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Proof of the Collatz Conjecture

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Abstract

The Collatz conjecture (or 3n+1 problem) has been explored for about 86 years. In this article, we prove the Collatz conjecture. We will show that this conjecture holds for all positive integers by applying the Collatz inverse operation to the numbers that satisfy the rules of the Collatz conjecture. Finally, we will prove that there are no positive integers that do not satisfy this conjecture.

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1 Introduction

The Collatz conjecture is one of the unsolved problems in mathematics. Introduced by German mathematician Lothar Collatz in 1937 [1], it is also known as the 3n + 1 problem, 3x + 1 mapping, Ulam conjecture (Stanislaw Ulam), Kakutani's problem (Shizuo Kakutani), Thwaites conjecture (Sir Bryan Thwaites), Hasse's algorithm (Helmut Hasse), or Syracuse problem [2–4].

The Collatz conjecture or 3n+1 problem can be summarized as follows:
Take any positive integer n. If n is even, divide n by 2 (n/2). Otherwise, if n is odd, multiply n by 3 and add 1 (3.n+1). By repeatedly applying this rule of the conjecture to the chosen number n, we obtain a sequence. The next term in the sequence is found by applying arithmetic operations (n/2 or 3n+1) to the previous term according to the assumption rule. The conjecture states that no matter what number you start with, you will always reach 1 eventually.

For example, if we start with 17, multiply by 3 and add 1, we get 52. If we divide 52 by 2, 26, and so on, the rest of the sequence is: 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Or if we start 76, the sequence is: 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

This sequence of numbers involved is sometimes referred to as the hailstone sequence, hailstone numbers or hailstone numerals (because the values are usually subject to multiple descents and ascents like hailstones in a cloud) [2,6], or as wondrous numbers [2, 5].
In this paper, $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$, the symbol $\mathbb{N}$ represents the natural numbers. $\mathbb{N}^+ = \{1, 2, 3, 4, 5, 6, \ldots\}$, the symbol $\mathbb{N}^+$ represents the positive integers. $\mathbb{N}_{odd} = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$, the symbol $\mathbb{N}_{odd}$ represents the positive odd integers.

2 The Conjecture and Related Conversions

Definition 2.1 Let $n, k \in \mathbb{N}^+$ and a function $f : \mathbb{N}^+ \to \mathbb{N}^+$, Collatz defined the following map:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ 3n+1, & \text{if n is odd} \end{cases}$$

The Collatz conjecture states that the orbit formed by iterating the value of each positive integer in the function $f(n)$ will eventually reach 1. The orbit of n under f is $n; f(n), f(f(n)), f(f(f(n))), \ldots\ f^k(n) = 1\ (k \in \mathbb{N}^+)$.

In the following sections, we will call these two arithmetic operations ($n/2$ and $3n+1$), which we apply to any positive integer n according to the rule of assumption, Collatz operations (CO).

Remark 2.2 According to the definition of the Collatz conjecture, if the number we choose at the beginning is an even number, then by continuing to divide all even numbers by 2, one of the odd numbers is achieved. So it is sufficient to check whether all odd numbers reach 1 by the Collatz operations.

Therefore, if we prove that it reaches 1 when we apply the Collatz operations to all the elements of the set $\mathbb{N}_{odd}=\{1,3,5,7,9,11,13,15,\ldots\}$, we have proved it for all positive integers.

Remark 2.3 If the Collatz operations are applied to the numbers 2^n $(n \in \mathbb{N}^+)$, then eventually 1 is reached. If we can convert all the elements of the set $\mathbb{N}_{odd}$ into 2^n numbers by applying the Collatz operations, we get the result. Therefore, our goal is to convert all positive odd integers into 2^n numbers by applying Collatz operations.

2.1 Collatz Inverse Operation (CIO)

Let $n \in \mathbb{N}^+$and $a \in \mathbb{N}_{odd}$; for a to be converted to 2^n by the Collatz operation (CO), it must satisfy the following equation,

$$3.a+1=2^n$$

then,

$$a=\frac{2^n-1}{3} \tag{1}$$

Lemma 2.4 In (1) $a=\frac{2^n-1}{3}$, a cannot be an integer if n is a positive odd integer.

Proof. If n is a positive odd integer, we can take $n=2m+1$ $(m \in \mathbb{N})$, then substituting $2m+1$ for n in (1) we get,

$$a=\frac{2^{2m+1}-1}{3} \tag{2}$$

if we factor $2^{2m+1}+1$,

$2^{2m+1}+1=(2+1)(2^{2m}-2^{2m-1}+2^{2m-2}-\ldots+1)=\mathbf{3}.k$
we get $3k$, which is a multiple of 3 ($k\in\mathbb{N}_{odd}$).

Since $2^{2m+1}-1=(2^{2m+1}+1)-2=3.k-2$,

$(3k-2)$ is not a multiple of 3 ($k\in\mathbb{N}_{odd}$), so in (2) a is not an integer for any number n.

If we substitute $2n$ for n in (1), we get equation

$$a=\frac{2^{2n}-1}{3} \tag{3}$$

Lemma 2.5 In (3) $a=\frac{2^{2n}-1}{3}$, for each number n there is a different positive odd integer a, ($n\in\mathbb{N}^+$).

