\frac{3189594752}{4209975}h^2t^{12}-\frac{1440011852\ 8}{7016625}h^2t^{11}+\frac{262144}{214708725}h^2t^{18}\\ -\frac{910}{9}h^2t^3-\frac{2176293171\ 86}{29469825}h^2t^7-\frac{255794977}{113400}h^2t^5\\ +\frac{2441898170\ 8}{5740875}h^2t^{10}+\frac{630836401}{321489000}h^2t-\frac{753694637}{5210265060}h^2\\ -\frac{89194496}{13395375}h^2t^{15}-\frac{1055620096}{4975425}h^2t^{13}-\frac{7751701981}{1148175}h^2t^9\end{array}\right)f_{n+3}$$

$$\beta_{\frac{13}{4}}(t)=\left(\begin{array}{l}\frac{1404923456}{2338875}h^2t^{11}-\frac{2309452578\ 8}{9823275}h^2t^8-\frac{56532992}{273648375}t^{16}h^2\\ -\frac{4250229231\ 2}{30405375}h^2t^6+\frac{748360792}{382725}h^2t^9-\frac{6037952194}{1085471887\ 5}h^2t\\ +\frac{16497868}{402026625}h^2+\frac{1120}{39}h^2t^3-\frac{1686433792}{127702575}h^2t^{14}\\ -\frac{732652}{3861}h^2t^4+\frac{1343488}{103378275}h^2t^{17}-\frac{939156608}{4209975}h^2t^{12}\\ -\frac{1421589248}{1148175}h^2t^{10}+\frac{48311842}{75075}h^2t^5+\frac{3862016}{61425}h^2t^{13}\\ +\frac{1277771776}{638512875}h^2t^{15}+\frac{6943734244}{3274425}h^2t^7-\frac{1048576}{2791213425}h^2t^{18}\end{array}\right)f_{n+\frac{13}{4}}$$

$$\beta_{\frac{7}{2}}(t)=\left(\begin{array}{l}-\frac{464922461}{1091475}h^2t^7-\frac{20578304}{49116375}h^2t^{15}+\frac{3103491973}{6548850}h^2t^8\\ +\frac{719672003}{6512831325}h^2t+\frac{1123806248}{4465125}h^2t^{10}-\frac{1998848}{723647925}h^2t^{17}\\ +\frac{116887552}{4256725}h^2t^{14}+\frac{524288}{6512831325}h^2t^{18}+\frac{9273344}{212837625}t^{16}h^2\\ -\frac{7181312}{552825}h^2t^{13}-\frac{40}{7}h^2t^3+\frac{40917248}{893025}h^2t^{12}\\ +\frac{2385079}{63063}h^2t^4-\frac{135094997}{1051050}h^2t^5-\frac{182637824}{1488375}h^2t^{11}\\ -\frac{151450421}{382725}h^2t^9+\frac{1985972823\ 2}{70945875}h^2t^6-\frac{265625566}{3256415662\ 5}hh2\end{array}\right)f_{n+\frac{7}{2}}$$

$$\beta_{\frac{15}{4}}(t)=\begin{pmatrix}\frac{286925608}{5740875}h^2t^9-\frac{172828288}{29469825}h^2t^{12}-\frac{1564672}{273648375}t^{16}h^2\\-\frac{1048576}{97692469875}h^2t^{18}+\frac{767645504}{49116375}h^2t^{11}-\frac{226988032}{638512875}h^2t^{14}\\+\frac{5692686122}{354729375}h^2t^5-\frac{916085248}{28704375}h^2t^{10}+\frac{143183564}{2679075}h^2t^7\\+\frac{33025436}{32564156625}h^2-\frac{9554044}{2027025}h^2t^4+\frac{360448}{986792625}h^2t^{17}\\+\frac{32}{45}h^2t^3+\frac{41588224}{24877125}h^2t^{13}-\frac{111833020088}{3192564375}h^2t^6\\-\frac{585898612}{9823275}h^2t^8+\frac{8044544}{147349125}h^2t^{15}-\frac{6707365394}{488462349375}h^2t\end{pmatrix}f_{n+\frac{15}{4}}$$

$$\beta_4(t)=\begin{pmatrix}\frac{35118025721}{17027010000}h^2t^6-\frac{32768}{9823275}h^2t^{15}+\frac{10384288}{29469825}h^2t^{12}\\-\frac{55808}{552825}h^2t^{13}-\frac{1}{24}h^2t^3-\frac{2065639}{654885}h^2t^7\\-\frac{612224}{654885}h^2t^{11}-\frac{2271089}{765450}h^2t^9+\frac{64413479}{80157924000}h^2t\\-\frac{2812393}{47366046000}h^2+\frac{13783552}{638512875}h^2t^{14}-\frac{13215487}{14014000}h^2t^5\\+\frac{96256}{273648375}t^{16}h^2-\frac{16384}{723647925}h^2t^{17}+\frac{65536}{97692469875}h^2t^{18}\\+\frac{54576553}{28704375}h^2t^{10}+\frac{277382447}{78586200}h^2t^8+\frac{1195757}{4324320}h^2t^4\end{pmatrix}f_{n+4}$$

Evaluating (5) at $t=1$ and other non-interpolating points to obtain the discrete schemes

$$\begin{aligned}y_{n+4}&=-2y_{n+1}+3y_{n+2}+\frac{30965483189}{28945917000}f_{n+2}h^2-\frac{567394727}{2894591700}f_{n+3}h^2-\frac{17406541001}{130256626500}f_{n+1}h^2+\\&\frac{1643814131}{521026506000}f_{n+4}h^2+\frac{7306054372}{3618239625}f_{n+\frac{9}{4}}h^2+\frac{2526885388}{2170943775}f_{n+\frac{11}{4}}h^2+\frac{930929732}{2504935125}f_{n+\frac{13}{4}}h^2+\\&\frac{92541292}{1206079875}f_{n+\frac{15}{4}}h^2+\frac{13300244}{328930875}f_{n+\frac{3}{4}}h^2-\frac{29964920152}{32564156625}f_{n+\frac{5}{2}}h^2+\frac{2084312332}{1302566265}f_{n+\frac{7}{4}}h^2\\&+\frac{715246282}{10854718875}f_{n+\frac{7}{2}}h^2-\frac{17577916}{2170943775}f_{n+\frac{1}{2}}h^2+\frac{32868652}{32564156625}f_{n+\frac{1}{4}}h^2-\frac{209083298}{402026625}f_{n+\frac{3}{2}}h^2+\\&\frac{5488459124}{10854718875}f_{n+\frac{5}{4}}h^2-\frac{311411}{5262894000}h^2f_n\end{aligned}\qquad(6)$$

