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Proof of the Collatz Conjecture

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Abstract

The Collatz conjecture (or 3n+1 problem) has been explored for about 86 years. In this article, we prove the Collatz conjecture. We will show that this conjecture holds for all positive integers by applying the Collatz inverse operation to the numbers that satisfy the rules of the Collatz conjecture. Finally, we will prove that there are no positive integers that do not satisfy this conjecture.

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1 Introduction

The Collatz conjecture is one of the unsolved problems in mathematics. Introduced by German mathematician Lothar Collatz in 1937 [1], it is also known as the 3n + 1 problem, 3x + 1 mapping, Ulam conjecture (Stanislaw Ulam), Kakutani's problem (Shizuo Kakutani), Thwaites conjecture (Sir Bryan Thwaites), Hasse's algorithm (Helmut Hasse), or Syracuse problem [2–4].

In this paper, $\mathbb{N}$ $=\{0, 1, 2, 3, 4, 5, \ldots\}$, the symbol $\mathbb{N}$ represents the natural numbers. $\mathbb{N}^+ =\{1, 2, 3, 4, 5, 6, \ldots\}$, the symbol $\mathbb{N}^+$ represents the positive integers. $\mathbb{N}_{odd} =\{1, 3, 5, 7, 9, 11, 13, \ldots\}$, the symbol $\mathbb{N}_{odd}$ represents the positive odd integers.

2 The Conjecture and Related Conversions

Definition 2.1 Let $n, k \in \mathbb{N}^+$ and a function $f : \mathbb{N}^+ \to \mathbb{N}^+$, Collatz defined the following map:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ 3n+1, & \text{if n is odd} \end{cases}$$

The Collatz conjecture states that the orbit formed by iterating the value of each positive integer in the function $f(n)$ will eventually reach 1. The orbit of n under f is $n; f(n), f(f(n)), f(f(f(n))), \ldots\ f^k(n) = 1\ (k \in \mathbb{N}^+)$.
In the following sections, we will call these two arithmetic operations ($n/2$ and $3n + 1$), which we apply to any positive integer n according to the rule of assumption, Collatz operations (CO).

Remark 2.2 According to the definition of the Collatz conjecture, if the number we choose at the beginning is an even number, then by continuing to divide all even numbers by 2, one of the odd numbers is achieved. So it is sufficient to check whether all odd numbers reach 1 by the Collatz operations.

Therefore, if we prove that it reaches 1 when we apply the Collatz operations to all the elements of the set $\mathbb{N}_{odd} =\{1, 3, 5, 7, 9, 11, 13, 15, \ldots\}$, we have proved it for all positive integers.

Remark 2.3 If the Collatz operations are applied to the numbers 2^n $(n \in \mathbb{N}^+)$, then eventually 1 is reached. If we can convert all the elements of the set $\mathbb{N}_{odd}$ into 2^n numbers by applying the Collatz operations, we get the result.

2.1 Collatz Inverse Operation (CIO)

Let $n \in \mathbb{N}^+$and $a \in \mathbb{N}_{odd}$; for a to be converted to 2^n by the Collatz operation (CO), it must satisfy the following equation,

$$3.a + 1 = 2^n$$

then,

$$a = \frac{2^n - 1}{3} \tag{1}$$

Lemma 2.4 In (1) $a = \frac{2^n-1}{3}$, a cannot be an integer if n is a positive odd integer.
Proof. If n is a positive odd integer, we can take $n = 2m + 1$ ($m \in \mathbb{N}$), then substituting $2m + 1$ for n in (1) we get,

$$a = \frac{2^{2m+1} - 1}{3} \tag{2}$$

if we factor $2^{2m+1} + 1$,
$2^{2m+1} + 1 = (2 + 1)(2^{2m} - 2^{2m-1} + 2^{2m-2} - \ldots + 1) = \mathbf{3}.k$ ($k \in \mathbb{N}_{odd}$).
Since $(2^{2m+1} + 1)$ is a multiple of 3, $(2^{2m+1} - 1)$ is not a multiple of 3. So in (1) a is not an integer for any number n.