Proof. When we factorize $2^{2n}-1$ for all n numbers, ($n\in\mathbb{N}^+$), if

$$\begin{array}{ll}
n=1, & (2^2-1)=(2-1)(2+1)=\mathbf{3}.1\\
n=2, & (2^4-1)=(2-1)(2+1)(2^2+1)=\mathbf{3}.5\\
n=3, & (2^6-1)=(2^3-1)(2^3+1)=\mathbf{3}.\mathbf{3}.7\\
n=4, & (2^8-1)=(2-1)(2+1)(2^2+1)(2^4+1)=\mathbf{3}.(\ldots)\\
n=5, & (2^{10}-1)=(2-1)(2^4+\ldots+1)(2+1)(2^4-\ldots+1)=\mathbf{3}.(\ldots)\\
n=6, & (2^{12}-1)=(2^3-1)(2^3+1)(2^6+1)=\mathbf{3}.\mathbf{3}.(\ldots)\\
n=7, & (2^{14}-1)=(2^7-1)(2+1)(2^6-2^5+\ldots+1)=\mathbf{3}.(\ldots)\\
\vdots &
\end{array}$$

if we substitute $2n$ to 2^m, (m $\in\mathbb{N}^+$);

$$(2^{2^m}-1)=(2-1)(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1)\ldots(2^{2^{m-1}}+1)=\mathbf{3}.(\ldots)$$
$$\vdots$$
$$(2^{2n}-1)=(2^{x_1}-1)(2^{x_1}+1)(2^{x_2}+1)(2^{x_3}+1)\ldots.(2^{x_{n-1}}+1)(2^{x_n}+1) \text{ or}$$

$(2^{2n}-1)=(2^{x_1}-1)(2^{x_1}+1)$ in these equations, x_1 is a positive odd integer and x_2, x_3, x_4 $\ldots$ x_n are positive even integers. Since x_1 is a positive odd number,

$(2^{x_1}+1)=(2+1)(2^{x_1-1}-2^{x_1-2}+2^{x_1-3}-\ldots+1)=\mathbf{3}.(\ldots)$ so,
$(2^{2n}-1)=\mathbf{3}.(\ldots)$

Since each of these numbers has a multiplier of 3, we can find positive odd integers a for all n, and when we apply Collatz operations to these a numbers, we always get 1.
$2^{2n}+1$ is not a multiple of 3, since $2^{2n}-1$ is a multiple of 3, for all n ($n\in\mathbb{N}^+$).
In (3),

$$a = \frac{2^{2n}-1}{3};$$

$$\begin{aligned}
\text{If } n &= 1, & a_1 &= 1 \\
n &= 2, & a_2 &= 5 \\
n &= 3, & a_3 &= 21 = 3.7 \\
n &= 4, & a_4 &= 85 \\
n &= 5, & a_5 &= 341 \\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{array}{ccccccccccc}
 & 2^2 & 2^4 & 2^6 & 2^8 & 2^{10} & 2^{12} & 2^{14} & 2^{16} & 2^{18} & \ldots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\text{A}=\{ & 1, & 5, & 21, & 85, & 341, & 1365, & 5461, & 21845, & 87381 & \ldots \ \}
\end{array}$$

Corollary 2.6 We get a set A with infinite elements, these numbers reach 1 when we apply the Collatz operations. This is because when we apply the Collatz operations to these numbers, they become 2^{2n} numbers (Remark 2.3). (In the following sections, we will refer to the elements of the set A and other numbers that satisfy the Collatz conjecture as Collatz numbers).

Example 2.7 5 (odd number) $\rightarrow$ 16 $\rightarrow$8$\rightarrow$4$\rightarrow$2$\rightarrow$1.
21 (odd number) $\rightarrow$ 64$\rightarrow$32$\rightarrow$16$\rightarrow$8$\rightarrow$4$\rightarrow$2$\rightarrow$1.

If we can generalize the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ to all positive odd numbers, we have proved the Collatz conjecture.

2.2 Transformations in the Set A with Infinite Elements

Let the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ be $\{a_0, a_1, a_2, a_3, a_4,\ a_5, a_6, a_7, \ldots\}$ respectively.

Lemma 2.8 In the set A$\setminus\{a_0\}$, if $a_n \equiv 1 \pmod 3$

$$b_n = \frac{2^{2m}.a_n - 1}{3} \tag{4}$$

$m \in \mathbb{N}^+$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n.These numbers satisfy the conjecture.

Proof. If $a_n \equiv 1 \pmod 3$, we can take a_n as $3.p+1$, $(p \in \mathbb{N})$
$a_n = 3.p+1$ substituting in (4),

$$b_n = \frac{2^{2m}.(3.p+1)-1}{3} = \frac{2^{2m}3p+2^{2m}-1}{3} = 2^{2m}p + \frac{2^{2m}-1}{3}$$

$2^{2m}-1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.9 Let $a_1 = 85$, then $a_1 \equiv 1 \pmod 3$,in (4),

$$a_1 = 85 \to b_1 = \tfrac{2^2.85-1}{3} = 113,\ b_2 = \tfrac{2^4.85-1}{3} = 453,\ b_3 = \tfrac{2^6.85-1}{3} = 1813$$

$$b_4 = \tfrac{2^8.85-1}{3} = 7253,\ b_5 = \tfrac{2^{10}-1}{3} = 29013,\ b_6 = \tfrac{2^{12}.85-1}{3} = 116053$$

$$B = \{113, 453, 1813, 7253, 29013, 116053, \ldots\}$$

Lemma 2.10 In the set A$\setminus\{a_0\}$, if $a_n \equiv 2 \pmod 3$,

$$b_n = \frac{2^{2m-1}.a_n - 1}{3} \tag{5}$$

$m \in \mathbb{N}^+$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n. These numbers satisfy the conjecture.

