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Vector Groupoids

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Abstract

The main purpose of this paper is to study the vector groupoids. Several properties of them are established.

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1 Introduction

A groupoid is an algebraic structure introduced by H. Brandt (Math. Ann., 96 (1926), 360–366). A groupoid can be thought as a set with a partially defined multiplication, for which the usual properties of a group hold whenever they make sense. C. Ehresmann (Oèuvres complètes. Dunod, Paris, 1950) added further structures (topological and differentiable as well as algebraic) to groupoids. Groupoids and its generalizations (topological and Lie groupoids, sympectic groupoids etc.) are mathematical structures that have proved to be useful in many areas of science [topology ([2]), analysis ([10], [9]), geometry ([3],

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[12]), algebraic and geometric combinatorics ([7], [11]), dynamics of networks ([5])].

In this paper we define the concept of vector groupoid. This is an algebraic structure which combines the concepts of groupoid and vector space such that these are compatible. The motivation for defining the vector groupoid comes from the important properties possessed by a vector space on which it exists a compatible structure of groupoid. Specifically, the set of units of the groupoid is a vector subspace and the structure functions are linear maps. A vector groupoid has all the properties of the groupoids and those of the vector spaces, but it has its characteristic properties, too. The new concept of vector groupoid has applications in geometry and other areas.

In Section 2 we discuss the groupoids and useful properties of them are presented. The study of vector groupoids is realized in Section 3.

2 Groupoids

We recall the minimal necessary backgrounds on groupoids for our developments (see [1], [4], [6], [8] and references therein for more details).

Definition 2.1. ([3]) *A groupoid G over G_0 is a pair (G, G_0) of nonempty sets such that $G_0 \subseteq G$ endowed with two surjective maps $\alpha, \beta : G \to G_0$, a partially binary operation $m : G_{(2)} := \{(x,y) \in G \times G \mid \beta(x) = \alpha(y)\} \to G,\ (x,y) \longmapsto m(x,y) := x \cdot y$, where $G_{(2)}$ is the set of composable pairs and a map $i : G \to G,\ x \longmapsto i(x) := x^{-1}$, which verify the following conditions:*

(G1) (associativity): $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, when the products $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ are defined;

(G2) (units): for each $x \in G \Rightarrow (\alpha(x), x),\ (x, \beta(x)) \in G_{(2)}$ and we have $\alpha(x) \cdot x = x \cdot \beta(x) = x$;

(G3) (inverses): for each $x \in G \Rightarrow (x, x^{-1}),\ (x^{-1}, x) \in G_{(2)}$ and we have $x^{-1} \cdot x = \beta(x),\ \ x \cdot x^{-1} = \alpha(x)$.

A groupoid G over G_0 with the *structure functions* α (*source*), β (*target*), m (*multiplication*), i (*inversion*) is denoted by $(G, \alpha, \beta, m, i, G_0)$ or (G, G_0). G_0 is called the *unit set* of G. The map (α, β) defined by:

$$(\alpha, \beta) : G \to G_0 \times G_0, \quad (\alpha, \beta)(x) := (\alpha(x), \beta(x)),\ x \in G,$$

is called the *anchor map* of G. For each $u \in G_0$, the set $\alpha^{-1}(u)$ (resp. $\beta^{-1}(u)$) is called $\alpha-$*fibre* (resp. $\beta-$*fibre*) of G at $u \in G_0$.

A groupoid (G, G_0) is said to be *transitive*, if its anchor map is surjective.

We write sometimes xy for $m(x,y)$, if $(x,y) \in G_{(2)}$.

In the following proposition we summarize some basic rules of algebraic calculation in a groupoid obtained directly from definitions.