If we substitute $2n$ for n in (1), we get equation

$$a = \frac{2^{2n} - 1}{3} \tag{3}$$

Lemma 2.5 In (3) $a = \frac{2^{2n}-1}{3}$, for each number n there is a different positive odd integer a, ($n \in \mathbb{N}^+$).

Proof. When we factorize $2^{2n} - 1$ for $\forall n \in \mathbb{N}^+$,

$(2^{2n} - 1) = (2^{x_1} - 1)(2^{x_1} + 1)(2^{x_2} + 1)(2^{x_3} + 1)\ldots.(2^{x_{n-1}} + 1)(2^{x_n} + 1)$ or
$(2^{2n} - 1) = (2^{x_1} - 1)(2^{x_1} + 1)$ in these equations, x_1 is a positive odd integer and x_2, x_3, x_4 ... x_n are positive even integers. Since x_1 is a positive odd number,
$(2^{x_1} + 1) = (2 + 1)(2^{x_1-1} - 2^{x_1-2} + 2^{x_1-3} - \ldots + 1) = \mathbf{3}.(\ldots)$ so,
$(2^{2n} - 1) = \mathbf{3}.(\ldots)$

Since each of these numbers has a multiplier of 3, we can find positive odd integers a for all n, and when we apply Collatz operations to these a numbers, we always get 1. $2^{2n} + 1$ is not a multiple of 3, since $2^{2n} - 1$ is a multiple of 3, for $\forall n \in \mathbb{N}^{+}$. In (3), If we replace n with positive integers, we get the set A.

$$a = \frac{2^{2n} - 1}{3};$$

A= { 1,5,21,85,341,1365,5461,21845,87381,... } (Collatz Numbers)

If we can generalize the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ to all positive odd numbers, we have proved the Collatz conjecture.

2.2 Transformations in the Set A with Infinite Elements

Let the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ be $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots\}$ respectively.

Lemma 2.6 In the set A$\backslash\{a_0\}$, if $a_n \equiv 1 \pmod 3$

$$b_n = \frac{2^{2m}.a_n - 1}{3} \tag{4}$$

$m \in \mathbb{N}^{+}$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n.These numbers satisfy the conjecture.

Proof. If $a_n{\equiv}1 \pmod 3$, we can take a_n as $3.p + 1$, $(p \in \mathbb{N})$
$a_n = 3.p + 1$ substituting in (4),

$$b_n = \frac{2^{2m}.(3.p + 1) - 1}{3} = \frac{2^{2m}3p + 2^{2m} - 1}{3} = 2^{2m}p + \frac{2^{2m} - 1}{3}$$

$2^{2m} - 1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.7 Let $a_1 = 85$, then $a_1{\equiv}1 \pmod 3$,in (4),

$B = \{113, 453, 1813, 7253, 29013, 116053, \ldots\}$

Lemma 2.8 In the set A$\backslash\{a_0\}$, if $a_n \equiv 2 \pmod 3$,

$$b_n = \frac{2^{2m-1}.a_n - 1}{3} \tag{5}$$

$m \in \mathbb{N}^{+}$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n. These numbers satisfy the conjecture.

Proof. If $a_n \equiv 2 \pmod 3$, we can take a_n as $3.p + 2$ ($p \in \mathbb{N}$)

$a_n = 3.p + 2$ substituting in (5),

$$b_n = \frac{2^{2m-1}.(3p+2)-1}{3} = \frac{2^{2m-1}.3p + 2^{2m} - 1}{3} = 2^{2m-1}p + \frac{2^{2m}-1}{3}$$

$2^{2m} - 1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.9 Let $a_1 = 5$, then $a_1 \equiv 2 \pmod 3$;

$$B = \{3,\ 13,\ 53,\ 213,\ 853,\ 3413,\ 13653,\ 54613,\ \ldots\}$$

Lemma 2.10 In the set A$\setminus\{a_0\}$, if $a_n \equiv 0 \pmod 3$,

$$b_n = \frac{2^m.a_n - 1}{3} \tag{6}$$

$m \in \mathbb{N}^+$, there is no such integer b_n.