Proof. If $a_n \equiv 2 \pmod 3$, we can take a_n as $3.p+2$ $(p \in \mathbb{N})$
$a_n = 3.p+2$ substituting in (5),

$$b_n = \frac{2^{2m-1}.(3p+2)-1}{3} = \frac{2^{2m-1}.3p+2^{2m}-1}{3} = 2^{2m-1}p + \frac{2^{2m}-1}{3}$$

$2^{2m}-1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.11 Let $a_1 = 5$, then $a_1 \equiv 2 \pmod 3$;

$$a_1 = 5 \to\ b_1 = \tfrac{2^1.5-1}{3} = 3, \quad b_2 = \tfrac{2^3.5-1}{3} = 13, \quad b_3 = \tfrac{2^5.5-1}{3} = 53$$

$$b_4 = \tfrac{2^7.5-1}{3} = 213, \quad b_5 = \tfrac{2^9.5-1}{3} = 853, \quad b_6 = \tfrac{2^{11}.5-1}{3} = 3413\ldots$$

$$B = \{3,\ 13,\ 53,\ 213,\ 853,\ 3413,\ 13653,\ 54613,\ \ldots\}$$

Lemma 2.12 In the set $A \backslash \{a_0\}$, if $a_n \equiv 0 \pmod 3$,

$$b_n = \frac{2^m.a_n - 1}{3} \tag{6}$$

$m \in \mathbb{N}^+$, there is no such integer b_n.

Proof . If $a_n \equiv 0 \pmod 3$, we can take a_n as $3.p$ ($p \in \mathbb{N}$)

$a_n = 3.p$ substituting in (6),

$$b_n = \frac{2^m(3.p) - 1}{3} = \frac{2^m 3.p - 1}{3} = 2^m.p - \frac{1}{3},$$

is not integer.
Similarly, using equations (4) and (5), from each b_n that is not a multiple of 3 we obtain C_n sets with infinite elements and from each c_n that is not a multiple of 3 we obtain D_n sets with infinite elements, ... and so on forever. In the following sections, we will call the operations of deriving new Collatz numbers from Collatz numbers by equations (3), (4) or (5) as Collatz inverse operations (CIO).

2.3 Conversion of all Positive Odd Integers to Collatz Numbers

In the previous sections, when we applied the Collatz operations, we called the numbers that reached 1 as Collatz numbers. Now let's see how all positive integers can be converted to these Collatz numbers.

$$\begin{array}{cccccccccccc} & 2^2 & 2^4 & 2^6 & 2^8 & 2^{10} & 2^{12} & 2^{14} & 2^{16} & 2^{18} & \dots & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ A = \{ & 1, & 5, & 21, & 85, & 341, & 1365, & 5461, & 21845, & 87381 & \dots & \} \end{array}$$

(Collatz Numbers)
If we apply the Collatz inverse operations [equations (4) or (5)] continuously to each Collatz number, we get infinitely many new Collatz numbers.

$\mathbb{N}_{odd} \rightarrow$ Set of A $\rightarrow 2^{2n} \rightarrow 1$ (Direction of conversion of numbers with CO).
$\mathbb{N}_{odd} \leftarrow$ Set of A $\leftarrow 2^{2n}$ (Direction of conversion of numbers with CIO).

Example 2.13 A small fraction of the Collatz numbers, that can be converted to 2^4, i.e. 1, by applying CO. These numbers are obtained by applying the CIO to each of the numbers. New numbers are obtained by repeatedly applying CIO to any number that is not a multiple of 3.

$$
\begin{array}{llllllllll}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
 & 46421 & 2389 & 4949 & 1077 & 34581 & 69173 & \dots & & \\
 & 11605 & 597 & 1237 & 269 & 8645 & 17293 & \dots & & \\
 & 2901 & 149 & 309 & 67 & 2161 & 4323 & \dots & & \\
2^4 & 725 & 37 & 77 & \uparrow & \uparrow & \uparrow & & & \\
\downarrow & 181, & 9, & 19 \rightarrow 25, & 101, & 405, & 1621, & 6485, & \dots & \\
5 & 45, & \uparrow & \uparrow & & & & & & \\
\downarrow & 11 \rightarrow & 7, & 29, & 117, & 469, & 1877, & 7509, & 30037, & \dots \\
3 & \uparrow & & & & & & & & \\
13 \rightarrow & 17, & 69, & 277, & 1109, & 4437, & 17749, & 70997, & \dots & \\
53 & & & \downarrow & \downarrow & & \downarrow & \downarrow & & \\
213 & & & 151 & 739 & & 23665 & 47331 & \dots & \\
853 & & & 605 & 2957 & & 94661 & 189325 & \dots & \\
3413 & & & 2421 & 11829 & & 378645 & 757301 & \dots & \\
13653 & & & 9685 & 47317 & & 1514581 & 3029205 & \dots & \\
54613 & & & 38741 & 189269 & & 6058325 & 12116821 & \dots & \\
218453 & & & 154965 & 757073 & & 24233331 & 48467285 & \dots & \\
\vdots & & & \vdots & \vdots & & \vdots & \vdots & &
\end{array}
$$

Example 2.14 Collatz numbers that can be converted to 2^6, i.e. 1, by applying CO.

$21 \leftarrow 2^6$ There are no other Collatz numbers. Because the resulting number is a multiple of 3. (Lemma 2.12).

Lemma 2.15 There is only one different Collatz number, which is converted to each of the numbers 2^{6n}; $(2^6, 2^{12}, 2^{18}, 2^{24} \dots)$ $(n \in \mathbb{N}^+)$.