Proposition 2.2. *([6]) Let $(G, \alpha, \beta, m, i, G_0)$ be a groupoid. Then:*
(i) $\alpha(u) = \beta(u) = u, \quad u \cdot u = u$ *and* $i(u) = u, \ \forall u \in G_0$;
(ii) $\alpha\,(x \cdot y) = \alpha\,(x)$ *and* $\beta\,(x \cdot y) = \beta\,(y)\,, \ \forall\,(x,y) \in G_{(2)}$;
(iii) $\alpha\,(x^{-1}) = \beta\,(x)\,, \quad \beta\,(x^{-1}) = \alpha\,(x)$ *and* $(x^{-1})^{-1} = x, \ \forall x \in G$;
(iv) **(cancellation law)** *If* $(x, y_i), \, (y_i, z) \in G_{(2)}, i = 1, 2,$ *then:*
(a) $x \cdot y_1 = x \cdot y_2 \ \Rightarrow \ y_1 = y_2$; $\quad (b)$ $y_1 \cdot z = y_2 \cdot z \ \Rightarrow \ y_1 = y_2$.
(v) If $(x,y) \in G_{(2)}$, *then* $(y^{-1}, x^{-1}) \in G_{(2)}$ *and* $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

For any $u \in G_0$, the set

$$G(u) := \alpha^{-1}(u) \cap \beta^{-1}(u) = \{x \in G \mid \alpha(x) = \beta(x) = u \ \}$$

is a group under the restriction of the multiplication m to $G(u)$, called the *isotropy group at* u of G.

Proposition 2.3. ([6]) *Let* $(G, \alpha, \beta, m, i, G_0)$ *be a groupoid. Then:*
(i) $\alpha \circ i = \beta, \quad \beta \circ i = \alpha$ *and* $i \circ i = Id_G$.
(ii) $\varphi : G(\alpha(x)) \to G(\beta(x)), \ \varphi(z) := x^{-1}zx$ *is an isomorphism of groups.*
(iii) If (G, G_0) *is transitive, then all isotropy groups are isomorphic.*

Example 2.4. (i) Any group G having e as unity, is a groupoid over $G_0 = \{e\}$ with the structure functions α, β, m, i given by $\alpha(x) = \beta(x) = e, \ i(x) = x^{-1}$ for all $x \in G$ and $m(x,y) = xy$ for all $x, y \in G$.

(ii) Any set X can be endowed with a *null groupoid* structure. For this we take: $\alpha = \beta = i = Id_X$ and we define $x \cdot x = x, \ \forall x \in X$.

(iii) The Cartesian product $G := X \times X$ has a structure of groupoid over $\Delta_X = \{(x,x) \in X \times X \mid x \in X\}$ by taking the structure functions as follows:

$\widetilde{\alpha}(x,y) := (x,x)$, $\widetilde{\beta}(x,y) := (y,y)$; the elements (x,y) and (y',z) are composable in $G := X \times X$ iff $y' = y$ and we define $(x,y)\cdot(y,z) = (x,z)$ and the inverse of (x,y) is defined by $(x,y)^{-1} := (y,x)$. This is called the *pair groupoid.* Its unit set is $G_0 := \Delta_X$.

Example 2.5. The symmetry groupoid $\mathcal{SG}(X)$. Let $X \neq \emptyset$ be a set. Consider

$G := \mathcal{SG}(A,X) = \{f : A \to A \mid \emptyset \neq A \subseteq X,\ f \text{ is bijective }\}$ and

$G_0 := \{Id_A \mid \emptyset \neq A \subseteq X\}$, where Id_A is the identity map on A.

Let $G_{(2)} := \{(f,g) \in G\times G | D(f) = D(g)\}$, where $D(f)$ denotes the domain of f. The structure functions $\alpha, \beta : G \to G_0$, $i : G \to G$ and $m : G_{(2)} \to G$ are given by $\alpha(f) := Id_{D(f)}$, $\beta(f) := Id_{D(f)}$, $i(f) := f^{-1}$ and $m(f,g) := f\circ g$.

Then (G, G_0) is a groupoid, called the *groupoid of bijective functions from the subsets A of X onto A* or the *symmetry groupoid of the set X.*

The isotropy group $G(u)$ at $u = Id_A$ is the symmetry group of the set A.