Proof . If $a_n \equiv 0 \pmod 3$, we can take a_n as $3.p$ ($p \in \mathbb{N}$)

$a_n = 3.p$ substituting in (6),

$$b_n = \frac{2^m(3.p)-1}{3} = \frac{2^m 3.p - 1}{3} = 2^m.p - \frac{1}{3},$$

is not integer.

In the following sections, we will call the operations of deriving new Collatz numbers from Collatz numbers by equations (3), (4) or (5) as Collatz inverse operations (CIO).

2.3 Conversion of all Positive Odd Integers to Collatz Numbers

In the previous sections, when we applied the Collatz operations, we called the numbers that reached 1 as Collatz numbers. Now let's see how all positive integers can be converted to these Collatz numbers.

A= { 1,5,21,85,341,1365,5461,21845,87381 ... } (Collatz Numbers)

If we apply the Collatz inverse operations [equations (4) or (5)] continuously to each Collatz number, we get infinitely many new Collatz numbers.

$\mathbb{N}_{odd} \rightarrow$ Set of A $\rightarrow 2^{2n} \rightarrow 1$ (Direction of conversion of numbers with CO).
$\mathbb{N}_{odd} \leftarrow$ Set of A $\leftarrow 1$ (Direction of conversion of numbers with CIO).

All positive numbers are obtained by repeatedly applying the Collatz inverse operations to each element of the set A and the Collatz numbers generated from these numbers.

Lemma 2.11 If we apply the Collatz inverse operations $(\frac{2^m.a_n-1}{3})$ $(m \in \mathbb{N}^+)$ to the different Collatz numbers, we obtain new Collatz numbers that are all different from each other.

Proof. Let a_1 and a_2 be arbitrary Collatz numbers and $a_1 \neq a_2$, when we apply the Collatz inverse operations to each of them, the resulting numbers are b_1 and b_2. If $b_1 = b_2$ then,

$b_1 = \frac{2^m.a_1-1}{3} = \frac{2^t.a_2-1}{3} = b_2$ then $2^m.a_1 = 2^t.a_2$ for odd positive integers (a_1 and a_2), must be $a_1 = a_2$ and $m = t$ (contradiction), so if $a_1 \neq a_2$ then $b_1 \neq b_2$.

Corollary 2.12 In set theory, the cardinality of a set S represents the number of elements in the set, and is denoted by $|S|$. The aleph numbers ($\aleph$) indicate the cardinality (size) of well-ordered infinite element sets. $\aleph_0$ is the notation for the cardinality of the set of natural numbers, the next larger cardinality is $\aleph_1$, then $\aleph_2$ and so on. The cardinality of a set is $\aleph_0$ if and only if there is a one-to-one correspondence (bijection) between all elements of the set and all natural numbers. Since there is a one-to-one correspondence between the infinite sets in Figure 1 and the set of natural numbers, the cardinality of each set is $\aleph_0$ [6].

The cardinality of the continuum is $2^{\aleph_0} = \aleph_1$. The order and operations between the cardinality of the sets are as follows: $|\mathbb{N}|$=$\aleph_0$, $\aleph_1$= cardinality of the "smallest" uncountably infinite sets;

$\aleph_0 < \aleph_1 < \aleph_2 < \ldots$
$\aleph_0 + \aleph_0 + \aleph_0 + \ldots = \aleph_0.\aleph_0 = \aleph_0$
$\aleph_0.\aleph_0.\aleph_0 = \aleph_0$
$\aleph_0.\aleph_0.\aleph_0 \ldots \aleph_0.\aleph_0 = \aleph_0^k = \aleph_0$ (k is a finite positive integer)
$\aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{\aleph_0}$

The elements of the set A (Lemma 2.5) are the Collatz numbers. We get new Collatz numbers by applying Collatz inverse operations [equation (4) or (5)] to each element of this set A. From these new infinite Collatz numbers, infinitely many new Collatz numbers are formed by applying the Collatz inverse operations (CIO) again and again, and this goes on endlessly.

