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A New Hilbert-Type integral inequality with the Homogeneous Kernel of Degree -2 and with the Integral in Whole Plane

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Abstract

In this paper, by estimating the weight function, we give a new Hilbert-type inequality with the homogeneous kernel of degree -2 and with the integral in whole plane.As applications,we consider the equivalent form and some particular results.

Mathematics Subject Classification: 26D15
Keywords: Hilbert-type integral inequality, weight function, Hölder's inequality, equivalent form

1 Introduction

If $f(x), g(x) \geq 0,$ such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty,$ then [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \qquad (1.1)$$

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where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which has been extended by Hardy-Riesz:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q}, \quad (1.2)$$

where the constant factor $\dfrac{\pi}{\sin(\pi/p)}$ also is the best possible.

Hilbet's inequality is important in analysis and its applications. Recently, Yang and Xie gave (1.2) some strengthened versions [2-18].

Now, by obtaining the weight function, we give a new Hilbert-type inequality with the integral in whole plane.

In the following we suppose that $c > b > 0$, $ac > b^2$.

2 Some Lemmas

Lemma 2.1. *If* $k_1 := \int_0^\infty \frac{min\{t,1\}dt}{\max\{t,1\}(at^2+2bt+c)}$, $k_2 := \int_0^\infty \frac{min\{t,1\}dt}{\max\{t,1\}(at^2-2bt+c)}$, *then*

$$k_1 = \frac{1}{2a} \ln \frac{a+2b+c}{c} + \frac{1}{2c} \ln \frac{a+2b+c}{a} - \frac{b}{a\sqrt{ac-b^2}} \left(\frac{\pi}{2} - \arctan \frac{b}{\sqrt{ac-b^2}}\right)$$

$$k_2 = \frac{1}{2a} \ln \frac{a-2b+c}{c} + \frac{1}{2c} \ln \frac{a-2b+c}{a} + \frac{b}{a\sqrt{ac-b^2}} \left(\frac{\pi}{2} + \arctan \frac{b}{\sqrt{ac-b^2}}\right)$$

and

$$k := k_1 + k_2 = \frac{1}{2a} \ln \frac{(a+c)^2 - 4b^2}{c^2} + \frac{1}{2c} \ln \frac{(a+c)^2 - 4b^2}{a^2} +$$
$$+ \frac{b}{a\sqrt{ac-b^2}} \left(\arctan \frac{a-b}{\sqrt{ac-b^2}} - \arctan \frac{a+b}{\sqrt{ac-b^2}} + 2\arctan \frac{b}{\sqrt{ac-b^2}}\right)$$
$$+ \frac{b}{c\sqrt{ac-b^2}} \left(\arctan \frac{a+b}{\sqrt{ac-b^2}} - \arctan \frac{a-b}{\sqrt{ac-b^2}}\right).$$

Proof We have

$$k_1 = \int_0^\infty \frac{min\{t,1\}dt}{\max\{t,1\}(at^2+2bt+c)} = \int_0^1 \frac{tdt}{t^2+2bt+c} + \int_1^\infty \frac{dt}{t(at^2+2bt+c)}$$
$$= [\frac{1}{2a}\ln|at^2+2bt+c| - \frac{b}{a\sqrt{ac-b^2}}\arctan\frac{at+b}{\sqrt{ac-b^2}}]_0^1$$
$$+ [\frac{1}{2c}\ln\frac{t^2}{at^2+2bt+c} - \frac{b}{c\sqrt{ac-b^2}}\arctan\frac{at+b}{\sqrt{ac-b^2}}]_1^\infty$$
$$= \frac{1}{2a}\ln\frac{a+2b+c}{c} - \frac{b}{a\sqrt{ac-b^2}}(\arctan\frac{a+b}{\sqrt{ac-b^2}} - \arctan\frac{b}{\sqrt{ac-b^2}})$$
$$+ \frac{1}{2c}\ln\frac{a+2b+c}{a} - \frac{b}{c\sqrt{ac-b^2}}(\frac{\pi}{2} - \arctan\frac{a+b}{\sqrt{ac-b^2}})$$
$$= \frac{1}{2a}\ln\frac{a+2b+c}{c} + \frac{1}{2c}\ln\frac{a+2b+c}{a} - \frac{b}{a\sqrt{ac-b^2}}(\arctan\frac{a+b}{\sqrt{ac-b^2}}$$
$$- \arctan\frac{b}{\sqrt{ac-b^2}}) - \frac{b}{c\sqrt{ac-b^2}}(\frac{\pi}{2} - \arctan\frac{a+b}{\sqrt{ac-b^2}}).$$