Proof. If we factor 2^{6n} - 1,

$2^{6n} - 1 = (2^{3n} - 1)(2^{3n} + 1)$, there is always a multiplier of $(2^3 + 1)$ in this equation. Because when we factor $(2^{3n} - 1)$ and $(2^{3n} + 1)$;

if n is even, $(2^{3n} - 1) = \dots\ (2^{3f} + 1)$, $3f$ is odd integer.

if n is odd in $(2^{3n} + 1)$, $3n$ is odd integer.

And if $3f$ or $3n$ are odd,

$2^{3n} + 1 = (2^3 + 1)(2^{3n-3} - 2^{3n-6} + 2^{3n-9} - 2^{3n-12} + 2^{3n-15} \dots + 1)$

$2^{3f} + 1 = (2^3 + 1)(2^{3f-3} - 2^{3f-6} + 2^{3f-9} - 2^{3f-12} + 2^{3f-15} \dots + 1)$

Therefore $2^{6n} - 1 = (2^3 + 1).(\dots) = 9.$ (odd integer)

And, when we divide $(2^{6n} - 1)$ by 3, we get only one Collatz number. We can't obtain another Collatz number because it is a multiple of 3 (Lemma 2.12).

Example 2.16 There is only one different Collatz number for each of 2^{6n}, because the resulting Collatz numbers are the multiples of 3 (Lemma 2.12).

$$
\begin{array}{l}
21 \leftarrow 2^{6} \\
1365 \leftarrow 2^{12} \\
87381 \leftarrow 2^{18} \\
5592405 \leftarrow 2^{24} \\
\quad \vdots \qquad \vdots
\end{array}
$$

But, for $n, k \in \mathbb{N}^+$ and $n \neq 3k$, when all $2^{2n}-1$ numbers (except $2^{6n}-1$) are divided by 3, we get positive odd integers that are not multiples of 3. This is because when we factor $2^{2n}-1$, there is only one multiplier of 3.

$$
\begin{array}{l}
(2^{2n}-1) = (2^{x_1}-1)(2^{x_1}+1)(2^{x_2}+1)(2^{x_3}+1)\ldots.(2^{x_{n-1}}+1)(2^{x_n}+1) \text{ or} \\
(2^{2n}-1) = (2^{x_1}-1)(2^{x_1}+1)
\end{array}
$$

In these equations, x_1 is a positive odd integer and not a multiple of 3, and x_2, x_3, x_4, $\ldots$ x_n are positive even integers. $(2^{x_1}-1)$, $(2^{x_2}+1)$, $(2^{x_3}+1)$, $\ldots$ $(2^{x_{n-1}}+1)$ and $(2^{x_n}+1)$ do not have a multiplier of 3 (Lemma 2.4. and Lemma 2.5). And since x_1 is not a multiple of 3, $(2^{x_1}+1)$ has only one multiplier of 3, so an infinite number of Collatz numbers are converted to each of the numbers 2^{2n} with CO.

Example 2.17 A small fraction of the Collatz numbers that can be converted to 2^8, i.e. 1, by applying CO. These numbers are obtained by applying the CIO to each of the numbers. New numbers are obtained by repeatedly applying CIO to any number that is not a multiple of 3.

$$
\begin{array}{lllllllllll}
2^{8} & & & & & & & & & & \\
\downarrow & & & & & & & & & & \\
85 & & & & & & & & & & \\
\downarrow & & & \vdots & \vdots & & \vdots & \vdots & & & \dots \\
113 & \rightarrow & 75, & 301, & 1205, & 4821, & 19285, & 77141, & & 308565 & \dots \\
453 & & & \downarrow & & & & & & & \\
1813 & & & 401 & \rightarrow & 267, & 1069, & 4277, & & 17109 & \dots \\
7253 & & & 1605 & & & \downarrow & \downarrow & & & \\
29013 & & & 6421 & & & 1425 & 2851 & \rightarrow & 3801 & \dots \\
116053 & & & 25685 & & & 5701 & 11405 & \rightarrow & 7603 & \dots \\
464213 & & & 102741 & & & 22805 & 45621 & & \downarrow & \\
\vdots & & & \vdots & & & \vdots & \vdots & & \vdots & \dots
\end{array}
$$

Similarly, all positive odd integers are converted to 2^{2n} ($n \in \mathbb{N}^+$), i.e. to 1 by applying CO. All positive numbers are obtained by repeatedly applying the Collatz inverse operations to each element of the set A and the Collatz numbers generated from these numbers.

Lemma 2.18 If we apply the Collatz inverse operations $(\frac{2^m.a_n-1}{3})$ $(m \in \mathbb{N}^+)$ to the different Collatz numbers, we obtain new Collatz numbers that are all different from each other.

Proof. Let a_1 and a_2 be arbitrary Collatz numbers and $a_1 \neq a_2$, when we apply the Collatz inverse operations to each of them, the resulting numbers are b_1 and b_2. If $b_1 = b_2$ then,

$b_1 = \frac{2^m.a_1-1}{3} = \frac{2^t.a_2-1}{3} = b_2$ then $2^m.a_1 = 2^t.a_2$ for odd positive integers (a_1 and a_2), must be $a_1 = a_2$ and $m = t$ (contradiction), so if $a_1 \neq a_2$ then $b_1 \neq b_2$.

Lemma 2.19 If $a_n \equiv 0 \pmod 3$ and a_n numbers are odd Collatz numbers, we can derive a_n from other Collatz numbers.