In particular, the symmetry groupoid of $X = \{x_1, x_2, \ldots, x_n\}$, is called the *symmetry groupoid of degree n* and is denoted by $\mathcal{SG}_n$. Its unit set is $\mathcal{SG}_{n,0} = \{Id_A \mid \emptyset \neq A \subseteq X\}\}$. The cardinals of these finite sets are given by:

$$\mid \mathcal{SG}_n \mid \;=\; \sum_{k=1}^{n} k!\binom{n}{k}, \qquad \mid \mathcal{SG}_{n,0} \mid \;=\; 2^n - 1.$$

Definition 2.6. ([3]) *By morphism of groupoids or groupoid morphism between the groupoids $(G, \alpha, \beta, m, i, G_0)$ and $(G', \alpha', \beta', m', i', G'_0)$, we mean a map $f : G \to G'$ which verifies the following conditions:*

(i) $\forall\ (x,y) \in G_{(2)} \implies (f(x), f(y)) \in G'_{(2)}$;

(ii) $f(m(x,y)) = m'(f(x), f(y)),\ \forall\ (x,y) \in G_{(2)}$.

Proposition 2.7. *If $f : G \longrightarrow G'$ is a morphism of groupoids, then:*

(a) $f(u) \in G'_0, \quad \forall\ u \in G_0$; *(b)* $f\left(x^{-1}\right) = (f(x))^{-1},\ \forall\ x \in G$.

From Proposition 2.7(a) follows that a groupoid morphism $f : G \to G'$ induces a map $f_0 : G_0 \to G'_0$ taking $f_0(u) := f(u)$, $(\forall) u \in G_0$, i.e. the map f_0 is the restriction of f to G_0. We say that $(f, f_0) : (G, G_0) \to (G', G'_0)$ is a morphism of groupoids.

A groupoid morphism (f, f_0) is said to be *isomorphism of groupoids* or *groupoid isomorphism*, if f and f_0 are bijective maps.

Proposition 2.8. ([6]) *The pair* $(f, f_0) : (G, \alpha, \beta, G_0) \longrightarrow (G', \alpha', \beta', G'_0)$ *where* $f : G \longrightarrow G'$ *and* $f_0 : G_0 \longrightarrow G'_0$, *is a groupoid morphism if and only if the following conditions are verified:*

(i) $\alpha' \circ f = f_0 \circ \alpha$ *and* $\beta' \circ f = f_0 \circ \beta$;

(ii) $f\left(m\left(x, y\right)\right) = m'\left(f\left(x\right), f\left(y\right)\right), \quad \forall\ (x, y) \in G_{(2)}$.

Definition 2.9. ([6]) A groupoid morphism $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ satisfying the following condition:

$$\forall\ x, y \in G \ \text{ such that } \ (f(x), f(y)) \in G'_{(2)} \quad \Rightarrow \quad (x, y) \in G_{(2)} \tag{1}$$

will be called *homomorphism of groupoids.*

Example 2.10. Let be the symmetry groupoid $\mathcal{SG}_n$ and the multiplicative group $\{+1, -1\}$ (regarded as groupoid over $\{+1\}$). We define the map

$$sgn^\sharp : \mathcal{SG}_n \to \{+1, -1\},\ f \in \mathcal{SG}_n \longmapsto sgn^\sharp(f) := sgn(f),$$

where $sgn(f)$ is the signature of the permutation f of degree $k = |D(f)|$.

We have that $sgn^\sharp : \mathcal{SG}_n \to \{+1, -1\}$ *is a groupoid morphism.*

Indeed, let $f, g \in G_{(2)}$, where $G = \mathcal{SG}(A, X)$ such that $D(f) = D(g) := A_k := \{x_{j_1}, \ldots, x_{j_k}\} \subseteq X,\ 1 \leq k \leq n$. Then f and g are permutations of A_k and $f \circ g$ is also a permutation of A_k. Also, we have $sgn(f \circ g) = sgn(f) \cdot sgn(g)$. Hence $sgn^\sharp(m(f, g)) = sgn^\sharp(f) \cdot sgn^\sharp(g)$. Therefore the conditions from Definition 2.6 hold.