$$
\begin{aligned}
y_n &= 2y_{n+1} - y_{n+2} - \frac{133706865361}{86837751000} f_{n+2}h^2 - \frac{753694637}{5210265060} f_{n+3}h^2 - \frac{1284218459}{6202696500} f_{n+1}h^2 - \\
&\frac{2812393}{47366046000} f_{n+4}h^2 + \frac{4406821148}{3618239625} f_{n+\frac{9}{4}}h^2 + \frac{1764299092}{4652022375} f_{n+\frac{11}{4}}h^2 + \frac{16497868}{402026625} f_{n+\frac{13}{4}}h^2 \\
&+ \frac{33025436}{32564156625} f_{n+\frac{15}{4}}h^2 + \frac{1101971788}{2960377875} f_{n+\frac{3}{4}}h^2 - \frac{4989017864}{6512831325} f_{n+\frac{5}{2}}h^2 + \frac{10661614276}{6512831325} f_{n+\frac{7}{4}}h^2 \\
&- \frac{265625566}{32564156625} f_{n+\frac{7}{2}}h^2 + \frac{428756404}{6512831325} f_{n+\frac{1}{2}}h^2 + \frac{2498771668}{32564156625} f_{n+\frac{1}{4}}h^2 - \frac{37974262334}{32564156625} f_{n+\frac{3}{2}}h^2 \\
&+ \frac{33787997092}{32564156625} f_{n+\frac{5}{4}}h^2 + \frac{547902499}{173675502000} h^2 f_n
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
y_{n+\frac{1}{4}} &= \frac{7}{4} y_{n+1} - \frac{3}{4} y_{n+2} + \frac{755787703620221}{541999890432000} f_{n+2}h^2 + \frac{328843390751}{27099994521600} f_{n+3}h^2 + \\
&\frac{19181431546723}{87107125248000} f_{n+1}h^2 + \frac{618663001}{126701273088000} f_{n+4}h^2 - \frac{7135620342143}{67749986304000} f_{n+\frac{9}{4}}h^2 - \\
&\frac{1861357623131}{58071416832000} f_{n+\frac{11}{4}}h^2 - \frac{18625175527}{5444195328000} f_{n+\frac{13}{4}}h^2 - \frac{3781723471}{45166657536000} f_{n+\frac{15}{4}}h^2 + \\
&\frac{14487698032823}{135499972608000} f_{n+\frac{3}{4}}h^2 + \frac{159402025138247}{2438999506944000} f_{n+\frac{5}{2}}h^2 - \frac{22738925289191}{243899950694400} f_{n+\frac{7}{4}}h^2 \\
&+ \frac{6043616323}{8934064128000} f_{n+\frac{2}{7}}h^2 + \frac{11150738059523}{162599967129600} f_{n+\frac{1}{2}}h^2 + \frac{2543479927081}{609749876736000} f_{n+\frac{1}{4}}h^2 + \\
&\frac{181310835779}{860317286400} f_{n+\frac{3}{2}}h^2 + \frac{6319257921037}{101624979456000} f_{n+\frac{5}{4}}h^2 - \frac{5884349021}{98545434624000} h^2 f_n
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
y_{n+\frac{1}{2}} &= +\frac{3}{2} y_{n+1} - \frac{1}{2} y_{n+2} - \frac{56230741813}{3705077376000} f_{n+2}h^2 - \frac{6612484171}{41`68212048000} f_{n+3}h^2 + \frac{911467059983}{8336424096000} f_{n+1}h^2 \\
&- \frac{8405993}{133382785536000} f_{n+4}h^2 + \frac{447757879}{32564156625} f_{n+\frac{9}{4}}h^2 + \frac{292187003}{69470200800} f_{n+\frac{11}{4}}h^2 + \frac{232176739}{521026506000} f_{n+\frac{13}{4}}h^2 \\
&+ \frac{11291663}{1042053012000} f_{n+\frac{15}{4}}h^2 + \frac{68640434803}{1042053012000} f_{n+\frac{3}{4}}h^2 - \frac{71655121877}{8336424096000} f_{n+\frac{5}{2}}h^2 + \frac{52307218153}{1042053012000} f_{n+\frac{7}{4}}h^2 \\
&- \frac{122009947}{1389404016000} f_{n+\frac{7}{2}}h^2 + \frac{13242910981}{2778808032000} f_{n+\frac{1}{2}}h^2 - \frac{7112293}{52102650600} f_{n+\frac{1}{4}}h^2 + \\
&\frac{1915624261}{41682120480} f_{n+\frac{3}{2}}h^2 + \frac{18416662469}{173675502000} f_{n+\frac{5}{4}}h^2 + \frac{27847133}{6062853888000} h^2 f_n
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
y_{n+\frac{3}{4}} &= \frac{5}{4}y_{n+1} - \frac{1}{4}y_{n+2} + \frac{17592978993677}{2276399539814400}f_{n+2}h^2 + \frac{2080698083}{4064999178240}f_{n+3}h^2 + \\
&\frac{32256274862369}{569099884953600}f_{n+1}h^2 + \frac{1221042517}{6208362381312000}f_{n+4}h^2 - \frac{8239037194859}{1707299654860800}f_{n+\frac{9}{4}}h^2 - \\
&\frac{5877374390551}{428249137152000}f_{n+\frac{11}{4}}h^2 - \frac{2075083721}{14592304742400}f_{n+\frac{13}{4}}h^2 - \frac{660153889}{194011324416000}f_{n+\frac{15}{4}}h^2 + \\
&\frac{1624196457203}{284549942476800}f_{n+\frac{3}{4}}h^2 + \frac{49116908637091}{17072996548608000}f_{n+\frac{5}{2}}h^2 + \frac{5912404501}{666913927680}f_{n+\frac{7}{4}}h^2 + \\
&\frac{94848204179}{3414599309721600}f_{n+\frac{7}{2}}h^2 - \frac{191708170583}{682919861944320}f_{n+\frac{1}{2}}h^2 + \frac{25027479793}{1219499753472000}f_{n+\frac{1}{4}}h^2 + \\
&\frac{630543567365149}{17072996548608000}f_{n+\frac{3}{2}}h^2 + \frac{5308458265351}{1219499753472000}f_{n+\frac{5}{4}}h^2 - \frac{59649043717}{68291986194432000}h^2 f_n
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
y_{n+\frac{5}{4}} &= \frac{3}{4}y_{n+1} + \frac{1}{4}y_{n+2} - \frac{5529907422869}{3793999233024000}f_{n+2}h^2 + \frac{195013561399}{1707299654860800}f_{n+3}h^2 - \\
&\frac{4973650034113}{1067062284288000}f_{n+1}h^2 + \frac{2999764459}{68291986194432000}f_{n+4}h^2 - \frac{1973559814123}{2134124568576000}f_{n+\frac{9}{4}}h^2 \\
&- \frac{866399746313}{2845499242768000}f_{n+\frac{11}{4}}h^2 - \frac{135821871329}{4268249137152000}f_{n+\frac{13}{4}}h^2 - \frac{6495832069}{8536498274304000}f_{n+\frac{15}{4}}h^2 \\
&+ \frac{244606431137}{776045297664000}f_{n+\frac{3}{4}}h^2 + \frac{10556243950933}{17072996548608000}f_{n+\frac{5}{2}}h^2 - \frac{28794331847599}{1707299654860800}f_{n+\frac{7}{4}}h^2 \\
&+ \frac{35382316339}{5690998849536000}f_{n+\frac{7}{2}}h^2 - \frac{8203685387}{227639953981440}f_{n+\frac{1}{2}}h^2 + \frac{132065419}{41040857088000}f_{n+\frac{1}{4}}h^2 - \\
&\frac{522741222280587}{17072996548608000}f_{n+\frac{3}{2}}h^2 - \frac{58644246932731}{1422749712384000}f_{n+\frac{5}{4}}h^2 - \frac{932052347}{6208362381312000}h^2 f_n
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
y_{n+\frac{3}{2}} &= \frac{1}{2}y_{n+1} + \frac{1}{2}y_{n+2} - \frac{3878452073}{1587890304000}f_{n+2}h^2 + \frac{59715949}{1667284819200}f_{n+3}h^2 - \frac{947002951}{347351004000}f_{n+1}h^2 \\
&+ \frac{27613}{1905468364800}f_{n+4}h^2 - \frac{120758749}{1042053012000}f_{n+\frac{9}{4}}h^2 - \frac{11916173}{130256626500}f_{n+\frac{11}{4}}h^2 - \\
&\frac{168661}{16540524000}f_{n+\frac{13}{4}}h^2 - \frac{3937}{1578868200}f_{n+\frac{15}{4}}h^2 + \frac{39083753}{260513253000}f_{n+\frac{3}{4}}h^2 \\
&+ \frac{689035189}{4168212048000}f_{n+\frac{5}{2}}h^2 - \frac{3344561417}{104205301200}f_{n+\frac{7}{4}}h^2 + \frac{16830041}{8336424096000}f_{n+\frac{7}{2}}h^2 - \\
&\frac{6507983}{416821204800}f_{n+\frac{1}{2}}h^2 + \frac{1377581}{1042053012000}f_{n+\frac{1}{4}}h^2 - \frac{2467753699}{44108064000}f_{n+\frac{3}{2}}h^2 \\
&- \frac{33250195141}{1042053012000}f_{n+\frac{5}{4}}h^2 - \frac{3978437}{66691392768000}h^2 f_n
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
y_{n+\frac{7}{4}} &= \frac{1}{4} y_{n+1} + \frac{3}{4} y_{n+2} - \frac{18543998932117}{3793999233024000} f_{n+1} h^2 + \frac{9930126109}{474249904128000} f_{n+3} h^2 - \\
& \frac{1764138977609}{1219499753472000} f_{n+1} h^2 + \frac{65081837}{6208362381312000} f_{n+4} + \frac{308558086279}{948499808256000} f_{n+\frac{9}{4}} h^2 - \\
& \frac{2127943229}{50812489728000} f_{n+\frac{11}{4}} h^2 - \frac{5127067769}{776045297664000} f_{n+\frac{13}{4}} h^2 - \frac{27897469}{158083301376000} f_{n+\frac{15}{4}} h^2 + \\
& \frac{21626520031}{237124952064000} f_{n+\frac{3}{4}} h^2 + \frac{328746311431}{17072996548608000} f_{n+\frac{5}{2}} h^2 - \frac{173340112544107}{4268249137152000} f_{n+\frac{7}{4}} h^2 \\
& + \frac{7864634993}{5690998849536000} f_{n+\frac{7}{2}} h^2 - \frac{4500040789}{43776914272000} f_{n+\frac{1}{2}} h^2 + \frac{7800576841}{8536498274304000} f_{n+\frac{1}{4}} h^2 - \\
& \frac{6672411527191}{210777735168000} f_{n+\frac{3}{2}} h^2 - \frac{44241906055591}{2845499424768000} f_{n+\frac{5}{4}} h^2 - \frac{322793573}{7587998466048000} h^2 f_n
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
y_{n+\frac{9}{4}} &= -\frac{1}{4} y_{n+1} + \frac{5}{4} y_{n+2} + \frac{122601171114509}{2276399539814400} f_{n+2} h^2 - \frac{117339013171}{1707299654860800} f_{n+3} h^2 + \\
& \frac{398034001579}{284549942476800} f_{n+1} h^2 - \frac{1158050701}{68291986194432000} f_{n+4} h^2 + \frac{2038799962907}{284549942476800} f_{n+\frac{9}{4}} h^2 \\
& + \frac{2123899032331}{8536498274304000} f_{n+\frac{11}{4}} h^2 + \frac{252219697}{15808330137600} f_{n+\frac{13}{4}} h^2 + \frac{2663078729}{853649874204000} f_{n+\frac{15}{4}} h^2 - \\
& \frac{139750766039}{1707299654860800} f_{n++\frac{3}{4}} h^2 - \frac{15565084462919}{17072996548608000} f_{n+\frac{5}{2}} h^2 + \frac{82124249350387}{1707299654860800} f_{n+\frac{7}{4}} h^2 - \\
& \frac{9451808407}{3414599309721600} f_{n+\frac{7}{2}} h^2 + \frac{30367290743}{3414599309721600} f_{n+\frac{1}{2}} h^2 - \frac{3321980719}{4268249137152000} f_{n+\frac{1}{4}} h^2 + \\
& \frac{525228995550983}{17072996548608000} f_{n+\frac{3}{2}} h^2 + \frac{16811515342079}{1067062284288000} f_{n+\frac{1}{4}} h^2 + \frac{74636111}{2069454127104000} h^2 f_n
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
y_{n+\frac{5}{2}} &= -\frac{1}{2} y_{n+1} + \frac{3}{2} y_{n+2} + \frac{426261380299}{3705077376000} f_{n+2} h^2 + \frac{8843}{300736800} f_{n+3} h^2 + \frac{23271778529}{8336424096000} f_{n+1} h^2 \\
& - \frac{566149}{66691392768000} f_{n+4} h^2 + \frac{3673651643}{57891834000} f_{n+\frac{9}{4}} h^2 - \frac{109308439}{347351004000} f_{n+\frac{11}{4}} h^2 - \frac{857}{1821771000} f_{n+\frac{13}{4}} h^2 \\
& + \frac{4567}{38594556000} f_{n+\frac{15}{4}} h^2 - \frac{18605651}{115783668000} f_{n+\frac{3}{4}} h^2 + \frac{46192450537}{8336424096000} f_{n+\frac{5}{2}} h^2 + \frac{19890116999}{208410602400} f_{n+\frac{7}{4}} h^2 \\
& - \frac{410233}{694702008000} f_{n+\frac{7}{2}} h^2 + \frac{9471127}{555761606400} f_{n+\frac{1}{2}} h^2 - \frac{108151}{74432358000} f_{n+\frac{1}{4}} h^2 + \frac{3172691173}{51459408000} f_{n+\frac{3}{2}} h^2 \\
& + \frac{5582813}{177219900} f_{n+\frac{5}{4}} h^2 + \frac{486527}{7410154752000} h^2 f_n
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
y_{n+\frac{11}{4}} &= -\frac{3}{4}y_{n+1}+\frac{7}{4}y_{n+2}+\frac{95440706420861}{541999890432000}f_{n+2}h^2-\frac{16079881931}{30487493836800}f_{n+3}h^2+\\
&\frac{5180271591317}{1219499753472000}f_{n+1}h^2-\frac{65081837}{886908911616000}f_{n+4}h^2+\frac{154223001812611}{1219499753472000}f_{n+\frac{9}{4}}h^2+\\
&\frac{1254365952403}{203249958912000}f_{n+\frac{11}{4}}h^2+\frac{15583724251}{174214250496000}f_{n+\frac{13}{4}}h^2+\frac{107652793}{76218734592000}f_{n+\frac{15}{4}}h^2-\\
&\frac{14277350629}{55431806976000}f_{n+\frac{3}{4}}h^2+\frac{150279932205437}{2438999506944000}f_{n+\frac{5}{2}}h^2+\frac{8683131661927}{60974987673600}f_{n+\frac{7}{4}}h^2-\\
&\frac{10978512277}{812999835648000}f_{n+\frac{7}{2}}h^2+\frac{934999229}{32519993425920}f_{n+\frac{1}{2}}h^2-\frac{625952183}{243899950694400}f_{n+\frac{1}{4}}h^2+\\
&\frac{32352183503461}{348428500992000}f_{n+\frac{3}{2}}h^2+\frac{19140459636113}{406499917824000}f_{n+\frac{5}{4}}h^2+\frac{1177291261}{9755998027776000}h^2 f_n
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
y_{n+3} &= -y_{n+1}+2y_{n+2}+\frac{20405207897}{86837751000}f_{n+2}h^2+\frac{119652077}{21709437750}f_{n+3}h^2+\\
&\frac{119652077}{21709437750}f_{n+1}h^2+\frac{4847}{47366046000}f_{n+4}h^2+\frac{6223208984}{32564156625}f_{n+\frac{9}{4}}h^2+\\
&\frac{2057641864}{32564156625}f_{n+\frac{11}{4}}h^2-\frac{1089064}{3618239625}f_{n+\frac{13}{4}}h^2-\frac{78392}{32564156625}f_{n+\frac{15}{4}}h^2-\\
&\frac{1089064}{3618239625}f_{n+\frac{3}{4}}h^2+\frac{4004671091}{32564156625}f_{n+\frac{5}{2}}h^2+\frac{6223208984}{23564156625}f_{n+\frac{7}{4}}h^2\\
&+\frac{978413}{32564156625}f_{n+\frac{7}{2}}h^2+\frac{978413}{32564156625}f_{n+\frac{1}{2}}h^2-\frac{78392}{32564156625}f_{n+\frac{1}{4}}h^2+\\
&\frac{4004671091}{32564156625}f_{n+\frac{3}{2}}h^2+\frac{2057641864}{32564156625}f_{n+\frac{5}{4}}h^2+\frac{4847}{47366046000}h^2 f_n
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
y_{n+\frac{13}{4}} &= -\frac{5}{4}y_{n+1}+\frac{9}{4}y_{n+2}+\frac{108992552878771}{36133326028800}f_{n+2}h^2+\frac{803934091589}{12646664110080}f_{n+3}h^2+\\
&\frac{1053014247689}{142274971238400}f_{n+1}h^2-\frac{2424456497}{3251999342592000}f_{n+4}h^2+\frac{30597510091}{123502579200}f_{n+\frac{9}{4}}h^2\\
&+\frac{116204054872553}{948499808256000}f_{n+\frac{11}{4}}h^2+\frac{216733872821}{40649991782400}f_{n+\frac{13}{4}}h^2+\frac{1845739111}{105388867584000}f_{n+\frac{15}{4}}h^2-\\
&\frac{32782995559}{63233320550400}f_{n+\frac{3}{4}}h^2+\frac{1085014937854303}{5690998849536000}f_{n+\frac{5}{2}}h^2+\frac{26640345785371}{113819976990720}f_{n+\frac{7}{4}}h^2\\
&-\frac{92255367131}{379399923302400}f_{n+\frac{7}{2}}h^2+\frac{4957572287}{75879984660480}f_{n+\frac{1}{2}}h^2-\frac{4561188473}{711374856192000}f_{n+\frac{1}{4}}h^2+\\
&\frac{1571791978151}{10037035008000}f_{n+\frac{3}{2}}h^2+\frac{36805185400801}{474249904128000}f_{n+\frac{5}{4}}h^2+\frac{74636111}{229939347456000}h^2 f_n
\end{aligned}
\tag{18}
$$