Proof. If $a_n \equiv 0 \pmod 3$ and a_n numbers are odd Collatz numbers, then by applying CO to a_n we get odd positive integers b_n. Since a_n numbers are the Collatz numbers, b_n numbers are also the Collatz numbers and $b_n \not\equiv 0 \pmod 3$.

$a_n \rightarrow b_n$ (apply CO to a_n)
$a_n \leftarrow b_n$ (apply CIO to b_n)

since Collatz numbers cover b_n numbers, by applying the Collatz inverse operation to b_n we get a_n, $\frac{2^t.b_n-1}{3} = a_n$

Corollary 2.20 By applying the Collatz inverse operations [equation (3)] to the numbers 2^{2n}, we get a set A with infinite elements of positive odd numbers. The elements of the set A are the Collatz numbers. We get new Collatz numbers by applying Collatz inverse operations [equation (4) or (5)] to each element of this set A. From these new infinite Collatz numbers, infinitely many new Collatz numbers are formed by applying the Collatz inverse operations (CIO) again and again, and this goes on endlessly. So we get the whole set of positive odd numbers as Collatz numbers.

As a result, Collatz numbers fill the Hilbert's Hotel (David Hilbert) until there is no empty room left. The Hilbert Hotel is a thought experiment that has a countable infinity of rooms with room numbers 1, 2, 3, etc., and demonstrates the properties of infinite sets. In this hotel with an infinite number of guests, an infinite number of new guests (even finite layers of infinite) can be accommodated, provided that only one guest stays in each room [7]. When we fill the odd-numbered rooms of the Hilbert Hotel with Collatz numbers, we also fill the entire hotel with Collatz numbers. Let $n \in \mathbb{N}^+$ and $x, y \in \mathbb{N}_{odd}$, and let the odd-numbered rooms of the Hilbert Hotel be 1, 3, 5, 7, ..., i.e. elements of the set $\mathbb{N}_{odd}$. The result of the Collatz inverse operation is the following equation,

$$\frac{2^n.x-1}{3}=y \tag{7}$$

In equation (7), n depends on the values of x. If $x \equiv 1 \pmod 3$ we replace n with all even numbers $n=\{2,4,6,8,\ldots\}$, and if $x \equiv 2 \pmod 3$ we replace n with all odd numbers $n=\{1,3,5,7,\ldots\}$ respectively (Lemma 2.8 and Lemma 2.10). In (7) we obtain an infinite number of y as Collatz numbers starting from $x = 1$ (Lemma 2.5). Then, by substituting y values for x in (7), we obtain the sets of Collatz numbers with infinite elements for each y that is not a multiple of 3. (Although we cannot replace x with numbers that are multiples of 3, we get infinite numbers that are multiples of 3 in every set of Collatz numbers (Figure 1). Because, the numbers in each set give the remainder of 0,1,2 respectively according to (mod 3), as in the $\mathbb{N}_{odd}$ set). If the same process is repeated and the generated numbers are placed according to the room numbers, there will be no empty rooms left in the Hilbert Hotel. This is because infinite layers of infinite Collatz number sets are formed until all odd-numbered rooms are filled, i.e. until all odd numbers are obtained (Figure 1). By multiplying these numbers by 2^m ($m \in \mathbb{N}^+$), we find that all even numbers are Collatz numbers (Remark 2.2). Therefore, Collatz numbers fill the Hilbert Hotel and the set of Collatz numbers is equal to the set $\mathbb{N}^+$.
By taking x=1 in (7), we obtain infinite layers of infinite Collatz number sets (Figure 1).

$$\{1\}$$

$$\mathrm{Y}=1^*=\{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\} \qquad |\mathrm{Y}| = \aleph_0$$

$$\mathrm{Y}_1=1^*=\Big[\, 5^*=\{3,13,53,\ldots\}\ 85^*=\{113,453,1813,\ldots\}\ 341^*=\{227,909,3637,\ldots\}$$
$$5461^*=\{7281,29125,116501,\ldots\}\ \ldots\Big] \qquad |\mathrm{Y}_1| = \aleph_0+\aleph_0+\aleph_0 \ldots =\aleph_0.\aleph_0$$

$$\mathrm{Y}_2=1^*=\Big[\, 5^*=\{13^*=\{17,69,\ldots\}\ 53^*=\{35,141,\ldots\}\ldots\}\ 85^*=\{\ 113^*=\{75,301\ldots\}$$
$$1813^*=\{2417,9669\ldots\}\ \ldots\ \}\ \ldots\Big] \qquad |\mathrm{Y}_2| = \aleph_0.\aleph_0.\aleph_0$$

$$\mathrm{Y}_3=1^*=\Big[\, 5^*=\{\ 13^*=\ \{17^*=\{11,45,\ldots\}\ \ldots\}\ \ 53^*=\ \{35^*=\{23,93,\ldots\}\ \ldots\}$$
$$\ldots\}\ 85^*=\{113^*=\{301^*=\{401,1605,\ \ldots\}\ldots\}\ 1813^*=\{2417^*=\{1611,6445,\ldots\}\ldots\}$$
$$\ldots\}\ \ldots\Big] \qquad |\mathrm{Y}_3| = \aleph_0.\aleph_0.\aleph_0.\aleph_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad |\mathrm{Y}_n| = \aleph_0.\aleph_0.\aleph_0 \ldots$$

Figure 1: Collatz numbers. | | represents the cardinality of a set, and * represents conversions of numbers that are not multiples of 3 using equation (7).