The map $sgn^\sharp : \mathcal{SG}_n \to \{+1, -1\}$ *is not a groupoid homomorphism.* Indeed, for $X = \{x_1, x_2, x_3, x_4\}$ consider

$$f = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} x_1 & x_3 & x_4 \\ x_4 & x_3 & x_1 \end{pmatrix}.$$

Then $sgn^\sharp(f) = +1,\ sgn^\sharp(g) = -1$ and $(sgn^\sharp(f), sgn^\sharp(g)) \in \{+1, -1\} \times \{+1, -1\}$. But f and g are not composable in $\mathcal{SG}_4$, since $D(f) \neq D(g)$.

3 Vector groupoids

Let V be a vector space over a field K.

Definition 3.1. *In what follows, by vector groupoid, we mean a V_0 groupoid $(V, \alpha, \beta, i, m, V_0)$ which verifies the following conditions:*

(3.1.1) V_0 *is a vector subspace of* V*;*

(3.1.2) $\alpha, \beta : V \to V_0$ *are linear maps;*

(3.1.3) $i : V \longrightarrow V$ *is a linear map such that* $x + i(x) = \alpha(x) + \beta(x),\ \forall x \in V$*;*

(3.1.4) *The multiplication* $m : V_{(2)} = \{(x, y) \in V \times V \mid \alpha(y) = \beta(x)\} \longrightarrow V$, $(x, y) \rightarrowtail\ m(x, y) := xy$, *satisfies the following conditions:*

1. $x(y + z - \beta(x)) = xy + xz - x,\ \forall x, y, z \in V$ *such that* $\alpha(y) = \beta(x) = \alpha(z)$*;*

2. $x(ky + (1-k)\beta(x)) = k(xy) + (1-k)x,\ \forall x, y \in V$ *such that* $\alpha(y) = \beta(x)$*;*

3. $(y + z - \alpha(x))x = yx + zx - x, \forall x, y, z \in V$ *such that* $\alpha(x) = \beta(y) = \beta(z)$*;*

4. $(ky + (1-k)\alpha(x))x = k(yx) + (1-k)x,\ \forall x, y \in V$ *such that* $\alpha(x) = \beta(y)$.

From Definition 3.1 follows the following corollary.

Corollary 3.2. *Let* $(V, \alpha, \beta, m, i, V_0)$ *be a vector groupoid. Then:*

(i) *The source and target* $\alpha, \beta : V \to V_0$ *are linear epimorphisms.*

(ii) *The inversion* $i : V \to V$ *is a linear automorphism.*

(iii) *The fibres* $\alpha^{-1}(0)$ *and* $\beta^{-1}(0)$ *and the isotropy group* $V(0) := \alpha^{-1}(0) \cap \beta^{-1}(0)$ *are vector subspaces of the vector space* V.

Example 3.3. Let V be a vector space. If we define $\alpha_0,\ \beta_0,\ i_0 : V \longrightarrow V$, $\alpha_0(x) = \beta_0(x) = 0,\ i_0(x) = -x$, and the multiplication $m_0(x, y) = x + y$, then $(V, \alpha_0, \beta_0, m_0, i_0, V_0 = \{0\})$ is a vector groupoid called *vector groupoid with a single unit*. We will denote this vector groupoid by $(V, +)$. Therefore, *each vector space V can be regarded as vector groupoid over $V_0 = \{0\}$.*

Example 3.4. A vector space V has a structure of null groupoid over V (see Example 2.4(ii)). In this case the structure functions are $\alpha = \beta = i = Id_V$ and $x \cdot x = x$ for all $x \in V$. We have that $V_0 = V$ and the maps α, β, i are linear. Since $x + i(x) = x + x$ and $\alpha(x) + \beta(x) = x + x$ imply that (3.1.3) holds. It is easy to verify the conditions (3.1.4) from Definition 3.1. Then V is a vector groupoid, called the *null vector groupoid* associated to V.

