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On a five and six parameters generalization of the Gamma function

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Abstract

The paper presents introduction to gamma function $\Gamma(k)$, generalization and extension of the gamma function. Further generalization of the extended gamma function with five parameters is proposed, and their applications in different fields are presented. This new generalized function open possibilities to represent mixture random phenomena in distribution theory. The extension of gamma function opens a new direction of research for statistical quality control.

Mathematics Subject Classification: 62P99
Keywords: Special function; Gamma function; Beta function; Generalized; Extended; Distribution theory

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1 Introduction to Gamma Function

In 1729, it was the birth of the gamma function in a correspondence between Leonhard Euler (1707-1783) and Christian Goldbach (1690-1764). Euler attempted to find a desirable continuous function, which provides the value of $k!$, for positive integers. Gamma function is a special transcendental function, which find many applications in mathematics, physics and engineering [1]. It is represented by the capital Greek letter $\Gamma(k)$, and its existence is due to fusion of several mathematical streams. There were two main streams that resulted in the invention of gamma function namely, interpolation theory and integral calculus [1-3]. The Gamma function is defined for a positive real part of a complex number (k) [4] as,

$$\Gamma(k) = \int_0^\infty x^{k-1}\; e^{-x}\; dx, \quad \mathrm{Re}\,(k) > 0 \tag{1}$$

On solving equation (1) with respect to x, we get

$$\begin{aligned}\Gamma(k) &= (k-1)\int_0^\infty x^{k-2} e^{-x}\, dx \\ &= (k-1)\;\Gamma(k-1)\end{aligned} \tag{2}$$

The equation (2) may also be written as,

$$\Gamma(k+1) = k\;\Gamma(k) \tag{3}$$

Equations (2 & 3) are same, and is known as recurrence property of the gamma function [4], which is true for all real numbers except the non positive integers. When subsitituted $k = 1, 2$ and 3 in equation(1), then $\Gamma(1) = \int_0^\infty e^{-x} dx \Rightarrow 1$, $\Gamma(2) = \int_0^\infty x\;\; e^{-x}\; dx \Rightarrow \int_0^\infty e^{-x}\; dx \Rightarrow 1$, and $\Gamma(3) = 2$ respectively. In the similar manner, the factorial equivalence may be proved as $\Gamma(k{+}1) = k\,!$

For an integer k and $0 < k < 1$,

$$\Gamma(k)\;\Gamma(1-k) \;=\; \frac{\pi}{\sin(\pi k)} \tag{4}$$

Proof. Using the Euler's infinite product [4], $\lim\limits_{m\to\infty} \frac{(m-1)!(m)^k}{k(k+1)(k+2)...(k+m-1)} = k! = \Gamma(k)$,

$$\begin{aligned}
\Gamma(k)\ \Gamma(1-k) &= \lim_{m\to\infty} \frac{(m-1)!\ (m)^k}{k\ (k+1)\ (k+2)...\ (k+m-1)} \lim_{m\to\infty} \frac{(m-1)!\ (m)^{1-k}}{(1-k)\ (2-k)\ (3-k)...\ (m-k)} \\
&= \lim_{m\to\infty} \frac{\left[(m-1)!\right]^2\ m}{k\ (1^2-k^2)\ (2^2-k^2)...\ ((m-1)^2-k^2)(m-k)} \\
&= \lim_{m\to\infty} \left[k\left(1-\frac{k^2}{1^2}\right)\left(1-\frac{k^2}{2^2}\right)...\left(1-\frac{k^2}{(m-1)^2}\right)\right]^{-1} \\
&= \left[k\prod_{m=1}^{\infty}\left(1-\frac{k^2}{m^2}\right)\right]^{-1} \\
&= \frac{\pi}{\sin(\pi k)}
\end{aligned}$$

This completes the proof. □

For an integer k and $k > 0$,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{5}$$

Proof. From equation (1),

$$\Gamma(k) = \int\limits_0^\infty x^{k-1}\ e^{-x}\ dx$$

Substituting $x = u^2$ and $dx = 2\ u\ du$ then

$$\Gamma(k) = \int\limits_0^\infty (u^2)^{k-1}\ e^{-u^2}\ 2\ u\ du \Rightarrow 2\int\limits_0^\infty u^{2k-1}\ e^{-u^2}\ du$$

A special value gamma may be derived, when $2k - 1 = 0$ i.e., $k = \frac{1}{2}$. On substituting $k = \frac{1}{2}$, we obtain

$$\Gamma(\frac{1}{2}) = 2\int\limits_0^\infty u^{2(\frac{1}{2})-1}\ e^{-u^2}\ du \Rightarrow 2\int\limits_0^\infty e^{-u^2}\ du \tag{6}$$