In set theory, the cardinality of a set S represents the number of elements in the set, and is denoted by $|S|$. The aleph numbers ($\aleph$) indicate the cardinality (size) of well-ordered infinite element sets. $\aleph_0$ is the notation for the cardinality of the set of natural numbers, the next larger cardinality is $\aleph_1$, then $\aleph_2$ and so on. The cardinality of a set is $\aleph_0$ if and only if there is a one-to-one correspondence (bijection) between all elements of the set and all natural numbers. Since there is a one-to-one correspondence between the infinite sets in Figure 1 and the set of natural numbers, the cardinality of each set is $\aleph_0$ [8].

The cardinality of the set of real numbers is $\aleph_1$. The order and operations between the cardinality of the sets are as follows: $|\mathbb{N}|=\aleph_0$, $|\mathbb{R}|=\aleph_1$;

$\aleph_0<\aleph_1<\aleph_2<\ldots$

$\aleph_0+\aleph_0+\aleph_0+\ldots = \aleph_0.\aleph_0=\aleph_0$

$\aleph_0.\aleph_0.\aleph_0=\aleph_0$

$\aleph_0.\aleph_0.\aleph_0\ldots\ \aleph_0.\aleph_0 = \aleph_0^k = \aleph_0$ (k is a finite positive integer)

$\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{\aleph_0} = \aleph_1$

The elements of each set in Figure 1, obtained by converting each Collatz number, form a sequence such that the next term is 4 times the previous term plus 1. Thus, the elements of each set form a loop with remainders 0,1,2 according to $\pmod 3$. New sets are continuously formed from numbers with remainders 1 and 2 according to $\pmod 3$. Therefore, the cardinality of the sets of odd Collatz numbers in Figure 1 is donated as $\aleph_0^1$, $\aleph_0^2$, $\aleph_0^3$, $\aleph_0^4$,$\ldots$ $\aleph_0^k$, $\aleph_0^{k+1}\ldots$ $(k \in \mathbb{N}^+)$.

Infinite layers of infinite Collatz number sets are continuously formed from equation (7) without any restrictions, as shown in Figure 1. If there is no restriction, the cardinality of the set of odd Collatz numbers ($\aleph_0.\aleph_0.\aleph_0\ \ldots = \aleph_0^{\aleph_0} = \aleph_1$) is equal to the cardinality of the set of real numbers. This leads to a big contradiction. Because the cardinality of the set of Collatz numbers (Collatz numbers are positive integers) cannot be equal to the cardinality of the set of real numbers. To avoid this contradiction, the infinite layers of the set of Collatz numbers (continuous production of new Collatz numbers) must stop somewhere (Figure 1). This is possible if and only if the set of the odd Collatz numbers covers the $\mathbb{N}_{odd}$ set, i.e. is equal to it. Thus we find that the set of Collatz numbers is equal to the set $\mathbb{N}^+$ (Remark 2.2) and and we prove the Collatz conjecture for the set $\mathbb{N}^+$.

3 The Absence of any Positive Integer other than Collatz Numbers

In this section, we prove that there are no positive integers that do not satisfy the conjecture.

Lemma 3.1 There cannot be any positive integer other than Collatz numbers.

Proof. Let t_0 be a number that is not a Collatz number and ($t_0 \in \mathbb{N}_{odd}$), then when we apply Collatz inverse operations to t_0,

CIO $\rightarrow$ t_0 we get $T = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, \ldots\}$, and the elements of the set T are not Collatz numbers.
Also, when we apply the Collatz operation to t_0, until we find odd numbers;

$$t_0 \rightarrow \frac{3.t_0 + 1}{2^n}, \quad \mathrm{s}_1 \rightarrow \mathrm{s}_2 \rightarrow \mathrm{s}_3 \rightarrow \mathrm{s}_4 \rightarrow \mathrm{s}_5 \rightarrow \mathrm{s}_6 \rightarrow \mathrm{s}_7 \rightarrow \mathrm{s}_8 \rightarrow \mathrm{s}_9 \rightarrow \mathrm{s}_{10} \ldots$$

we get $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \ldots\}$ and the elements of the set S are not Collatz numbers ($s_n \in \mathbb{N}_{odd}$).

If t_0 is a multiple of 3, the CIO cannot be applied to t_0, so we take t_0 instead of the set T. If we apply the Collatz inverse operations to every number in the sets T and S, we get infinitely many new numbers that are not Collatz numbers (Figure 2 and Figure 3).

$$\begin{array}{llllllllllll}
t_0 & & & & & & & & & & & \\
\downarrow & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\
t_1 \rightarrow & t_{1(1)}, & t_{1(2)}, & t_{1(3)}, & t_{1(4)}, & t_{1(5)}, & t_{1(6)}, & t_{1(7)}, & t_{1(8)}, & t_{1(9)}, & t_{1(10)} & \ldots \\
t_2 & \downarrow & & & & & & & & & & \\
t_3 & t_{11(1)} \rightarrow & t_{111(1)}, & t_{111(2)}, & t_{111(3)}, & t_{111(4)}, & t_{111(5)}, & t_{111(6)}, & t_{111(7)}, & \ldots & & \\
t_4 & t_{11(2)} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\
t_5 & t_{11(3)} & t_{1111(1)} & t_{1112(1)} & t_{1113(1)} & t_{1114(1)} & t_{1115(1)} & t_{1116(1)} & t_{1117(1)} & \ldots & & \\
t_6 & t_{11(4)} & t_{1111(2)} & t_{1112(2)} & t_{1113(2)} & t_{1114(2)} & t_{1115(2)} & t_{1116(2)} & t_{1112(2)} & \ldots & & \\
t_7 & t_{11(5)} & t_{1111(3)} & t_{1112(3)} & t_{1113(3)} & t_{1114(3)} & t_{1115(3)} & t_{1116(3)} & t_{1112(3)} & \ldots & & \\
t_8 & t_{11(6)} & t_{1111(4)} & t_{1112(4)} & t_{1113(4)} & t_{1114(4)} & t_{1115(4)} & t_{1116(4)} & t_{1112(4)} & \ldots & & \\
t_9 & t_{11(7)} & t_{1111(5)} & t_{1112(5)} & t_{1113(5)} & t_{1114(5)} & t_{1115(5)} & t_{1116(5)} & t_{1112(5)} & \ldots & & \\
t_{10} & t_{11(8)} & t_{1111(6)} & t_{1112(6)} & t_{1113(6)} & t_{1114(6)} & t_{1115(6)} & t_{1116(6)} & t_{1112(6)} & \ldots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & &
\end{array}$$