As a result, Collatz numbers fill the Hilbert's Hotel (David Hilbert) until there is no empty room left. The Hilbert Hotel is a thought experiment that has a countable infinity of rooms with room numbers 1, 2, 3, etc., and demonstrates the properties of infinite sets. In this hotel with an infinite number of guests, an infinite number of new guests (even finite layers of infinite) can be accommodated, provided that only one guest stays in each room [5]. When we fill the odd-numbered rooms of the Hilbert Hotel with Collatz numbers, we also fill the entire hotel with Collatz numbers. Let $n \in \mathbb{N}^+$ and $x, y \in \mathbb{N}_{odd}$, and let the odd-numbered rooms of the Hilbert Hotel be 1, 3, 5, 7, ..., i.e. elements of the set $\mathbb{N}_{odd}$. The result of the Collatz inverse operation is the following equation,

$$\frac{2^n.x-1}{3} = y \tag{7}$$

In equation (7), n depends on the values of x. If $x \equiv 1 \pmod 3$ we replace n with all even numbers n =$\{2,4,6,8,\dots\}$, and if $x \equiv 2 \pmod 3$ we replace n with all odd numbers n =$\{1,3,5,7,\dots\}$ respectively (Lemma 2.6 and Lemma 2.8). In (7) we obtain an infinite number of y values as Collatz numbers starting from $x = 1$ (Lemma 2.5). Then, by substituting y values for x in (7), we obtain the Collatz number sets with infinite elements for each y that is not a multiple of 3. [Although we cannot replace x with numbers that are multiples of 3, we get infinite numbers that are multiples of 3 in each Collatz number sets (Figure 1). Because, the numbers in each set give the remainder of 0,1,2 respectively according to $\pmod 3$, as in the $\mathbb{N}_{odd}$ set]. If the same process is repeated and the generated numbers are placed according to the room numbers, there will be no empty rooms left in the Hilbert Hotel. This is because infinite layers of disjoint Collatz number sets (Collatz number set is a countably infinite set of positive odd integers) are formed without limit by equation (7), and these sets fill all odd-numbered rooms, i.e. we get all positive odd integers (Figure 1). By multiplying these numbers by 2^m ($m \in \mathbb{N}^+$), we find that all even numbers are Collatz numbers (Remark 2.2). Therefore, Collatz numbers fill the Hilbert Hotel and the set of Collatz numbers is equal to the set $\mathbb{N}^+$. Starting with $x = 1$ in (7) and continuing the process to infinity, we get infinite layers of disjoint Collatz number sets (Figure 1).

$$\{1\}$$

$$\mathrm{Y}_0 = 1^* = [\{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \dots\}] \quad |\mathrm{Y}_0| = 1$$

$$\mathrm{Y}_1 = 1^* = \Big[\, 5^* = \{3,13,53,\dots\}\ 85^* = \{113,453,1813,\dots\}\ 341^* = \{227,909,3637,\dots\}\ 5461^* = \{7281,29125,116501,\dots\}\ \dots \Big] \quad |\mathrm{Y}_1| = \aleph_0$$

$$\mathrm{Y}_2 = 1^* = \Big[\, 5^* = \{13^* = \{17,69,\dots\}\ 53^* = \{35,141,\dots\}\dots\}\ 85^* = \{\ 113^* = \{75,301\dots\}\ 1813^* = \{2417,9669\dots\}\ \dots\ \}\ \dots \Big] \quad |\mathrm{Y}_2| = \aleph_0 + \aleph_0 + \aleph_0\ \dots\ = \aleph_0.\aleph_0$$

Y_3= 1*= $\Big[$ 5*={ 13*= {17*={11,45,...} ...} 53*= {35*={23,93,...} ...} ...} 85*={113*={301*={401,1605, ...}...} 1813*={2417*={1611,6445,...}...} ...} ... $\Big]$ $|Y_3| = \aleph_0.\aleph_0.\aleph_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad |Y| = \aleph_0.\aleph_0.\aleph_0 \ldots$