In order to solve equation (6), let us square it and then transforming into polar coordinates,

$$\left[\Gamma(\frac{1}{2})\right]^2 = \left[\Gamma(\frac{1}{2})\right]\left[\Gamma(\frac{1}{2})\right] \Rightarrow \left[2\int_0^\infty e^{-u^2}\ du\right]\left[2\int_0^\infty e^{-v^2}\ dv\right]$$
$$= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\ du\ dv \tag{7}$$

The bivariate transformation $u = r\cos\theta$, $v = r\sin\theta$ will transform the integral problem from cartesian coordinates to polar coordinates (r, θ), such that $0 \le r \le \infty$ and $0 \le \theta \le \frac{\pi}{2}$ for the first quadrant. The Jacobian of the transformation is given as,

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} \Rightarrow \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \Rightarrow r\cos^2 x + r\sin^2 x \Rightarrow r$$

Therefore, equation (7) can be written as,

$$= 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-(r^2)}\ r\ dr\ d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} d\theta \int_0^\infty e^{-(r^2)}\ r\ dr \tag{8}$$

Equation (8) may be solved by again substituting $u = -r^2$and $du = -2\ r\ dr$ then

$$\left[\Gamma(\frac{1}{2})\right]^2 = 4\left[\frac{\pi}{2}\right]\left[-\frac{1}{2}\int_0^{-\infty} e^u\ du\right] \Rightarrow \pi$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \pi$$
$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

This completes the proof. $\square$

The above result may also be obtained, when $k = \frac{1}{2}$ is substituted in equation (4). Further, when $k = \frac{3}{2}$, then using equation (2), it is obtained as $\Gamma(\frac{3}{2}) = \frac{1}{2}\ \sqrt{\pi}$.

Remark When $k = n + \frac{1}{2}$ then using equation (2),

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\ \Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\ \left(n-\frac{3}{2}\right)\ \Gamma\left(n-\frac{3}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\ \left(n-\frac{3}{2}\right)...\left(\frac{3}{2}\right)\ \left(\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}\right) \\
&= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)...\left(\frac{3}{2}\right)\ \left(\frac{1}{2}\right)\ \sqrt{\pi} \\
&= \left(\frac{(2n-1)(2n-3)...(3)(1)}{2^{\frac{n+1}{2}}}\right)\sqrt{\pi} \\
&= \frac{(2n-1)(2n-2)(2n-3)(2n-4)...(3)(2)(1)}{2^{n}(2n-2)(2n-4)...2}\sqrt{\pi} \\
&= \frac{(2n-1)!}{2^{n-1}2^{n}(n-1)(n-2)...1}\sqrt{\pi} \\
&= \frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi} \qquad (9)
\end{aligned}$$

2 Generalization of Gamma function

2.1 Three Parameters Generalization

The generalization of gamma function was first introduced by Kobayashi in 1991 [5-9]. He introduced the generalized gamma function as $\Gamma_r(k,n)$ with three parameters $r, k, n > 0$ as

$$\Gamma_r\ (k,n) = \int_0^\infty x^{k-1}\ \ (x+n)^{-r}\ \ e^{-x}\ dx, \quad \forall\ r,k,n > 0 \qquad (10)$$

Equation (10) finds many applications in diffraction theory and corrosion problems in machines [6]. As a particular case, this formula gives the original gamma function but in different form.

Case 1 When $r = 0$ then equation (10) becomes

$$\Gamma_0\ (k, n) = \int_0^\infty x^{k-1}\ e^{-x}\ dx \Rightarrow \Gamma(k) \tag{11}$$

Equation (11) may be treated as the original gamma function for $r = 0$.

Case 2 When $n = 0$ then equation (10) becomes

$$\Gamma_r\ (k, 0) = \int_0^\infty x^{(k-r)-1}\ e^{-x}\ dx \Rightarrow \Gamma(k - r) \tag{12}$$

Case 3 For $r \in \mathbb{Z}$ then equation (10) may be written as

$$\Gamma_r(k, n) = \begin{cases} \displaystyle\sum_{a=0}^{r} \frac{\Gamma(k+a-r)}{\binom{r}{a}\ n^a}, & r \in \mathbb{Z}^+ \\ \displaystyle\sum_{a=0}^{-r} \binom{-r}{a}\ n^a\ \Gamma(k - a - r), & r \in \mathbb{Z}^- \end{cases}$$