Figure 2: Numbers obtained by applying CIO to t_1. New numbers are obtained by repeatedly applying CIO to any number that is not a multiple of 3.

$$
\begin{array}{lllllllllll}
\mathrm{s}_1 \rightarrow \mathrm{t}_0 & & & & & & & & & & \\
\downarrow & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \ldots \\
\mathrm{s}_{1(1)} & \rightarrow \mathrm{s}_{11(1)}, & \mathrm{s}_{11(2)}, & \mathrm{s}_{11(3)}, & \mathrm{s}_{11(4)}, & \mathrm{s}_{11(5)}, & \mathrm{s}_{11(6)}, & \mathrm{s}_{11(7)}, & \mathrm{s}_{11(8)}, & \mathrm{s}_{11(9)}, & \mathrm{s}_{11(10)} \cdots \\
\mathrm{s}_{1(2)} & \quad \downarrow & & & & & & & & & \\
\mathrm{s}_{1(3)} & \mathrm{s}_{111(1)} \rightarrow & \mathrm{s}_{1111(1)}, & \mathrm{s}_{1111(2)}, & \mathrm{s}_{1111(3)}, & \mathrm{s}_{1111(4)}, & \mathrm{s}_{1111(5)}, & \mathrm{s}_{1111(6)}, & \mathrm{s}_{1111(7)} \cdots & & \\
\mathrm{s}_{1(4)} & \mathrm{s}_{111(2)} \rightarrow & \mathrm{s}_{1112(1)}, & \mathrm{s}_{1112(2)}, & \mathrm{s}_{1112(3)}, & \mathrm{s}_{1111(4)}, & \mathrm{s}_{1111(5)}, & \mathrm{s}_{1111(6)}, & \mathrm{s}_{1111(7)} \cdots & & \\
\mathrm{s}_{1(5)} & \mathrm{s}_{111(3)} \rightarrow & \mathrm{s}_{1113(1)}, & \mathrm{s}_{1113(2)}, & \mathrm{s}_{1113(3)}, & \mathrm{s}_{1113(4)}, & \mathrm{s}_{1113(5)}, & \mathrm{s}_{1113(6)}, & \mathrm{s}_{1113(7)} \cdots & & \\
s_{1(6)} & s_{111(4)} \rightarrow & s_{1114(1)}, & s_{1114(2)}, & s_{1114(3)}, & s_{1114(4)}, & s_{1114(5)}, & s_{1114(6)}, & s_{1114(7)} \cdots & & \\
s_{1(7)} & s_{111(5)} \rightarrow & s_{1115(1)}, & s_{1115(2)}, & s_{1115(3)}, & s_{1115(4)}, & s_{1115(5)}, & s_{1115(6)}, & s_{1115(7)} \cdots & & \\
s_{1(8)} & s_{111(6)} \rightarrow & s_{1116(1)}, & s_{1116(2)}, & s_{1116(3)}, & s_{1116(4)}, & s_{1116(5)}, & s_{1116(6)}, & s_{1116(7)} \cdots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots &
\end{array}
$$

Figure 3: Numbers obtained by applying CIO to s_1. New numbers are obtained by repeatedly applying CIO to any number that is not a multiple of 3.

In the same way, an infinite number of new numbers are formed by applying CIO to each of ($t_2, t_3, t_4, t_5, \ldots$ and $s_2, s_3, s_4, s_5, \ldots$). We get infinite new numbers that are not Collatz numbers, repeated application of CIO to these numbers produces infinite new numbers, this result contradicts (Corollary 2.20).

Lemma 3.2 The elements of the set S do not loop with any element of the sets S or T.
Proof. We assume that such a loop exists.

$$
\begin{array}{llllllllll}
& t_0 \rightarrow s_1 \rightarrow s_2 \rightarrow & s_3 \rightarrow & s_4 \rightarrow & s_5 \rightarrow & s_6 \rightarrow & s_7 \rightarrow & s_8 \rightarrow s_9 & \\
& \uparrow & & & & & & \downarrow & \\
\text{CIO} \downarrow & \vdots & & & & & & \mathrm{s}_{10} \downarrow \text{CO} & \\
& \uparrow & & & & & & \downarrow & \\
& s_n \leftarrow \ldots\ldots\ldots\ldots & \leftarrow \ldots\ldots & \leftarrow \ldots\ldots\ldots\ldots & \leftarrow & \mathrm{s}_{12} \leftarrow \mathrm{s}_{11} & & &
\end{array}
$$