$$\begin{aligned} y_{n+\frac{7}{2}} = {} & -\frac{3}{2} y_{n+1} + \frac{5}{2} y_{n+2} + \frac{249940512719}{741015475200} f_{n+2}h^2 + \frac{196077331267}{1667284819200} f_{n+3}h^2 + \\ & \frac{795637543}{119091772800} f_{n+1}h^2 + \frac{28777757}{6062853888000} f_{n+4}h^2 + \frac{7020424877}{208410602400} f_{n+\frac{9}{4}}h^2 \\ & + \frac{4982542483}{24810786000} f_{n+\frac{11}{4}}h^2 + \frac{13633991831}{208410602400} f_{n+\frac{13}{4}}h^2 - \frac{3318379}{23683023000} f_{n+\frac{15}{4}}h^2 - \\ & \frac{612217}{104205301200} f_{n+\frac{3}{4}}h^2 + \frac{240114818893}{1042053012000} f_{n+\frac{5}{2}}h^2 + \frac{1956514271}{6512831325} f_{n+\frac{7}{4}}h^2 \\ & + \frac{205663813}{42750892800} f_{n+\frac{7}{2}}h^2 - \frac{11878303}{277880803200} f_{n+\frac{1}{2}}h^2 + \frac{7528847}{1042053012000} f_{n+\frac{1}{4}}h^2 + \\ & \frac{1466138577067}{8336424096000} f_{n+\frac{3}{2}}h^2 + \frac{34383204839}{347351004000} f_{n+\frac{5}{4}}h^2 - \frac{31793101}{66691392768000} h^2 f_n \end{aligned} \tag{19}$$