Example 3.5. Let V be a vector space over a field K. We consider the pair groupoid $(V \times V, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{m}, \widetilde{i}, \Delta_V)$ associated to V (see Example 2.4(iii)). We have that $V \times V$ is a vector space over K and the source $\widetilde{\alpha}$ and target $\widetilde{\beta}$ are linear maps. Also, the inversion $\widetilde{i} : V \times V \to V \times V$ is a linear map and $(x, y) + \widetilde{i}(x, y) = \widetilde{\alpha}(x, y) + \widetilde{\beta}(x, y)$. By a direct computation we verify that the relations 3.1.4(1) - 3.1.4(4) from Definition 3.1 hold. Hence $V \times V$ is a vector groupoid called the *pair vector groupoid* associated to V.

Example 3.6. The vector groupoid $V^2(p, q)$. Let V be a vector space over a field K and let $p, q \in K$ such that $pq = 1$. The maps $\alpha, \beta, i : V^2 \longrightarrow V^2$, $\alpha(x, y) := (x, px)$, $\beta(x, y) := (qy, y)$, $i(x, y) := (qy, px)$ together with the multiplication map given on $V^2_{(2)} := \{((x, y), (qy, z)) \mid x, y, z \in V\} \subset V^2 \times V^2$, by $(x, y) \cdot (qy, z) := (x, z)$ determine on V^2 a structure of vector groupoid. This is called the *pair vector groupoid of type* (p, q) and it is denoted by $V^2(p, q)$.

If $p = q = 1$, then the vector groupoid $V^2(1, 1)$ coincides with the pair vector groupoid $V \times V$.

If n is a prime number and $p, q \in \mathbb{Z}_n$, such that $pq = 1$, then $\mathbb{Z}^2_n(p, q)$ is called the *modular* or *cryptographic vector groupoid*.

Example 3.7. Let V be vector space. One consider $\alpha, \beta, i : V^3 \to V^3$ and $m : V^3_{(2)} = \{((x_1, x_2, x_3), (x_2, y_2, y_3)) | x_i, y_j \in V,\ i = 1, 2, 3,\ j = 2, 3\} \to V^3$ defined as follows:

$\alpha(x_1, x_2, x_3) := (x_1, x_1, 0),\ \beta(x_1, x_2, x_3) := (x_2, x_2, 0),$

$i(x_1, x_2, x_3) := (x_2, x_1, -x_3),$

$(x_1, x_2, x_3) \cdot (x_2, y_2, y_3) := (x_1, y_2, x_3 + y_3)$ and $V^3_0 = \{(x, x, 0) \mid x \in V\}$.

Then $(V^3, \alpha, \beta, m, i, V^3_0)$ is a vector groupoid.

In the following proposition, we give, in addition to those in Proposition 2.2, other rules of algebraic calculation in a vector groupoid.

Proposition 3.8. *Let* $(V, \alpha, \beta, m, i, V_0)$ *be a vector groupoid. Then:*
(i) $0 \cdot x = x, \ \forall \ x \in \alpha^{-1}(0)$ *and* $x \cdot 0 = x, \ \forall \ x \in \beta^{-1}(0)$*;*
(ii) For all $x, \ y \in \beta^{-1}(0)$*, we have* $x - \alpha(x) = y - \alpha(y) \Longrightarrow x = y$*;*
(iii) For all $x, y \in \alpha^{-1}(0)$*, we have* $x - \beta(x) = y - \beta(y) \Longrightarrow x = y$*.*

Proof. *(i)* If $x \in \alpha^{-1}(0)$, then $\alpha(x) = 0 = \beta(0)$. So $(0, x) \in V_{(2)}$ and, using the condition $(G2)$ from Definition 2.1, one obtains that $0 \cdot x = \alpha(x) \cdot x = x$.