Proof. For $r \in \mathbb{Z}^+$,

$$
\begin{aligned}
\Gamma_r\ (k,n) &= \int_0^\infty x^{k-1}\ (x+n)^{-r}\ e^{-x}\ dx \\
&= \int_0^\infty \frac{x^{k-1}\ e^{-x}}{(x+n)^r} dx \\
&= \int_0^\infty \frac{x^{k-1}\ e^{-x}\ dx}{\sum_{a=0}^{r} \begin{pmatrix} r \\ a \end{pmatrix}\ n^a\ x^{r-a}} \\
&= \sum_{a=0}^{r} \begin{pmatrix} r \\ a \end{pmatrix}^{-1}\ n^{-a}\ \int_0^\infty x^{k-r+a-1}\ e^{-x}\ dx \\
&= \sum_{a=0}^{r} \begin{pmatrix} r \\ a \end{pmatrix}^{-1}\ n^{-a}\ \Gamma(k-r+a) \\
&= \sum_{a=0}^{r} \frac{\Gamma(k-r+a)}{\begin{pmatrix} r \\ a \end{pmatrix}\ n^a}
\end{aligned}
$$

$$
\therefore \Gamma_r\ (k,n) = \sum_{a=0}^{r} \frac{\Gamma(k-r+a)}{\begin{pmatrix} r \\ a \end{pmatrix}\ n^a} \qquad \forall\ n \in \mathbb{Z}^+ \tag{13}
$$

For $r \in \mathbb{Z}^-$,

$$
\begin{aligned}
\Gamma_r\ (k,n) &= \int_0^\infty x^{k-1}\ (x+n)^{-r}\ e^{-x}\ dx \\
&= \int_0^\infty \sum_{a=0}^{-r} \begin{pmatrix} -r \\ a \end{pmatrix}\ n^a\ x^{-r-a}\ x^{k-1}\ e^{-x}\ dx \\
&= \sum_{a=0}^{-r} \begin{pmatrix} -r \\ a \end{pmatrix}\ n^a\ \int_0^\infty x^{k-r-a-1}\ e^{-x}\ dx \\
&= \sum_{a=0}^{-r} \begin{pmatrix} -r \\ a \end{pmatrix}\ n^a\ \Gamma(k-r-a)
\end{aligned}
$$

$$\therefore \Gamma_r\ (k,n) = \sum_{a=0}^{-r} \begin{pmatrix} -r \\ a \end{pmatrix}\ n^a\ \Gamma(k-r-a) \tag{14}$$

This completes the proof. □

Remark Equations (13 & 14) proves the generalization of Kobayashi's integral converges, and these summation helps in representing mixture distributions in distribution theory. In addition, these equations also provides the scope for further generalization.

2.2 Four Parameters Generalization

In 1996, Agarwal and Kalla considered a modified form of the Kobayashi's gamma function [6-8] with four parameters $r, k, n, \lambda > 0$. The generalized gamma function with four parameters proposed by Agarwal and Kalla in integral form is as,

$$\Gamma_r\ (k,n,\lambda) = \int_0^\infty x^{k-1}\ (x+n)^{-r}\ e^{-\lambda x}\ dx, \quad r,k,n,\lambda > 0 \tag{15}$$

For parameters r, k, n and $\lambda > 0$, the generalization of Agarwal and Kalla is represented in terms of Kobayashi gamma function as

$$\Gamma_r\ (k,n,\lambda) = \lambda^{r-k}\ \Gamma_r(k,\lambda n) \tag{16}$$

Proof. On substituting $y = \lambda x$ in equation (15) then,

$$\begin{aligned}
\Gamma_r\ (k,n,\lambda) &= \int_0^\infty \left(\frac{y}{\lambda}\right)^{k-1}\ \left(\frac{y}{\lambda}+n\right)^{-r}\ e^{-y}\ \frac{dy}{\lambda} \\
&= (\lambda)^{r-k} \int_0^\infty (y)^{k-1}\,(y+\lambda n)^{-r}\, e^{-y}\ dy \\
&= (\lambda)^{r-k}\ \Gamma_r(k,n\lambda)
\end{aligned}$$

This completes the proof. □

Property 1

$$\lim_{r-k\to+\infty} \Gamma_r\ (k,n,\lambda) = \lim_{r-k\to+\infty} (\lambda)^{r-k}\ \Gamma_r(k,n\lambda) = \begin{cases} 0, & 0<\lambda<1 \\ \lim\limits_{r-k\to+\infty} \Gamma_r(k,n), & \lambda=1 \\ +\infty, & \lambda>1 \end{cases}$$

i.e. When $\lambda = 1$, then equation (15) becomes,

$$\Gamma_r\ (k,n,1) = \int_0^\infty x^{k-1}\ (x+n)^{-r}\ e^{-x}\ dx \tag{17}$$

The resulted equation is same as Kobayashi generalized equation (10).