$$\begin{aligned} y_{n+\frac{15}{4}} = {} & -\frac{7}{4} y_{n+1} + \frac{11}{4} y_{n+2} + \frac{81397633020869}{147818151936000} f_{n+2}h^2 + \frac{2548202903923}{11086361395200} f_{n+3}h^2 + \\ & \frac{114979352213}{5279219712000} f_{n+1}h^2 - \frac{7542902099}{126701273088000} f_{n+4}h^2 + \frac{26740872502679}{110863613952000} f_{n+\frac{9}{4}}h^2 \\ & + \frac{684028286531}{3959414784000} f_{n+\frac{11}{4}}h^2 + \frac{187225001747}{1759739904000} f_{n+\frac{13}{4}}h^2 + \frac{1026630779}{246363586560} f_{n+\frac{15}{4}}h^2 - \\ & \frac{13677259667}{3464487936000} f_{n+\frac{3}{4}}h^2 + \frac{94446951697547}{221727227904000} f_{n+\frac{5}{2}}h^2 + \frac{508005559667}{2217272279040} f_{n+\frac{7}{4}}h^2 \\ & + \frac{2173887187553}{31675318272000} f_{n+\frac{7}{2}}h^2 + \frac{32329996453}{44345445580800} f_{n+\frac{1}{2}}h^2 - \frac{9749457439}{110863613952000} f_{n+\frac{1}{4}}h^2 + \\ & \frac{2962347508129}{10558439424000} f_{n+\frac{3}{2}}h^2 + \frac{8705555192123}{110863613952000} f_{n+\frac{5}{4}}h^2 + \frac{4489467503}{886908911616000} h^2 f_n \end{aligned} \tag{20}$$

Evaluating the first derivatives of (5) at $t = 0\left(\frac{1}{4}\right)4$ we obtained the discrete schemes

$$-\frac{1}{15630795180000}\frac{1}{h}\begin{pmatrix} 337423532226240\, f_{n+\frac{7}{4}}h^2 - 1727212807200\, f_{n+\frac{7}{2}}h^2 \\ +185892625689024\, f_{n+\frac{5}{4}}h^2 + 975350182367 h^2 f_n \\ -92902794560100\, f_{n+1}h^2 + 44571612340160\, f_{n+\frac{3}{4}h}h^2 \\ +214635692608\, f_{n+\frac{15}{4}}h^2 + 8694651159360\, f_{n+\frac{13}{4}}h^2 \\ +80485395412800\, f_{n+\frac{11}{4}}h^2 + 259197328628800\, f_{n+\frac{9}{4}}h^2 \\ -12560628405\, f_{n+4}h^2 - 30671265816620\, f_{n+3}h^2 \\ -329332115821770\, f_{n+2}h^2 - 162754446090624\, f_{n+\frac{5}{2}}h^2 \\ -156030795180000\, y_{n+2} - 274785998140960\, f_{n+\frac{3}{2}}h^2 \\ -9776049453120\, f_{n+\frac{1}{2}}h^2 + 7953504757440\, f_{n+\frac{1}{4}}h^2 \\ +15630795180000\, y_{n+1} \end{pmatrix} = y_n' \quad (21)$$

$$\frac{1}{128047474114560000}\frac{1}{h}\begin{pmatrix} 258231393162748790\, f_{n+\frac{7}{4}}h^2 - 1257998674653210\, f_{n+\frac{7}{2}}h^2 \\ +13163047617477590\, f_{n+\frac{5}{4}}h^2 + 102920509582785 h^2 f_n \\ -119460625398637810\, f_{n+1}h^2 + 10064821583611710\, f_{n+\frac{3}{4}}h^2 \\ +155517866693010\, f_{n+\frac{15}{4}}h^2 + 6371057618287810\, f_{n+\frac{13}{4}}h^2 \\ +59906106995287098\, f_{n+\frac{11}{4}}h^2 + 197506569886789770\, f_{n+\frac{9}{4}}h^2 \\ -9060035554241\, f_{n+4}h^2 - 22635204738163470\, f_{n+3}h^2 \\ -255840602999339520\, f_{n+2}h^2 - 122406028892389010\, f_{n+\frac{5}{2}}h^2 \\ +128047474114560000\, y_{n+2} - 241523451157573998\, f_{n+\frac{3}{3}}h^2 \\ -51155410792964070\, f_{n+\frac{1}{2}}h^2 - 9739823751673234\, f_{n+\frac{1}{4}}h^2 \\ -128047474114560000\, y_{n+1} \end{pmatrix} = y'_{n+\frac{1}{4}} \quad (22)$$

$$-\frac{1}{250092722880000}\frac{1}{h}\begin{pmatrix} 111337863493840 f_{n+\frac{7}{4}}h^2 - 408365032170 f_{n+\frac{7}{2}}h^2 \\ +10957372585936 f_{n+\frac{5}{4}}h^2 + 17648816143h^2 f_n \\ +20941620567730 f_{n+1}h^2 + 87979351532560 f_{n+\frac{3}{4}}h^2 \\ +5018838972 f_{n+\frac{15}{4}}h^2 + 2082426725360 f_{n+\frac{13}{4}}h^2 \\ +19937856248112 f_{n+\frac{11}{4}}h^2 + 67490752155440 f_{n+\frac{9}{4}}h^2 \\ -2909113999 f_{n+4}h^2 - 7459336997170 f_{n+3}h^2 \\ -87689268241920 f_{n+2}h^2 - 41232594657826 f_{n+\frac{5}{2}}h^2 \\ -250092722880000 y_{n+2} - 53443853103902 f_{n+\frac{3}{2}}h^2 \\ +214184282288000 y_{n+2} - 53443853103902 f_{n+\frac{3}{2}}h^2 \\ +21418428204330 f_{n+\frac{1}{2}}h^2 - 50083862336 f_{n+\frac{1}{4}}h^2 \\ +250092722880000 y_{n+1} \end{pmatrix} = y'_{n+\frac{1}{2}} \quad (23)$$