(iii) Let $x, y \in \alpha^{-1}(0)$ such that $x - \beta(x) = y - \beta(y)$. Then $\alpha(x) = \alpha(y) = 0$ and $x - y = \beta(x) - \beta(y)$. Since α is linear map and applying Proposition 2.2(i), one obtains that

$$0 = \alpha(x) - \alpha(y) = \alpha(x - y) = \alpha(\beta(x) - \beta(y)) = \beta(x) - \beta(y) = x - y,$$

and so $x = y$. Similarly, we prove that the assertions (ii) hold. $\square$

Proposition 3.9. *Let* $(V, \alpha, \beta, m, i, V_0)$ *be a vector groupoid. Then:*
(i) $t_\beta : \alpha^{-1}(0) \longrightarrow \beta^{-1}(0), \ t_\beta(x) := \beta(x) - x$ *is a linear isomorphism.*
(ii) $t_\alpha : \beta^{-1}(0) \longrightarrow \alpha^{-1}(0), \ t_\alpha(x) := \alpha(x) - x$ *is a linear isomorphism.*

Proof. *(i)* For $x_1, x_2 \in V$ and $k_1, k_2 \in K$ we have

$$\begin{aligned} t_\beta(k_1x_1 + k_2x_2) &= \beta(k_1x_1 + k_2x_2) - (k_1x_1 + k_2x_2) = \\ &= k_1(\beta(x_1) - x_1) + k_2(\beta(x_2) - x_2) = k_1t_\beta(x_1) + k_2t_\beta(x_2). \end{aligned}$$

Hence t_β is a linear map. Let now $x, y \in \alpha^{-1}(0)$ such that $t_\beta(x) = t_\beta(y)$. Applying Proposition 3.8(iii), one obtains $x = y$, and so t_β is injective. For any $y \in \beta^{-1}(0)$ we take $x = \alpha(y) - y$. Clearly $x \in \alpha^{-1}(0)$. We have

$$t_\beta(x) = \beta(\alpha(y) - y) - (\alpha(y) - y) = \alpha(y) - \beta(y) - \alpha(y) + y = y,$$

since $\beta(y) = 0$. Hence t_β is surjective. Therefore t_β is a linear isomorphism.

(ii) Similarly we prove that t_α is a linear isomorphism. $\square$

Definition 3.10. *Assume that* $(V_1, V_{1,0})$ *and* $(V_2, V_{2,0})$ *are vector groupoids. A groupoid morphism (resp. groupoid homomorphism)* $f : V_1 \to V_2$ *with property that* f *is a linear map, is called vector groupoid morphism (resp. vector groupoid homomorphism).*

A vector groupoid morphism $f : V_1 \to V_2$ with property that f is a bijective linear map is an *isomorphism of vector groupoids*.

In the following theorem we give a procedure for to introduce a structure of vector groupoid with a single unit on the isotropy group of a vector groupoid.

Theorem 3.11. *Let $(V, \alpha, \beta, m, i, V_0)$ be a vector groupoid and $u \in V_0$ any unit of V. The following assertions hold.*

(i) $V(u)$ endowed with the operations $\oplus_u : V(u) \times V(u) \to V(u)$ and $\otimes_u : K \times V(u) \to V(u)$ given by:

$$x \oplus_u y = x + y - u, \quad \forall\ x, y \in V(u) \tag{2}$$

$$k \otimes_u x = kx + (1-k)u, \quad \forall\ k \in K,\ x \in V(u), \tag{3}$$

has a structure of vector space over K.

(ii) The vector space $(V(u), \oplus_u, \otimes_u)$ together with the restrictions of maps α, β, i to $V(u)$ and the multiplication $\odot_u : V(u)_{(2)} = V(u) \times V(u) \to V(u)$ given by:

$$x \odot_u y = (x-u) \cdot (y-u) + u, \quad \forall\ x, y \in V(u) \tag{4}$$

has a structure of vector groupoid with a single unit.

(iii) $(\varphi, \varphi_0) : (V(0), \{0\}) \to (V(u), \{u\})$, where φ and φ_0 are given by

$$\varphi(x) = x + u, \quad \forall x \in V(0) \quad \text{and} \quad \varphi_0(0) = u \tag{5}$$

is an isomorphism of vector groupoids with a single unit.