The set of disjoint Collatz number sets:
Y= $\Big[$ {1,5,21,...}{3,13,53,...} {113,453,1813,...}{227,909,3637,...} {7281,29125, 116501,...} {17,69,277,...} {35,141,565,...} ... $\Big]$ $|Y| = \aleph_0.\aleph_0.\aleph_0 \ldots$

Figure 1: Collatz number sets. | | represents the cardinality of the set of Collatz number sets, and * represents conversions of numbers that are not multiples of 3 using equation (7).

In Figure 1, the infinite layers of Collatz number sets continue until they fill Hilbert's hotel, because they form continuously without any restriction; the restriction occurs only when the hotel is completely filled, i.e., when all positive odd numbers are obtained. Suppose buses with an infinite number of people on each arrive at the Hilbert Hotel. The following buses fill the Hilbert Hotel.

- $Y_0 = \{1, 5, 21, 85, 341, \ldots\}$ (infinite people, cardinality of buses: 1)
- $Y_1 = \aleph_0$ (1st layer, cardinality of buses: $\aleph_0$)
  - $\{(5,3),(5,13),(5,53),\ldots,$
    $(85,113),(85,453),\ldots,$
    $(341,227),(341,909),\ldots$
    $\ldots \quad \}$ (infinite buses each with infinite people)
- $Y_2 = \aleph_0 \cdot \aleph_0$ (2nd layer, cardinality of buses: $\aleph_0.\aleph_0$)
  - $\{(5,13,17),(5,13,69),\ldots,$
    $(85,113,75),(85,113,301),\ldots$
    $\ldots \quad \}$ (infinite ferries, each containing infinite buses, infinite people on each bus)
- $Y_3 = \aleph_0 \cdot \aleph_0 \cdot \aleph_0$ (3rd layer, cardinality of buses:$\aleph_0.\aleph_0.\aleph_0$)
  - $\{(5,13,17,11),(5,13,17,45),\ldots,$
    $(85,113,301,401),(85,113,301,1605),\ldots$
    $\ldots \quad \}$ (infinite oceans with infinite ferries on each, infinite buses on each ferry, infinite people on each bus)
- $Y = \aleph_0.\aleph_0.\aleph_0 \ldots$ (infinite layer, cardinality of buses: $\aleph_0.\aleph_0.\aleph_0 \ldots$)

Since there are different people on the buses, the buses represent disjoint Collatz number sets. The expression $\aleph_0.\aleph_0.\aleph_0\ldots$ is the cardinality of the set of disjoint Collatz number sets, it can also be thought of as the cardinality of sequences of disjoint Collatz number sets.As we move from each layer to the next layer, the cardinality of the set of disjoint Collatz number sets increases by a factor of $\aleph_0$, so $\aleph_0.\aleph_0.\aleph_0\ldots$ is the cardinality of the set of all disjoint Collatz sets.

The elements of each Collatz number set in Figure 1, obtained by converting each Collatz number, form a sequence such that the next term is 4 times the previous term plus 1. Thus, the elements of each Collatz number set form a loop with remainders 0,1,2 according to $\pmod 3$. New Collatz number sets are formed continuously to infinity from numbers with remainders 1 and 2 according to $\pmod 3$. Therefore, $\aleph_0^1$, $\aleph_0^2$, $\aleph_0^3$ exist (Figure 1), and for $\forall k \in \mathbb{N}^+$, if $\aleph_0^k$ exists, then $\aleph_0^{k+1}$ also exists. Thus, the cardinality of the set of disjoint Collatz number sets in Figure 1 is $\aleph_0.\aleph_0.\aleph_0\ldots$ ˙ Since all the elements of Collatz number sets form a cycle with remainders 0,1,2 with respect to $\pmod 3$, we get all positive odd numbers that are multiples of 3 from remainders 0 with respect to $\pmod 3$.