$$\frac{1}{128047474114560000}\frac{1}{h}\begin{pmatrix} 6715004464140150 f_{n+\frac{7}{4}}h^2 - 54515898425370 f_{n+\frac{7}{2}}h^2 \\ -12643563547698234 f_{n+\frac{5}{4}}h^2 + 1513308056513h^2 f_n \\ 40724544725245170 f_{n+1}h^2 - 12029913739099330 f_{n+\frac{3}{3}}h^2 \\ +6652854144642 f_{n+\frac{15}{4}}h^2 + 280368223615170 f_{n+\frac{13}{4}}h^2 \\ +2748299174844474 f_{n+\frac{11}{4}}h^2 + 9723851518385290 f_{n+\frac{9}{4}}h^2 \\ -383323270593 f_{n+4}h^2 - 1014790860110350 f_{n+3}h^2 \\ -13881082214499840 f_{n+2}h^2 - 5784033151782546 f_{n+\frac{5}{2}}h^2 \\ +128047474114560000 y_{n+2} - 29808348926230894 f_{n+\frac{3}{2}}h^2 \\ +464750182887450 f_{n+\frac{1}{2}}h^2 - 34868825641362 f_{n+\frac{1}{4}}h^2 \\ -128047474114560000 y_{n+1} \end{pmatrix} = y'_{n+\frac{3}{4}} \quad (24)$$

$$-\frac{1}{15630795180000}\frac{1}{h}\begin{pmatrix} 1713206099840 f_{n+\frac{7}{4}}h^2 - 2265363120 f_{n+\frac{7}{2}}h^2 \\ +3791159211648 f_{n+\frac{5}{4}}h^2 + 43882029h^2 f_n \\ +1485140358680 f_{n+1}h^2 - 82344804480 f_{n+\frac{3}{4}}h^2 \\ +274237056 f_{n+\frac{15}{4}}h^2 + 117631768320 f_{n+\frac{13}{4}}h^2 \\ +118307846784 f_{n+\frac{11}{4}}h^2 + 430495597440 f_{n+\frac{9}{4}}h^2 \\ -15693683 f_{n+4}h^2 - 43074747720 f_{n+3}h^2 \\ -5262930000090 f_{n+2}h^2 - 253060374608 f_{n+\frac{5}{2}}h^2 \\ -15530795180000 y_{n+2} + 1163060999856 f_{n+\frac{3}{2}}h^2 \\ +9917900880 f_{n+\frac{5}{2}}h^2 - 917736832 f_{n+\frac{1}{4}}h^2 \\ +15630795180000 y_{+1} \end{pmatrix} = y'_{n+1} \tag{25}$$

$$\frac{1}{12804744114560000}\frac{1}{h}\begin{pmatrix} -3709188560264330 f_{n+\frac{7}{4}}h^2 - 10091882573850 f_{n+\frac{7}{2}}h^2 \\ -5666451438540858 f_{n+\frac{5}{4}}h^2 + 152404340161h^2 f_n \\ +1756501052807950 f_{n+1}h^2 - 219511415371970 f_{n+\frac{3}{4}}h^2 \\ +1210717786514 f_{n+\frac{15}{4}}h^2 + 53006859304130 f_{n+\frac{13}{4}}h^2 \\ +552825692398650 f_{n+\frac{11}{4}}h^2 + 2221907779776650 f_{n+\frac{9}{4}}h^2 \\ -68780796865 f_{n+4}h^2 - 197058638717710 f_{n+3}h^2 \\ -3994249820060160 f_{n+2}h^2 - 1222243458039442 f_{n+\frac{5}{2}}h^2 \\ +128047474114560000 y_{n+2} - 21606308257624430 f_{n+\frac{3}{2}}h^2 \\ +3075567115290 f_{n+\frac{1}{2}}h^2 - 3056750179730 f_{n+\frac{1}{4}}h^2 \\ -128047474114560000 y_{n+1} \end{pmatrix} = y'_{n+\frac{5}{4}} \tag{26}$$

$$-\frac{1}{25009272288000}\frac{1}{h}\begin{pmatrix} 23339386192080 f_{n+\frac{7}{4}}h^2 - 9932368170 f_{n+\frac{7}{2}}h^2 \\ -18510704470320 f_{n+\frac{5}{4}}h^2 + 87771695h^2 f_n \\ -767077329870 f_{n+1}h^2 - 52120196080 f_{n+\frac{3}{4}}h^2 \\ +1172481520 f_{n+\frac{15}{4}}h^2 + 53208411120 f_{n+\frac{13}{4}}h^2 \\ +586261410096 f_{n+\frac{11}{4}}h^2 + 2512334133040 f_{n+\frac{9}{4}}h^2 \\ -65719407 f_{n+4}h^2 - 202702496690 f_{n+3}h^2 \\ -2995560095040 f_{n+2}h^2 - 1344431404770 f_{n+\frac{5}{2}}h^2 \\ -25009272288000 y_{n+2} - 2621277447646 f_{n+\frac{3}{2}}h^2 \\ +13003109610 f_{n+\frac{1}{2}} - 1581981168 f_{n+\frac{1}{4}}h^2 \\ +250092722880000 y_{n+1} \end{pmatrix} = y'_{n+\frac{3}{2}} \quad (27)$$

$$\frac{1}{128047474114560000}\frac{1}{h}\begin{pmatrix} 9006594797047670 f_{n+\frac{7}{4}}h^2 - 5512382685210 f_{n+\frac{7}{2}}h^2 \\ +7479891752344518 f_{n+\frac{5}{4}}h^2 + 57490025409h^2 f_n \\ +882721885790990 f_{n+1}h^2 - 80379829451970 f_{n+\frac{3}{4}}h^2 \\ +646582416786 f_{n+\frac{15}{4}}h^2 + 29818273396930 f_{n+\frac{13}{4}}h^2 \\ +342763542222906 f_{n+\frac{11}{4}}h^2 + 1751609283320970 f_{n+\frac{9}{4}}h^2 \\ -36084723137 f_{n+4}h^2 - 115376379517710 f_{n+3}h^2 \\ -4216798172136960 f_{n+2}h^2 - 827832193224338 f_{n+\frac{5}{2}}h^2 \\ +128047474114560000 y_{n+2} + 17753689133629074 f_{n+\frac{3}{2}}h^2 \\ +11139494539290 f_{n+\frac{1}{2}}h^2 - 1128664355218 f_{n+\frac{1}{4}}h^2 \\ 128047474114560000 y_{n+1} \end{pmatrix} = y'_{n+\frac{7}{4}} \quad (28)$$

$$-\frac{1}{15630795180000}\frac{1}{h}\left(\begin{aligned}&-326854299200 f_{n+\frac{7}{4}}h^2 - 504164640 f_{n+\frac{7}{2}}h^2\\&-10255588644448 f_{n+\frac{5}{4}}h^2 + 1494511h^2 f_n\\&-74425272740 f_{n+1}h^2 + 1851106240 f_{n+\frac{3}{4}}h^2\\&+57065024 f_{n+\frac{15}{4}}h^2 + 2853650240 f_{n+\frac{13}{4}}h^2\\&+37890769728 f_{n+\frac{11}{4}}h^2 + 281213986880 f_{n+\frac{9}{4}}h^2\\&-3094021 f_{n+4}h^2 - 11724222700 f_{n+3}h^2\\&-1836468710730 f_{n+2}h^2 - 104238723712 f_{n+\frac{5}{2}}h^2\\&-15630795180000 y_{n+2} - 1818003399968 f_{n+\frac{3}{2}}h^2\\&+34526400 f_{n+\frac{1}{2}}h^2 - 19436864 f_{n+\frac{1}{4}}h^2\\&+15630795180000 y_{n+1}\end{aligned}\right) = y'_{n+2} \qquad (29)$$