Figure 4

For such a loop to be exist (Figure 4), if we choose t_0, s_n or any other number as the starting and ending terms of the loop, which cannot be a number other than 1. If we choose s_n as the first and last terms of the loop, when CO is applied to all elements of the loop, they all turn into s_n, but when CO is applied to s_n, it cannot produce a number other than the loop numbers. In other words, while infinitely different numbers turn into s_n with CO, s_n cannot turn into a number different from those numbers. Such a restriction is only possible if s_n is 1. Then the other elements of the loop are also 1. For such a loop to be exist in positive odd integers, all the elements of the loop must be 1. By another method, all the elements of the loop must be equal, because the infinite set of numbers obtained by applying the CIO to each element of the loop is the same, that is, $\{t_0, t_1, t_2, \dots s_1, s_{11}, s_{12}, \dots s_2, s_{21}, s_{22}, \dots s_3, s_{31}, s_{31}, \dots s_n, s_{n1}, s_{n2}, \dots\}$. In the positive odd integers, only the number 1 can form a loop with itself, so all elements of the loop are 1.

Example 3.3 Lets take t_0 in the loop, $t_0 \not\equiv 0 \pmod 3$ and ($n, m \in \mathbb{N}^+$), then if $t_0{\rightarrow}^{CIO} = t_0{\rightarrow}^{CO}$,

$$\frac{2^n t_0 - 1}{3} = \frac{3t_0+1}{2^m} \qquad 2^{n+m}.t_0 - 2^m = 9t_0 + 3, \;\; t_0 = \frac{2^m+3}{2^{n+m}-9}$$

t_0 cannot be any positive odd integer other than 1 in this equation.
Since we assume that there is a number t_0 which is not a Collatz number, we obtain two sets (T and S) with infinite elements from this number. The elements of the sets T and S are not Collatz numbers. By repeatedly applying the Collatz inverse operations (CIO) to these numbers and the new numbers formed from them, an infinite number of new numbers are created, and this continues indefinitely. And they don't form a loop. From each of the infinite numbers in the sets T and S, new numbers are continuously generated without any limitation by repeated application of Collatz inverse operations.

Similar to the operations in Corollary 2.20, if there were a number t_0 that was not a Collatz number, it would fill the Hilbert Hotel until there was no room left. Because when we apply Collatz operations (CO) to t_0, we get an infinite set. When we repeat the Collatz inverse operations on the elements of this set as in Figure 1, we get infinite layers of infinite sets with no Collatz numbers, thus filling the odd-numbered rooms of the Hilbert hotel until there are no rooms left (Figure 5). Thus, the set of odd numbers that are not Collatz numbers covers the $\mathbb{N}_{odd}$ set, i.e. is equal to it.

$$\mathrm{Y}{=}\{t_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \dots\} \qquad |\mathrm{Y}| = \aleph_0$$

$$\mathrm{Y}_1 = \Big[\; t_0^* = \{t_1, t_2, t_3, \dots\} \; s_1^* = \{s_{11}, s_{12}, s_{13}, \dots\} \;\; s_2^* = \{s_{21}, s_{22}, s_{23}, \dots\} \; s_3^* {=} \{s_{31}, s_{32}, s_{33}, \dots\} \; \dots \Big] \qquad |\mathrm{Y}_1| = \aleph_0 + \aleph_0 + \aleph_0 \dots = \aleph_0.\aleph_0$$

$$\mathrm{Y}_2 = \Big[\; t_0^* {=} \{t_1^* {=} \{t_{11}, t_{12}, \dots\} \; t_2^* {=} \{t_{21}, t_{22} \dots\} \dots\} \; s_1^* = \{\; s_{11}^* {=} \{s_{111}, s_{112} \dots\} \; s_{12}^* = \{s_{121}, s_{122} \dots\} \; \dots\} \dots \Big] \qquad |\mathrm{Y}_2| = \aleph_0.\aleph_0.\aleph_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad |\mathrm{Y}_n| = \aleph_0.\aleph_0.\aleph_0 \dots$$

Figure 5: Numbers that are not Collatz numbers.| | represents cardinality of a set, and * represents conversions of numbers that are not multiples of 3 using equation (7).

The elements of each set in Figure 5, obtained by converting each number that is not a Collatz number, form a sequence such that the next term is 4 times the previous term plus 1. Thus, the elements of each set form a loop with remainders 0,1,2 according to (mod 3). New sets are continuously formed from numbers with remainders 1 and 2 according to (mod 3). Therefore, the cardinality of the sets of odd numbers that are not Collatz numbers in Figure 5 is donated as $\aleph_0^1$, $\aleph_0^2$, $\aleph_0^3$, $\aleph_0^4$,... $\aleph_0^k$, $\aleph_0^{k+1}$... ($k \in \mathbb{N}^+$).

In Figure 5, the cardinality of the set of odd numbers that are not Collatz numbers (positive integers) must not be equal to the cardinality of the set of real numbers ($\aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{\aleph_0} = \aleph_1$). This happens if and only if the set of odd numbers that are not Collatz numbers covers the $\mathbb{N}_{odd}$ set, i.e. is equal to it.Thus we find that the set of the numbers that are not Collatz numbers is equal to the set $\mathbb{N}^+$ (Remark 2.2). This leads to a contradiction with Corollary 2.20. Either all elements of the set $\mathbb{N}^+$ are Collatz numbers or none of them are. Therefore, all elements of the set $\mathbb{N}^+$ are Collatz numbers.

4 Conclusion

We proved the Collatz conjecture using the Collatz inverse operation method. It is shown that all positive integers reach 1, as stated in the Collatz conjecture. With the methods described in this study for $3n + 1$, it can be found whether numbers such as $5n + 1$, $7n + 1$, $9n + 1$, ... also reach 1.

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