Similarly,

$$k_2 = \frac{1}{2a}\ln\frac{a-2b+c}{c} + \frac{1}{2c}\ln\frac{a-2b+c}{a}$$
$$+ \frac{b}{a\sqrt{ac-b^2}}(\arctan\frac{a-b}{\sqrt{ac-b^2}} + \arctan\frac{b}{\sqrt{ac-b^2}})$$
$$+ \frac{b}{c\sqrt{ac-b^2}}(\frac{\pi}{2} - \arctan\frac{a-b}{\sqrt{ac-b^2}}).$$

And we have
$$k := k_1 + k_2 = \frac{1}{2a}\ln\frac{(a+c)^2-4b^2}{c^2} + \frac{1}{2c}\ln\frac{(a+c)^2-4b^2}{a^2} + \frac{b}{a\sqrt{ac-b^2}}(\arctan\frac{a-b}{\sqrt{ac-b^2}}$$
$$- \arctan\frac{a+b}{\sqrt{ac-b^2}} + 2\arctan\frac{b}{\sqrt{ac-b^2}}) + \frac{b}{c\sqrt{ac-b^2}}(\arctan\frac{a+b}{\sqrt{ac-b^2}} - \arctan\frac{a-b}{\sqrt{ac-b^2}}).$$
□

Lemma 2.2. *Define the weight functions as follow:*

$$w(x) := \int_{-\infty}^\infty \frac{\min\{|x|,|y|\}|x|dy}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)},$$

$$\widetilde{w}(y) := \int_{-\infty}^\infty \frac{\min\{|x|,|y|\}|y|dx}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}.$$

then $w(x) = \widetilde{w}(y) = k$.

Proof We only prove that $w(x) = k$ for $x \in (-\infty, 0)$.

$$w(x) = \int_{-\infty}^{0} \frac{\min\{|x|,|y|\}|x|dy}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)} + \int_{0}^{\infty} \frac{\min\{|x|,|y|\}|x|dy}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}$$

$$:= w_1 + w_2,$$

setting $y = tx$, then

$$w_1 = \int_{-\infty}^{0} \frac{\min\{(-x),(-y)\}(-x)dy}{\max\{(-x),(-y)\}(ay^2+2bxy+cx^2)} = \int_{0}^{\infty} \frac{\min\{1,t\}dt}{\max\{1,t\}(at^2+2bt+c)} = k_1.$$

similarly, setting $y = -tx$,

$$w_2 = \int_{0}^{\infty} \frac{\min\{(-x),y\}(-x)dy}{\max\{(-x),y\}(ay^2+2bxy+cx^2)} = \int_{0}^{\infty} \frac{\min\{1,t\}dt}{\max\{1,t\}(at^2-2bt+c^2)} = k_2,$$

and $w(x) = k$.

$$\widetilde{w}(y) = \int_{-\infty}^{\infty} \frac{\min\{|x|,|y|\}|y|dx}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}$$

$$= \int_{-\infty}^{0} \frac{\min\{|x|,|y|\}|y|dx}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)} + \int_{0}^{\infty} \frac{\min\{|x|,|y|\}|y|dx}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}$$

$$:= \widetilde{w}_1 + \widetilde{w}_2$$

for $y \in (-\infty, 0)$, setting $x = y/t$, then

$$\begin{aligned}\widetilde{w}_1 &= \int_{-\infty}^{0} \frac{\min\{-x,-y\}(-y)dx}{\max\{-x,-y\}(ay^2+2bxy+cx^2)} \\ &= \int_{-\infty}^{0} \frac{\min\{-y/t,-y\}(-y)d(y/t)}{\max\{-y/t,-y\}(ay^2+2b(y/t)y+c(y/t)^2)} \\ &= \int_{0}^{\infty} \frac{\min\{1,t\}dt}{\max\{1,t\}(at^2+2bt+c)} = k_1.\end{aligned}$$

similarly $\widetilde{w}_2 = k_2$, and $\widetilde{w}(x) = k$, using lemma 2.1, the lemma is proved. □