$$\frac{1}{128047474114560000}\frac{1}{h}\left(\begin{aligned}&22719044155204470 f_{n+\frac{7}{4}}h^2 - 7292218077210 f_{n+\frac{7}{2}}h^2\\&+7748213489723334 f_{n+\frac{5}{4}}h^2 + 49187909057h^2 f_n\\&+821113046162190 f_{n+1}h^2 - 68359638481090 f_{n+\frac{3}{4}}h^2\\&+820414468498 f_{n+\frac{15}{4}}h^2 + 41838464367810 f_{n+\frac{13}{4}}h^2\\&+611085279601722 f_{n+\frac{11}{4}}h^2 + 15464058614177770 f_{n+\frac{9}{4}}h^2\\&-44386839489 f_{n+4}h^2 - 17698521946510 f_{n+3}h^2\\&+340305501528737280 f_{n+2}h^2 - 2006681656442514 f_{n+\frac{5}{2}}h^2\\&128047474114560000 y_{n+2} + 16574839670410898 f_{n+\frac{3}{2}}h^2\\&+9359659147290 f_{n+\frac{1}{2}}h^2 - 954832303506 f_{n+\frac{1}{4}}h^2\\&-128047474114560000 y_{n+1}\end{aligned}\right) = y'_{n+\frac{9}{4}} \qquad (30)$$

$$-\frac{1}{250092722880000}\frac{1}{h}\begin{pmatrix} -50306579130160 f_{n+\frac{7}{4}}h^2 - 20517321450 f_{n+\frac{7}{2}}h^2 \\ -16388950925616 f_{n+\frac{5}{4}}h^2 + 40127247h^2 f_n \\ -11756889430350 f_{n+1}h^2 + 22067692560 f_{n+\frac{3}{4}}h^2 \\ +2184031728 f_{n+\frac{15}{4}}h^2 + 127396299760 f_{n+\frac{13}{4}}h^2 \\ +2708014954800 f_{n+\frac{11}{4}}h^2 - 71133631189200 f_{n+\frac{9}{4}}h^2 \\ -113363855 f_{n+4}h^2 - 611314597170 f_{n+3}h^2 \\ -55771438648320 f_{n+2}h^2 - 28134596531234 f_{n+\frac{5}{2}}h^2 \\ -250092722880000 y_{n+2} - 29411442574110 f_{n+\frac{3}{2}}h^2 \\ +2418156330 f_{n+\frac{1}{2}}h^2 - 570430960 f_{n+\frac{1}{4}}h^2 \\ +250092722880000 y_{n+1} \end{pmatrix} = y'_{n+\frac{5}{2}} \quad (31)$$

$$\frac{1}{128047474114560000}\frac{1}{h}\begin{pmatrix} 2224874565874790 f_{n+\frac{7}{4}}h^2 - 26908690653210 f_{n+\frac{7}{2}}h^2 \\ +7538151339547590 f_{n+\frac{5}{4}}h^2 + 81883982785h^2 f_n \\ +902795305362190 f_{n+1}h^2 - 91548224388290 f_{n+\frac{3}{4}}h^2 \\ +2748500293010 f_{n+\frac{15}{4}}h^2 + 18097005287810 f_{n+\frac{13}{4}}h^2 \\ +13757428470487098 f_{n+\frac{11}{4}}h^2 + 28179841998789770 f_{n+\frac{9}{4}}h^2 \\ -139301154241 f_{n+4}h^2 - 105076438163470 f_{n+3}h^2 \\ +34082953176660480 f_{n+2}h^2 + 37353315734810990 f_{n+\frac{5}{2}}h^2 \\ +128047474114560000 y_{n+2} + 16969250935226002 f_{n+\frac{3}{2}}h^2 \\ 13939159035930 f_{n+\frac{1}{2}}h^2 - 1518967673234 f_{n+\frac{1}{4}}h^2 \\ -128047474114560000 y_{n+1} \end{pmatrix} = y'_{n+\frac{11}{4}} \quad (32)$$

$$-\frac{1}{15630795180000}\frac{1}{h}\left(\begin{array}{l}
-3417635909760 f_{n+\frac{7}{4}}h^2 - 10387539120 f_{n+\frac{7}{2}}h^2 \\
-1105975941504 f_{n+\frac{5}{4}}h^2 + 14094173h^2 f_n \\
-43074747720 f_{n+1}h^2 - 7058419840 f_{n+\frac{3}{4}}h^2 \\
+955364992 f_{n+\frac{15}{4}}h^2 + 87049560960 f_{n+\frac{13}{4}}h^2 \\
+4778827306368 f_{n+\frac{11}{4}}h^2 - 4700346412160 f_{n+\frac{9}{2}}h^2 \\
-45481539 f_{n+4}h^2 - 1571289854120 f_{n+3}h^2 \\
-3146644421370 f_{n+2}h^2 - 3085303123536 f_{n+\frac{5}{2}}h^2 \\
-15630795180000 y_{n+2} - 1669181749072 f_{n+\frac{3}{2}}h^2 \\
+1795724880 f_{n+\frac{1}{2}}h^2 - 236608896 f_{n+\frac{1}{4}}h^2 \\
+15630795180000 y_{n+1}
\end{array}\right) = y'_{n+3} \qquad (33)$$

$$\frac{1}{128047474114560000}\frac{1}{h}\left(\begin{array}{l}
14746901920140150 f_{n+\frac{7}{4}}h^2 - 460902906425370 f_{n+\frac{7}{2}}h^2 \\
5342677857101766 f_{n+\frac{5}{4}}h^2 + 396426456512h^2 f_n \\
+1720527626754830 f_{n+1}h^2 - 318909588699330 f_{n+\frac{3}{4}}h^2 \\
+34560575754642 f_{n+\frac{15}{4}}h^2 11991372374015170 f_{n+\frac{13}{4}}h^2 \\
+20734540579644474 f_{n+\frac{11}{4}}h^2 + 17755648974385290 f_{n+\frac{9}{4}}h^2 \\
-1500204870593 f_{n+4}h^2 + 41430281391889650 f_{n+3}h^2 \\
+4396785571100160 f_{n+2}h^2 + 45555356403417454 f_{n+\frac{5}{2}}h^2 \\
+128047474114560000 y_{n+2} + 21531040628969106 f_{n+\frac{3}{2}}h^2 \\
+58363174887450 f_{n+\frac{1}{2}}h^2 - 6961104041362 f_{n+\frac{1}{4}}h^2 \\
-128047474114560000 y_{n+1}
\end{array}\right) = y'_{n+\frac{13}{4}} \qquad (34)$$

$$-\frac{1}{250092722880000}\frac{1}{h}\begin{pmatrix} -115284997152560\, f_{n+\frac{7}{4}}h^2 - 21425942416170\, f_{n+\frac{7}{2}}h^2 \\ -35740545763632\, f_{n+\frac{5}{4}}h^2 + 2883521839h^2 f_n \\ +6080945070130\, f_{n+1}h^2 - 2007150621680\, f_{n+\frac{3}{4}}h^2 \\ +501440742896\, f_{n+\frac{15}{4}}h^2 - 87904075428880\, f_{n+\frac{13}{4}}h^2 \\ -125376442101456\, f_{n+\frac{11}{4}}h^2 - 159132108490960\, f_{n+\frac{9}{4}}h^2 \\ -17674408303\, f_{n+4}h^2 - 22320012494770\, f_{n+3}h^2 \\ +28922269498560\, f_{n+2}h^2 + 22687979125022\, f_{n+\frac{5}{2}}h^2 \\ -250092722880000 y_{n+2} + 10476720678946\, f_{n+\frac{3}{2}}h^2 \\ +400850820330\, f_{n+\frac{1}{2}}h^2 - 49586339312\, f_{n+\frac{1}{4}}h^2 \\ +250092722880000 y_{n+1} \end{pmatrix} = y'_{n+\frac{7}{2}} \quad (35)$$