The elements of Collatz number sets obtained by equation 7 form a sequence in which each term is one more than four times the previous term. The same method is used to cover the set of positive odd integers. In the $\mathbb{N}_{odd}$ set, we create sets from each odd integer such that the next term is 1 more than 4 times the previous term.

$$\begin{aligned}
p_1 = &\quad \{\mathbf{1}, 5, 21, 85, \ldots\} \\
p_2 = &\quad \{\mathbf{3}, 13, 53, 213, \ldots\} \\
&\quad \{5, 21, 85, 341, \ldots\} \\
p_3 = &\quad \{\mathbf{7}, 29, 117, 469, \ldots\} \\
p_4 = &\quad \{\mathbf{9}, 37, 149, 597, \ldots\} \\
&\quad \vdots
\end{aligned}$$

The union of sets that are disjoint from sets of the form is equal to the set of positive odd integers. Since the other sets are subsets of the disjoint sets, we can ignore them.

$\mathbb{N}_{odd}$ =[p_1={1,5,21,85,...}p_2={3,13,53,213,...}p_3={7,29,117,469,...}p_4={9,37, 149,597,...} ...]= {p_1, p_2, p_3, p_4, p_5,...}
$\mathbb{N}_{odd}$= $\bigcup_{i=1}^{\infty} p_i$
Let $P = \{p_1, p_2, p_3, p_4 \ldots\}$, where P is an infinite subset of the set of natural

numbers, and $P \neq \mathbb{N}$. Suppose we list the disjoint sets in the $\mathbb{N}_{odd}$ set with the elements of the set P, assign an element of the set P to each disjoint set, i.e., equate each disjoint set with an element of the set P, and complete the list so that no elements in the $\mathbb{N}_{odd}$ and P sets are empty. Thus, the set of disjoint sets is equal to the set of P. The union of disjoint Collatz number sets obtained in Figure 1 is equal to the set $\mathbb{N}_{odd}$. This is because the cardinality of the set of disjoint Collatz number sets, by the inductive method described above, it was shown that $\aleph_0^1$, $\aleph_0^2$, $\aleph_0^3$ exist in Figure 1 and $\forall k \in \mathbb{N}^+$, if $\aleph_0^k$ exists, then $\aleph_0^{k+1}$ also exists. This result implies that we can list the disjoint Collatz number sets in the same way as in the set $\mathbb{N}_{odd}$, ensuring that no disjoint Collatz set or element of the P set is left out by assigning a unique element of the set P to each set. Therefore, we get exactly the same listing in the Collatz set as in the $\mathbb{N}_{odd}$ set.

Disjoint Collatz Number Sets (Figure 1):
Y= [$p_1 = \{1,5,21,\dots\}$ p_2=$\{3,13,53,\dots\}$ p_3=$\{113,453,1813,\dots\}$ p_4=$\{227,909, 3637,\dots\}$ p_5=$\{7281, 29125, 116501,\dots\}$ p_6=$\{17, 69, 277,\dots\}$ p_7=$\{35, 141, 565,\dots\}$ ...] = $\{p_1, p_2, p_3, p_4, p_5,\dots\}$
$\mathrm{Y} = \bigcup_{i=1}^{\infty} p_i$

The set Y is equal to the set of P, i.e., the $\mathbb{N}_{odd}$ set. The sets Y and $\mathbb{N}_{odd}$ are composed of the same disjoint sets and equal in numbers, i.e., they are equal sets. The number of disjoint Collatz sets cannot be less than the number of sets in the $\mathbb{N}_{odd}$ set because, as shown by induction, set formation is continuous and the number of sets cannot be greater because the Collatz numbers are elements of the $\mathbb{N}_{odd}$ set.