Property 2 When $n = 0$ and $\lambda = 1$, then equation (15) becomes,

$$\Gamma_r\ (k,0,1) = \int_0^\infty x^{k-1}\ (x)^{-r}\ e^{-x}\ dx \Rightarrow \int_0^\infty x^{k-r-1}\ e^{-x}\ dx \Rightarrow \Gamma(k-r)$$

As a particular case of Agarwal and Kalla generalization, this shows that, it is possible to represent the gamma function with difference of two parameters k and r.

Property 3 When $k = 1$ and $\lambda = 1$, then equation (15) becomes,

$$\Gamma_r\ (1,n,1) = \int_0^\infty x^{1-1}\ (x+n)^{-r}\ e^{-x}\ dx = \begin{cases} \sum\limits_{a=0}^{r} \dfrac{\Gamma(1-r+a)}{\begin{pmatrix} r \\ a \end{pmatrix} n^a}, & r\in\mathbb{Z}^+ \\ \sum\limits_{a=0}^{-r} \begin{pmatrix} -r \\ a \end{pmatrix} n^a\ \Gamma(1-r-a), & r\in\mathbb{Z}^- \end{cases}$$

Proof. The proof is straightforward, and is similar to the proof mentioned for equations (13 & 14). □

Property 4 For all $k,n,r,\lambda > 0$ and $k > r$ the improper integral function is $0 \le \Gamma_r\ (k,n,\lambda) < +\infty$

2.3 Five Parameters Generalization

The authors proposes a new generalization with five parameters $r \in \mathbb{R}$, and $k, n, m, \lambda > 0$, which is given as

$$\Gamma_r(k,n,m,\lambda) = \int_0^{+\infty} x^{k-1} \ [n + x^m]^{-r} \ e^{-\lambda x} \ dx \tag{18}$$

In order to solve this equation, substituting $y = \lambda x$, then

$$\begin{aligned}\Gamma_r(k,n,m,\lambda) &= \int_0^{+\infty} \left(\frac{y}{\lambda}\right)^{k-1} \ \left[n + \left(\frac{y}{\lambda}\right)^m\right]^{-r} \ e^{-y} \ \frac{dy}{\lambda} \\ &= \lambda^{mr-k} \int_0^{+\infty} (y)^{k-1} \ [n\lambda^m + y^m]^{-r} \ e^{-y} \ dy \\ &= \lambda^{mr-k} \ \Gamma_r(k, n\lambda^m, m, 1)\end{aligned}$$

This new generalization helps in representing a five parameters gamma function into four parameter gamma function. The difference between the Agarwal & Kalla generalized gamma function and this new generalized function is the weighted factor of four parameters (λ^{mr-k}). This new representation opens the scope for its application in different fields. The authors are looking forward to work on this new generalization in future.

2.4 Six Parameters Generalization

In 2004, one of the author (Bachioua L.) introduced extended generalized gamma function with six parameters [9] r, k, p, m, n, and λ, which is given as

$$\Lambda_r(k,p,m,n,\lambda) = \int_0^{\infty} x^{k-1} \, [x^m + n]^{-r} \, e^{-\lambda x^p} dx; \quad r \in \mathbb{R}, \ k,p,m,n,\lambda > 0 \tag{19}$$

Case 1 When $p = 1$ and $m = 1$, equation (19) becomes Agarwal & Kalla generalized equation (15) as,

$$\Lambda_r(k,1,1,n,\lambda) = \int_0^{\infty} x^{k-1} \, [x + n]^{-r} \, e^{-\lambda x} dx$$

Case 2 When $p = 1$, $m = 1$ and $\lambda = 1$, equation (19) becomes Kobayashi generalized equation (10) as,

$$\begin{aligned}\Lambda_r(k,1,1,n,1) &= \int_0^\infty x^{k-1}\left[x+n\right]^{-r} e^{-x}dx \\ &= \Gamma_r(k,n), \quad r,k,n>0\end{aligned}$$

Case 3 Similarly, when $r = 0$, then the above equation becomes the basic gamma function as,

$$\begin{aligned}\Lambda_0(k,1,1,n,1) &= \int_0^\infty x^{k-1}\left[x+n\right]^{-0} e^{-x}dx \\ &= \int_0^\infty x^{k-1}e^{-x}dx \\ &= \Gamma(k), \quad k>0\end{aligned}$$