Lemma 2.3. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ \varepsilon > 0,$ *and* $\max\{\frac{2\varepsilon}{p}, \frac{2\varepsilon}{q}\}) \in (0,1),$ *define both functions,* $\widetilde{f}$ *and* $\widetilde{g}$*, as follow:*

$$\widetilde{f}(x) = \begin{cases} x^{-2\varepsilon/p}, & \textit{if } x \in (1,\infty), \\ 0, & \textit{if } \ x \in [-1,1], \\ (-x)^{-2\varepsilon/p}, & \textit{if } x \in (-\infty,-1); \end{cases}$$

$$\widetilde{g}(x)=\begin{cases} x^{-2\varepsilon/q}, & if\ x\in(1,\infty),\\ 0, & if \quad x\in[-1,1],\\ (-x)^{-2\varepsilon/q}, & if\ x\in(-\infty,-1),\end{cases}$$

then

$$I(\varepsilon):=\varepsilon\left\{\int_{-\infty}^{\infty}|x|^{-1}\widetilde{f}^{p}(x)dx\right\}^{1/p}\left\{\int_{-\infty}^{\infty}|x|^{-1}\widetilde{g}^{q}(x)dx\right\}^{1/q}=1;$$

$$\widetilde{I}(\varepsilon):=\varepsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}\widetilde{f}(x)\widetilde{g}(y)}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dxdy=k+o(1)\quad(\varepsilon\to 0^+).$$

Proof Easily,

$$I(\varepsilon)=2\varepsilon\left\{\int_{1}^{\infty}x^{-1}x^{-2\varepsilon}dx\right\}^{1/p}\left\{\int_{1}^{\infty}x^{-1}x^{-2\varepsilon}dx\right\}^{1/q}=1;$$

Let $y=-Y$,using $\widetilde{f}(-x)=\widetilde{f}(x)$, $\widetilde{g}(-x)=\widetilde{g}(x)$, and

$$\widetilde{f}(-x)\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}\widetilde{g}(y)dy}{\max\{|x|,|y|\}(ay^2-2bxy+cx^2)}$$

$$=\widetilde{f}(x)\int_{-\infty}^{\infty}\frac{\min\{|x|,|Y|\}\widetilde{g}(Y)dY}{\max\{|x|,|Y|\}(aY^2+2bxY+cx^2)}$$

we have that $\widetilde{f}(x)\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}\widetilde{g}(y)dy}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}$ is an even function on x,then

$$\begin{aligned}\widetilde{I}(\varepsilon)\ &=2\varepsilon\int_{0}^{\infty}\widetilde{f}(x)\left(\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}\widetilde{g}(y)}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dy\right)dx\\ &=2\varepsilon\left[\int_{1}^{\infty}x^{-\frac{2\varepsilon}{p}}\left(\int_{-\infty}^{-1}\frac{\min\{|x|,|y|\}(-y)^{-\frac{2\varepsilon}{q}}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dy\right)dx\right.\\ &\qquad\left.+\int_{1}^{\infty}x^{-\frac{2\varepsilon}{p}}\left(\int_{1}^{\infty}\frac{\min\{|x|,|y|\}y^{-\frac{2\varepsilon}{q}}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dy\right)dx\right]\\ &:=I_1+I_2.\end{aligned}$$