The cardinality of the set of disjoint sets in the $\mathbb{N}_{odd}$ set is $\aleph_0$. We had found that the cardinality of the set of disjoint Collatz number sets is $\aleph_0.\aleph_0.\aleph_0\dots$ (Figure 1).

$$\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots \geq \aleph_0$$

therefore, the set of Collatz numbers definitely covers the set of $\mathbb{N}_{odd}$ numbers, which is its universal set, but it cannot exceed it, because Collatz numbers are positive odd numbers. The Collatz number set covers the $\mathbb{N}_{odd}$ set, and since the $\mathbb{N}_{odd}$ set covers the Collatz number set, they are equal. Thus we find that the set of Collatz numbers is equal to the set $\mathbb{N}^+$ (Remark 2.2) and and we prove the Collatz conjecture for the set $\mathbb{N}^+$.

3 The Absence of any Positive Integer other than Collatz Numbers

In this section, we prove that there are no positive integers that do not satisfy the conjecture.

Let s_1 be a number that is not a Collatz number and ($s_1 \in \mathbb{N}_{odd}$), then when we apply Collatz operations to s_1, until we find odd numbers;

$$s_1 \rightarrow \ \frac{3.s_1\ +1}{2^n}, \quad \mathrm{s}_2 \rightarrow \mathrm{s}_3 \rightarrow \mathrm{s}_4 \rightarrow \mathrm{s}_5 \rightarrow \mathrm{s}_6 \rightarrow \mathrm{s}_7 \rightarrow \mathrm{s}_8 \rightarrow \mathrm{s}_9 \rightarrow \mathrm{s}_{10} \ldots$$

we get $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \ldots\}$ and the elements of the set S are not Collatz numbers ($s_n \in \mathbb{N}_{odd}$).

Lemma 3.1 The elements of the set S do not any loop.
Proof. We assume that such a loop exists.

$$\begin{array}{lccccccccccl}
 & s_1 \rightarrow & s_2 \rightarrow & s_3 \rightarrow & s_4 \rightarrow & s_5 \rightarrow & s_6 \rightarrow & s_7 \rightarrow & s_8 \rightarrow & s_9 \rightarrow & s_{10} & \\
 & \uparrow & & & & & & & & & \downarrow & \\
\text{CIO} \downarrow & \vdots & & & & & & & & & \mathrm{s}_{11} & \downarrow \text{CO} \\
 & \uparrow & & & & & & & & & \downarrow & \\
 & s_n \leftarrow & \ldots\ldots\ldots\ldots & \leftarrow & \ldots\ldots & \leftarrow & \ldots\ldots\ldots\ldots & \leftarrow & \mathrm{s}_{13} \leftarrow & & \mathrm{s}_{12} &
\end{array}$$

Figure 4

For such a loop to be exist in positive odd integers (Figure 4), all the elements of the loop must be equal, because the infinite set of numbers obtained by applying the CIO to each element of the loop is the same, that is, $\{s_1, s_{11}, s_{12}, \ldots\ s_2, s_{21}, s_{22}, \ldots\ s_3, s_{31}, s_{31}, \ldots\ s_n, s_{n1}, s_{n2}, \ldots\}$. In the positive odd integers, only the number 1 can form a loop with itself, so all elements of the loop are 1.
Lets take s_1 in the loop, $s_1 \not\equiv 0 \pmod 3$ and ($n, m \in \mathbb{N}^+$), then if $s_1 \rightarrow^{CIO} = s_1 \rightarrow^{CO}$,

$$\frac{2^n s_1 - 1}{3} = \frac{3s_1+1}{2^m} \qquad 2^{n+m}.s_1 - 2^m = 9s_1 + 3, \ \ s_1 = \frac{2^m+3}{2^{n+m}-9}$$

s_1 cannot be any positive odd integer other than 1 in this equation.

Similar to the operations in Corollary 2.12, if there were a positive odd number s_1 that was not a Collatz number, it would fill the Hilbert Hotel until there was no room left. Because when we apply Collatz operations (CO) to s_1, we get an infinite set S_0. If we repeat the Collatz inverse operations on the elements of this set as in Figure 1, we get infinite layers of sets with no Collatz numbers, these sets fill the odd-numbered rooms of the Hilbert Hotel and there are no rooms left (Figure 5). Thus, the set of odd numbers that are not Collatz numbers covers the $\mathbb{N}_{odd}$ set, i.e. is equal to it.