Remark The equation (18) is a particular case of $\Lambda_r(k,1,m,n,\lambda)$. Thus, this extended form of gamma function provides an open problem to different direction and maximises the intersection of gamma and beta functions. The six parameters gamma function may be further studied to find out other applications.

$$\Lambda_r(k,p,m,n,\lambda;a,b,c) = \int_0^\infty (x-a)^{k-1}\left[n+(x-b)^m\right]^{-r} e^{-\lambda(x-c)^p}dx;$$
$$r\in\mathbb{R},\ \ k,p,m,n,\lambda>0, a,b,c\in\mathbb{R}$$

One may use Matlab software to see different graphs of this function and may find the use of parameter a, b and c.

3 Applications

In past three decades, several new special functions and their applications have been discovered. Gamma function and its other generalized forms find applications in theoretical physics, porbability theory, hydrological studies and combinatorics. The gamma function is used in applied physics such as fluid dynamics, astrophysics, quantum physics and statistical mechanics. Its various integral and series representation provides a powerful calculation tool in variety of contexts such as modeling the intervals of earthquake sequence, evaluating infinite products and integrals of expressions involving exponential term [10]. The gamma function apart from having wide range of powerful applicatons, also provides a convenient representation of many other special functions, representing beta function in terms of gamma function depicts the same. It also plays an important role in solving various hypergeometric identities.

In reliablity theory, the gamma function and its generalized forms is used to study the displacement phenomenon of the corrosion problems. It also finds application is studying metal fatigue and internal corrosion of machines [11]. It is important to note that the extension of gamma function provides a powerful method for solving definite integrals, which are encountered in practical engineering applications. However, there exists few applications of the extended gamma function in expenditure estimation. In addition, the extended gamma functions are used in weather forecast, which may improve the quality of forecasts and predicting correlated data [12]. The extension of gamma function opens a new direction of reasearch for statistical quality control.

4 Conclusion

Gamma function, its generalization and extension upto six parameters is studied. In addition to portray the gamma function, its generalization & extension; a new generalization of the gamma function with five parameters k, n, r, m and λ is introduced. The authors have experimented with Matlab software, in order to develop a program for obtaining 3D plots of gamma function. About 200 plots of gamma function in 3 dimension are obtained, some of these 3D plots are mentioned in Figure 1. The five and six parameters gamma function may be further studied to find other applications. This new generalized functions open possibilities to represent mixture random phenomena in distribution theory, and also opens a new direction of reasearch for statistical quality control. In view of the applications of gamma function, mentioned above, the extended form of gamma function provides an open problem to different direction and maximises the intersection of gamma and beta functions.

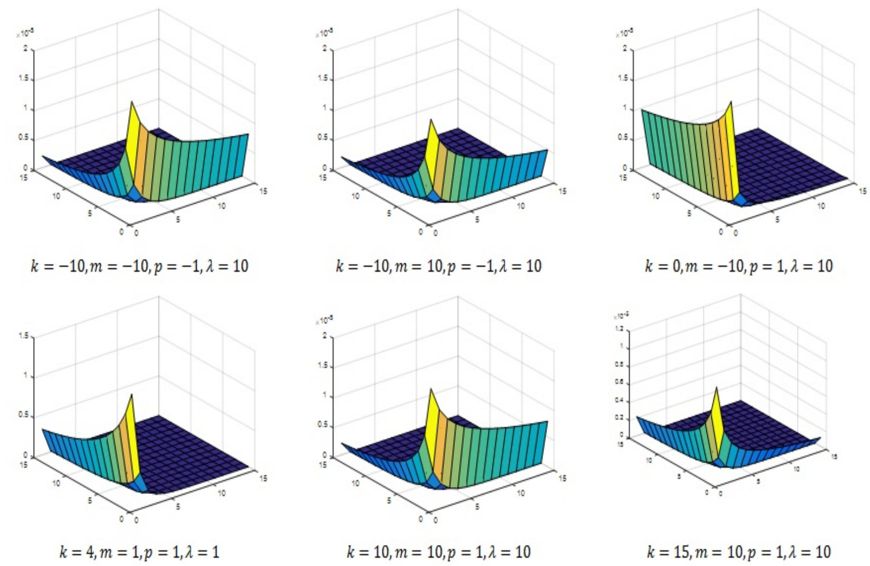

Figure 1: 3D Plots of gamma function with six parameters k, n, r, m, p and λ

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