$$\frac{1}{128047474114560000}\frac{1}{h}\begin{pmatrix} -173035916448264330\, f_{n+\frac{7}{4}}h^2 + 51159258069426150\, f_{n+\frac{7}{2}}h^2 \\ -51815129963340858\, f_{n+\frac{5}{4}}h^2 + 9073138740161h^2 f_n \\ +23340941404807950\, f_{n+1}h^2 - 6460959898337197 0\, f_{n+\frac{3}{4}}h^2 \\ +9739515501786514\, f_{n+\frac{15}{4}}h^2 - 10103362948695870\, f_{n+\frac{13}{4}}h^2 \\ -123539499142801350\, f_{n+\frac{11}{4}}h^2 - 233760739724223350\, f_{n+\frac{9}{4}}h^2 \\ -102907406396865\, f_{n+4}h^2 + 121066362065282290\, f_{n+3}h^2 \\ +285929306355939840\, f_{n+2}h^2 + 257270458634760558\, f_{n+\frac{5}{2}}h^2 \\ +128047474114560000 y_{n+2} + 138153036369575570\, f_{n+\frac{3}{2}}h^2 \\ +1261845951115290\, f_{n+\frac{1}{2}}h^2 - 155826116579730\, f_{n+\frac{1}{4}}h^2 \\ -128047474114560000 y_{n+1} \end{pmatrix} = y'_{n+\frac{15}{4}} \quad (36)$$

$$-\frac{1}{15630795180000}\frac{1}{h}\begin{pmatrix}-262184468941120 f_{n+\frac{7}{4}}h^2 + 9775579814880 f_{n+\frac{7}{2}}h^2 \\ -81473063507520 f_{n+\frac{5}{4}}h^2 + 12559028895h^2 f_n \\ +30585116321180 f_{n+1}h^2 - 8689946402880 f_{n+\frac{3}{4}}h^2 \\ -7953467129280 f_{n+\frac{15}{4}}h^2 - 44566907583680 f_{n+\frac{13}{4}}h^2 \\ -186880293783744 f_{n+\frac{11}{4}}h^2 - 340410672538560 f_{n+\frac{9}{4}}h^2 \\ -975351781877 f_{n+4}h^2 + 92816645064660 f_{n+3}h^2 \\ +325659178400310 f_{n+2}h^2 + 272862756017280 f_{n+\frac{5}{2}}h^2 \\ -15630795180000 y_{n+2} = 160832203966944 f_{n+\frac{3}{2}}h^2 \\ +1726743168960 f_{n+\frac{1}{2}}h^2 - 214598064448 f_{n+\frac{1}{4}}h^2 \\ +15630795180000 y_{n+1}\end{pmatrix} = y'_{n+4} \qquad (37)$$

Equations (6) – Equations (37) are combined together as an Integrators to solve second order delay differential equations directly.

3. Basic Properties of the Block Methods

Here the properties of the Block methods are presented in details

3.1 Order and Error Constant of the Block Method

Let the linear Operator defined on the method be $\varsigma[y(x);h]$, where

$$\Delta[y(x);h] = A^{(o)}Y_m^{(i)} - \sum_{i=0}^{k}\frac{jh}{i}y'_n - h^{(2-i)}[d_i f(y_n) + b_i F(y_m)], \qquad (38)$$

Expanding the form Y_m and $F(y_m)$ in Taylor Series and comparing coefficients of h, we obtained

$$\Delta[y(x);h] = C_0 y(x) + C_1 hy'(x) + \ldots \; C_p h^p y^p(x) + C_{p+1}h^{p+1}y^{p+1}(x) + C_{p+2}h^{p+2}y^{p+2}(x) + \ldots \qquad (39)$$

Theorem 1: The linear operator and the associated block method are said to be of order p if $C_0 = C_1 = \ldots C_p = C_{p+1} = 0, C_{p+2} \neq 0$ C_{p+2} is called the error constant. It implies that the local truncation error is given by

$$T_{n+k} = C_{p+2}h^{p+2}y^{p+2}(x) + O(h^{p+3}) \qquad (40)$$

Expanding the block method (38) in Taylor Series expansion and comparing the coefficients of h, the order of the block is of order $[17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17]^T$ with error constant

$$C_{p+2} = \begin{pmatrix} 7.34X10^{-15}, 1.84X10^{-14}, 2.92X10^{-14}, 4.00X10^{-14}, 5.08X10^{-14}, \\ 6.16X10^{-14}, 7.24X10^{-14}, 8.32X10^{-14}, 9.40X10^{-14}, 1.05X10^{-13}, \\ 1.16X10^{-13}, 1.26X10^{-13}, 1.37X10^{-13}, 1.48X10^{-13}, 1.59X10^{-13}, \\ 1.66X10^{-13} \end{pmatrix}^T \quad (41)$$

3.2 Consistency

Here, the developed method has been examined and found to have order p greater than one and it is also convergence Areo and Omole [15], Fatunla [14]. Hence, the method satisfies the necessary and sufficient conditions for consistency of a numerical method.

3.3 Stability Domain of the Block Methods

In line with the approach in Familua et al. [14], the figure below shows the stability domain of the Four –step twelve off step points block method.

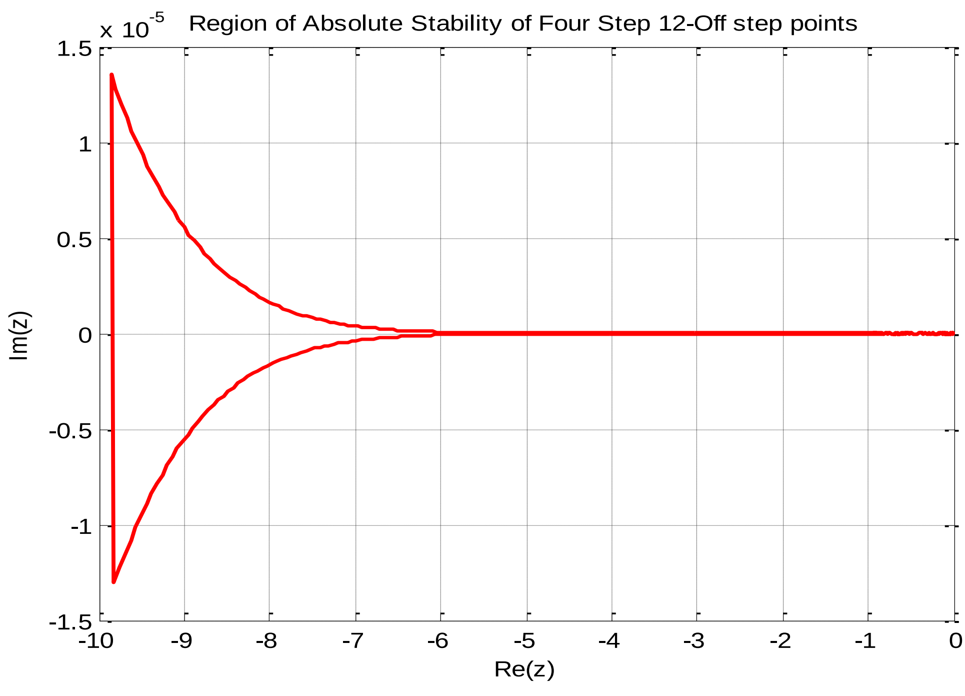

Figure 1: The Stability domain of Four-step twelve off-step points

4. Implementation of the Method

The methods were adopted on some delay differential equation of special second order to access the accuracy and efficiency of the methods.

4.1 Numerical Examples

The following problems were considered

Problem 1: Consider the linear delay equation

$$y''(x) = y(t-\tau) - \frac{1}{(1+x)^2} - In(1+x-\tau),\ x \ge x_0$$

$$y(x) = In(1+x),\ y'(x) = \frac{1}{(1+x)},\ x \ge x_0,\quad \tau = \frac{h}{10}$$

Exact solution: $y(x) = In(1+x)$

Problem 2: Consider the non-linear delay equation with single delay term

$$y''(x) = e^{-2\tau}\,\frac{y^2(x-\tau)}{y(x)},\ x \ge x_0$$

$$y(x) = e^{-x},\ y'(x) = -e^{-t},\ x \ge x_0,\quad \tau = \frac{h}{10}$$

Exact solution: $y(x) = e^{-x}$

Problem 3: Application Electrical Engineering problem namely, Matheiu's Equation.