$\mathrm{S}_0 = \{\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \ldots\}\}$ $\quad |\mathrm{S}_0| = 1$

$\mathrm{S}_1 = \Big[\ s_1^* = \{s_{11}, s_{12}, s_{13}, \ldots\}\ s_2^* = \{s_{21}, s_{22}, s_{23}, \ldots\}\ \ s_3^* = \{s_{31}, s_{32}, s_{33}, \ldots\}$
$s_4^* = \{s_{41}, s_{42}, s_{43}, \ldots\}\ \ \ldots\ \Big]$ $\quad |\mathrm{S}_1| = \aleph_0$

$\mathrm{S}_2 = \Big[\ s_1^* = \{s_{11}^* = \{s_{111}, s_{112}, \ldots\}\ s_{12}^* = \{s_{121}, s_{122}, \ldots\}\ \ldots\}\ s_2^* = \{s_{21}^* = \{s_{211}, s_{212}, \ldots\}$
$s_{22}^* = \{s_{221}, s_{222}, \ldots\}\ \ldots\}\ \ldots\ \Big]$ $\quad |\mathrm{S}_2| = \aleph_0 + \aleph_0 + \aleph_0\ \ldots\ = \aleph_0.\aleph_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad |\mathrm{S}| = \aleph_0.\aleph_0.\aleph_0\ \ldots$

The set of disjoint Sets that are not Collatz number sets:
$\mathrm{S} = \Big[\ \{s_1, s_2, s_3, \ldots\}\{s_{11}, s_{12}, s_{13}, \ldots\}\ \{s_{21}, s_{22}, s_{23}, \ldots\}\{s_{31}, s_{32}, s_{33}, \ldots\}$
$\{s_{41}, s_{42}, s_{43}, \ldots\}\ \{s_{111}, s_{112}, \ldots\}\ \{s_{121}, s_{122}, \ldots\}\ \ldots\ \Big]$ $\quad |\mathrm{S}| = \aleph_0.\aleph_0.\aleph_0\ \ldots$

Figure 5: Sets that are not Collatz number sets.| | represents cardinality of the set of sets, and * represents conversions of numbers that are not multiples of 3 using equation (7).

The elements of each set in Figure 5, obtained by converting each number that is not a Collatz number, form a sequence such that the next term is 4 times the previous term plus 1. Thus, the elements of each set form a loop with remainders 0,1,2 according to (mod 3). New sets are formed continuously to infinity from numbers with remainders 1 and 2 according to (mod 3).
Therefore, the cardinality of the set of disjoint sets that are not Collatz number sets in Figure 5 is $\aleph_0.\aleph_0.\aleph_0\ldots$

In Figure 5, all disjoint sets are transformed into the set S_0 by Collatz operations, and the set S_0 is then transformed into a positive odd number, such as s_n. The set S is equal to the set $\mathbb{N}_{odd}$. The number of disjoint sets that are not Collatz sets cannot be less than the number of sets in the $\mathbb{N}_{odd}$ set because, as shown by induction, set formation is continuous and the number of sets cannot be greater because the numbers that are not Collatz numbers are elements of the $\mathbb{N}_{odd}$ set (Corollary 2.12). So the set of the numbers that are not Collatz numbers covers the $\mathbb{N}_{odd}$ set, and since $\mathbb{N}_{odd}$ set covers this set, they are equal. Thus we find that the set of the numbers that are not Collatz numbers is equal to the set $\mathbb{N}^+$ (Remark 2.2). This leads to a contradiction with Corollary 2.12. Either all elements of the set $\mathbb{N}^+$ are Collatz numbers or none of them are. Therefore, all elements of the set $\mathbb{N}^+$ are Collatz numbers.

4 Conclusion

We proved the Collatz conjecture using the Collatz inverse operation method. It is shown that all positive integers reach 1, as stated in the Collatz conjecture. With the methods described in this study for $3n+1$, it can be found whether numbers such as $5n+1$, $7n+1$, $9n+1$, ... also reach 1.

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