Setting $y = tx$ then

$$
\begin{aligned}
I_1 &= 2\varepsilon\left[\int_1^{\infty} x^{-\frac{2\varepsilon}{p}}\left(\int_1^{\infty}\frac{\min\{|x|,|y|\}y^{-\frac{2\varepsilon}{q}}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dy\right)dx\right]\\
&= 2\varepsilon\left[\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{\frac{1}{x}}^{\infty}\frac{\min\{1,t\}t^{-\frac{2\varepsilon}{q}}}{\max\{1,t\}(at^2+2bt+c)}dt\right)dx\right]\\
&= 2\varepsilon\left[\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_0^{\infty}\frac{t^{-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2bt+c)}dt\right)dx\right]\\
&\quad - 2\varepsilon\left[\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_0^{\frac{1}{x}}\frac{\min\{1,t\}t^{-\frac{2\varepsilon}{q}}}{\max\{1,t\}(at^2+2bt+c)}dt\right)dx\right]\\
&= 2\varepsilon\left[\int_0^{\infty} x^{-1-2\varepsilon}\left(\int_1^{\infty}\frac{\min\{1,t\}t^{-\frac{2\varepsilon}{q}}}{\max\{1,t\}(at^2+2bt+c)}dt\right)dx\right.\\
&\quad \left.-\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_0^{\frac{1}{x}}\frac{\min\{1,t\}t^{-\frac{2\varepsilon}{q}}}{\max\{1,t\}(at^2+2bt+c)}dt\right)dx\right]\\
&= \int_0^{\infty}\frac{\min\{1,t\}t^{-\frac{2\varepsilon}{q}}}{\max\{1,t\}(at^2+2bt+c)}dt\\
&\quad - 2\varepsilon\int_0^{1}\frac{\min\{1,t\}t^{-\frac{2\varepsilon}{q}}}{\max\{1,t\}(at^2+2bt+c)}\left(\int_1^{\frac{1}{t}} x^{-1-2\varepsilon}dx\right)dt\\
&= k_1+\int_0^{\infty}\frac{\min\{1,t\}(t^{-\frac{2\varepsilon}{q}}-1)}{\max\{1,t\}(at^2+2bt+c)}dt+\int_0^{1}\frac{\min\{1,t\}(t^{-\frac{2\varepsilon}{q}}-t^{\frac{2\varepsilon}{p}})}{\max\{1,t\}(at^2+2bt+c)}dt\\
&= k_1+\eta(\varepsilon).
\end{aligned}
$$

there $\lim_{\varepsilon\to 0^+}\eta(\varepsilon)=0$, and we have $I_1 \to k_1 \ (\varepsilon \to 0^+)$.

Similarly $I_2 \to k_2 \ (\varepsilon \to 0^+)$. The lemma is proved. □

Lemma 2.4. *If* $p > 1, \frac{1}{p}+\frac{1}{q}=1,$ *and* $0 < \int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx < \infty,$ *We have*

$$J := \int_{-\infty}^{\infty}|y|^{p-1}\left(\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}f(x)}{\max\{|x|,|y|\}(y^2+2bxy+cx^2)}dx\right)^p dy$$

$$\le k^p\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx. \quad (2.1)$$

Proof By lemma 2.2,we find

$$\Big(\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}f(x)}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dx\Big)^p$$
$$=\Big(\int_{-\infty}^{\infty}f(x)(\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)})^{1/p}$$
$$(\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)})^{1/q}dx\Big)^p$$
$$\leq\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}f^p(x)dx$$
$$\Big(\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dx\Big)^{p-1}$$
$$=k^{p-1}|y|^{-p+1}\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}f^p(x)dx,$$
$$J \;\leq\; k^{p-1}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}f^p(x)dx\Big]dy$$
$$=k^{p-1}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dy\Big]f^p(x)dx$$
$$=k^p\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx.$$

□

3 Main Results

Theorem 3.1. *If* $p>1, \frac{1}{p}+\frac{1}{q}=1,$ *and both functions,* $f(x)$ *and* $g(x)$*, are nonnegative measurable functions, and satisfy*
$0<\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx<\infty$ *and* $0<\int_{-\infty}^{\infty}|x|^{-1}g^q(x)dx<\infty$*. Then,*

$$I^*:=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}f(x)g(y)}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dxdy$$
$$<k\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-1}g^q(x)dx\right)^{1/q}, \tag{3.1}$$

and

$$J=\int_{-\infty}^{\infty}|y|^{-1}(\int_{-\infty}^{\infty}\frac{\min\{|x|,|y|\}f(x)}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)}dx)^pdy$$

$$< k^p \int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx. \tag{3.2}$$

Inequalities (3.1)and (3.2) are equivalent,and where the constant factors k and k^p are the best possible.