In this section we apply our developed method to solve a well-known equation in engineering, the Matheiu's equation, which is defined as follows:

$$y''(t) + (\delta + a\cos t)y(t) + cy^3(t) = by(t-T) \tag{42}$$

Source: Morisson and Rand [16]

which is a nonlinear delay differential equation. where δ, a, b, c and T are parameters. δ is the frequency squared of the simple harmonic oscillator, and a is the amplitude of the parametric resonance, and b is the amplitude of delay which c is the amplitude of the cubic nonlinearity and T is the time delay. Equation (42) is a model for high speed milling, a kind of parametrically interrupted cutting as opposed to the self-interrupted cutting arising in an unstable turning process.

According to Morisson and Rand [16], various special cases of (42) have been studied, depending on which parameters is zero. when δ = a = b = 1 and c = 0 we obtained the following Linear Matheiu's equation:

$$y''(t) = (1+\cos t)y(t) = y(t-T);\ tE[0,10],\ y(t) = \sin(t),\ y'(t) = \cos(t),\ t < 0 \qquad (43)$$

where T = τ = h/10 is the delay term, the exact solution does not exist.

when δ = a = b = c = 1 we obtained the following Nonlinear Matheiu's equation:

$$y''(t) = (1+\cos t)y(t) + y^3(t) = y(t-T);\ y(t) = \sin(t),\ y'(t) = \cos(t),\ t < 0 \qquad (44)$$

where T = τ = h/10 is the delay term, the exact solution does not exist. Both the linear and nonlinear Mathieu equations are solved using the developed method and the results are presented in Table 3.

4.2 Table of Results

The results of the new methods and its comparison with other authors are displayed below.

Table 1: Showing results for Problem 1 using the proposed method

x	y-exact	y-computed	Error in new method	Error in [14]
0.1	0.095310179804324935	0.095310179804324852	8.32667268e-017	2.49800181e-16
0.2	0.182321556793954590	0.182321556793954650	5.55111512e-017	3.60822483e-16
0.3	0.262364264467491060	0.262364264467491120	5.55111512e-017	5.55111512e-16
0.4	0.336472236621212840	0.336472236621213010	0.00000000e+000	5.55111512e-16
0.5	0.405465108108164380	0.405465108108164500	0.00000000e+000	6.10622664e-16
0.6	0.470003629245735580	0.470003629245735630	1.11022302e-016	6.66133815e-16
0.7	0.530628251062170490	0.530628251062170490	1.11022302e-016	7.77156117e-16
0.8	0.587786664902119060	0.587786664902119280	0.00000000e+000	6.66133815e-16
0.9	0.641853886172394810	0.641853886172395030	0.00000000e+000	6.66133815e-16
1.0	0.693147180559945290	0.693147180559945620	1.11022302e-016	5.55111512e-16

Table 2: Showing results for Problem 2 using the proposed method (Non-Linear with single delay)

X	y-exact	y-computed	Error in new method	Error in [14]
0.1	0.904837418035959520	0.904837418035959740	2.22044605e-016	1.11022302e-16
0.2	0.818730753077981820	0.818730753077981040	7.77156117e-016	3.33066907e-16
0.3	0.740818220681717770	0.740818220681717100	6.66133815e-016	4.44089210e-16
0.4	0.670320046035639330	0.670320046035642440	3.10862447e-015	3.33066907e-16
0.5	0.606530659712633420	0.606530659712638750	5.32907052e-015	1.11022302e-16
0.6	0.548811636094026390	0.548811636094029940	3.44169138e-015	2.22044605e-16
0.7	0.496585303791409470	0.496585303791411690	2.16493490e-015	2.77555756e-16
0.8	0.449328964117221560	0.449328964117222230	6.66133815e-015	3.88578059e-16
0.9	0.406569659740599050	0.406569659740599720	6.10622664e-015	5.55111512e-17
1.0	0.367879441171442220	0.367879441171441780	5.55111512e-015	1.11022302e-16

Table 3: Showing results of application problem for Linear and Nonlinear using h=0.1

t	y-computed (Linear problem)	Time	y-computed (Nonlinear problem)	Time
0.1	0.066915931967836559	0.1249	0.106402490682701220	0.3252
0.2	0.119736074898535010	0.1538	0.213539379413318100	0.3790
0.3	0.159083017071007220	0.1806	0.321194476666256670	0.4668
0.4	0.179398173160589980	0.2056	0.430884668392530420	0.5242
0.5	0.197257104598293180	0.2099	0.560806687140919410	0.5265
0.6	0.194783177415669940	0.2119	0.696368628862776170	0.5340
0.7	0.172336315345482910	0.2126	0.836901935996008040	0.5349
0.8	0.126430711712440080	0.2134	0.987798359457115360	0.5358
0.9	0.077043234605272862	0.2143	1.154259182652009400	0.5374
1.0	0.004808573520858618	0.2150	1.335531260425935500	0.5462

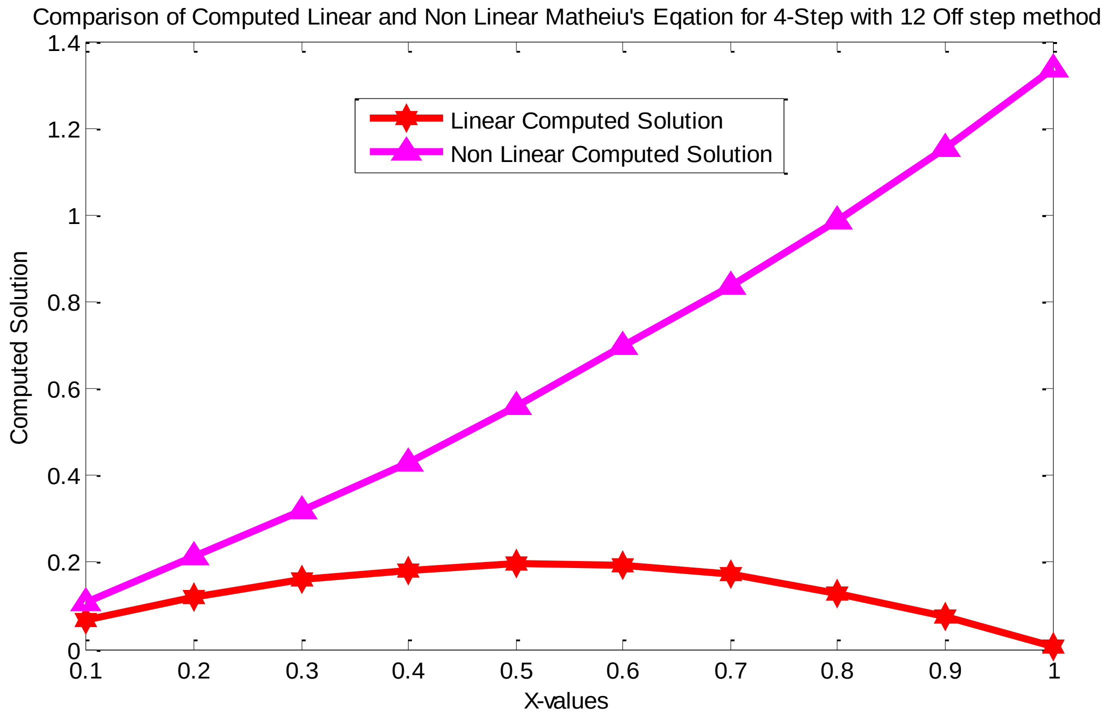

Figure 2: Showing the comparison of linear and nonlinear Mathieu's equation with h=0.1

4.3 Discussion of Results

The developed method of equations (6) - (37) were simultaneously implemented on three test problems. Examples 1 and 2 performed accurately with the exact solutions and also compete favourably with other author in the literature namely Familua et al. [14] as displayed in 'Table 1' and 'Table 2'. In example 3 which is an application problem. The application problem is linear and nonlinear problem in nature as the parameter changes. The Results are shown in 'Table 3'. 'Figure 1' shows the region in which the method is absolutely stable. 'Figure 2' shown comparisons of the linear and nonlinear application problems. The graph shows that the method performs better on linear problem than the nonlinear.

5. Conclusion

We want to conclude that this paper demonstrated a successful implementation of four-step twelve off-grid points for the solution of special second order DDEs. The method has great basic properties. The results were obtained in block forms which speed up the computational processes, less burden in the implementations and also increase the rate of convergence of the solutions. The results are displayed in 'Table 1-3'. Hence the method is recommended for solving second order DDEs.

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