Proof If (2.1) takes the form of equality for some $y \in (-\infty, 0) \cup (0, \infty)$, then there exists constants M and N, such that they are not all zero, and $Mf^p(x) = N$ a.e. in $(-\infty, \infty)$. If $M \neq 0$, then $|x|^{-1} f^p(x) = \frac{N}{M|x|}$ a.e. in $(-\infty, \infty)$ which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx < \infty$. In the same way, we claim that $N = 0$. This is too a contradiction and hence by (2.1), we have (3.2).
By Hölder's inequality with weight and (3.2), we have,

$$\begin{aligned} I^* &= \int_{-\infty}^{\infty} \left[|y|^{\frac{1}{q}} \int_{-\infty}^{\infty} \frac{\min\{|x|, |y|\} f(x)}{\max\{|x|, |y|\}(ay^2 + 2bxy + cx^2)} dx \right] \left[|y|^{-\frac{1}{q}} g(y) \right] dy \\ &\leq (J)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{-1} g^q(y)dy \right)^{1/q}. \end{aligned} \tag{3.3}$$

Using (3.2), we have (3.1).
Setting

$$g(y) = (\int_{-\infty}^{\infty} \frac{\min\{|x|, |y|\} f(x)}{\max\{|x|, |y|\}(ay^2 + 2bxy + cx^2)} dx)^{p-1},$$

then $J = \int_{-\infty}^{\infty} |y|^{-1} g^q(y)dy$ by (2.7) we have $J < \infty$. If $J = 0$ then (3.2) is proved; if $0 < J < \infty$, by (3.1), we obtain

$$0 < \int_{-\infty}^{\infty} |y|^{-1} g^q(y)dy = J = I^*$$

$$< k \left(\int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|)^{-1} g^q(x)dx \right)^{1/q},$$

$$\left(\int_{-\infty}^{\infty} |x|^{-1} g^q(x)dx \right)^{1/p} = J^{1/p} < k \left(\int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx \right)^{1/p}.$$

Inequalities (3.1)and (3.2) are equivalent.

If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with $h < k$), such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\min\{|x|, |y|\} f(x)}{\max\{|x|, |y|\}(ay^2 + 2bxy + cx^2)} dxdy$$

$$< h\left(\int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{-1} g^q(x)dx\right)^{1/q}. \qquad (3.4)$$

For $\varepsilon > 0$, by (3.4), using lemma 2.3, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\min\{|x|,|y|\}\widetilde{f}(x)\widetilde{g}(y)}{\max\{|x|,|y|\}(ay^2+2bxy+cx^2)} dxdy =$$

$$k + o(1) < \varepsilon h \left(\int_{-\infty}^{\infty} |x|^{-1} \widetilde{f}^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{-1} \widetilde{g}^q(x)dx\right)^{1/q} = h.$$

Hence we find, $k + o(1) < h$. For $\varepsilon \to 0^+$, it follows that $k \le h$, which contradicts the fact that $h < k$. Hence the constant k in (3.1) is the best possible. Thus we complete the prove of the theorem. □

Remark 3.2.

1) For $a = c = 1$, $b = \cos\theta$, $0 \le \theta \le \pi$, then $k = 2\ln(2\sin\theta) + (\pi - 2\theta)\cot\theta$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\min\{|x|,|y|\}f(x)}{\max\{|x|,|y|\}(y^2+2xy\cos\theta+x^2)} dxdy$$

$$< (2\ln(2\sin\theta)+(\pi-2\theta)\cot\theta) \left(\int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{-1} g^q(x)dx\right)^{1/q}. \qquad (3.5)$$

2) For $\theta = \pi/3$, inequality (3.5) reduces to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\min\{|x|,|y|\}f(x)}{\max\{|x|,|y|\}(y^2+xy+x^2)} dxdy$$

$$< (\ln 3 + \frac{\pi}{3\sqrt{3}}) \left(\int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{-1} g^q(x)dx\right)^{1/q}. \qquad (3.6)$$

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