Proof. (i) Using the linearity of the maps α and β we verify that the operations $\oplus_u$ and $\otimes_u$ given by (2) and (3) are well-defined. For instance, for $x, y \in V(u)$ we have $\alpha(x \oplus_u y) = \alpha(x + y - u) = \alpha(x) + \alpha(y) - \alpha(u) = u$, since $\alpha(x) = \alpha(y) = \alpha(u) = u$. Similarly, $\beta(x \oplus_u y) = u$. Hence $x \oplus_u y \in V(u)$. It is easy to verify that $(V(u), \oplus_u)$ is a commutative group. Its null vector is the element $u \in V(u)$. The opposite $\ominus_u x$ of $x \in V(u)$ is $\ominus_u x = 2u - x$.

For $x, y \in V(u)$ and $k, k_1, k_2 \in K$, the operation $\otimes_u$ verify the relations:

(a) $k \otimes_u (x \oplus_u y) = (k \otimes_u x) \oplus_u (k \oplus_u y)$; (b) $(k_1 + k_2) \otimes_u x = (k_1 \otimes_u x) \oplus_u (k_2 \otimes_u x)$;

(c) $k_1 \otimes_u (k_2 \otimes_u x) = (k_1 k_2) \otimes_u x$; (d) $1 \otimes_u x = x$ (here 1 is the unit of K).

Indeed, we have $k \otimes_u (x \oplus_u y) = k(x \oplus_u y) + (1-k)u = k(x+y) + (1-2k)u$ and $(k \otimes_u x) \oplus_u (k \oplus_u y) = (k \otimes_u x) + (k \oplus_u y) - u = k(x+y) + (1-2k)u$. Hence the equality (a) holds. In the same manner we prove that the equalities $(b) - (d)$ hold. Therefore $(V, \oplus_u, \otimes_u)$ is a vector space.

(ii) From (i) follows that the condition (3.1.1) from Definition 3.1 is satisfied. The restrictions of the linear maps α, β and i to $V(u)$ are linear maps. Also, for any $x \in V(u)$ we have $x \oplus_u i(x) = x + i(x) - u = \alpha(x) + \beta(x) - u = \alpha(x) \oplus_u \beta(x)$, since $x + i(x) = \alpha(x) + \beta(x)$. Therefore, (3.1.2) and (3.1.3) from Definition 3.1 hold.

Let $x, y \in V(u)$. The operation $\odot_u$ given by (4) is well-defined. Indeed, applying the linearity of maps α and β we have

$\alpha(x \odot_u y) = \alpha((x-u) \odot (y-u) + u) = \alpha((x-u) \odot (y-u)) + \alpha(u) = \alpha(x-u) + \alpha(u) = \alpha(x) = u$. Similarly, we have $\beta(x \odot_u y) = u$ and so $x \odot_u y \in V(u)$.

If $x, y, z \in V(u)$ then the following equality holds:

(e) $x \odot_u (y \oplus_u z \oplus_u (\ominus_u \beta(x))) = (x \odot_u y) \oplus_u (x \odot_u z) \oplus_u (\ominus_u x))$.

Indeed, since $\beta(x) = u$ and $\ominus_u x = 2u - x$, we have

$(e.1)$ $x \odot_u (y \oplus_u z \oplus_u (\ominus_u \beta(x))) = x \odot_u (y \oplus_u z \oplus_u (\ominus_u u)) = x \odot_u (y \oplus_u z \oplus_u u) =$
$= x \odot_u (y \oplus_u z) = (x-u) \odot (y \oplus_u z - u) + u = (x-u) \odot ((y-u) + (z-u)) + u$.

Replacing in the equality (3.1.4)(1) the elements $x, y, z \in V(u)$ respectively with $x-u, y-u, z-u \in V(u)$, we obtain the following equality

(f) $(x-u) \odot ((y-u) + (z-u)) = (x-u) \odot (y-u) + (x-u) \odot (z-u) - (x-u)$,

since $\beta(x-u) = 0$.

Using the relation (f), the equality $(e.1)$ becomes

$(e.2)$ $x \odot_u (y \oplus_u z \oplus_u (\ominus_u \beta(x))) = (x-u) \odot (y-u) + (x-u) \odot (z-u) + 2u - x$.

On the other hand we have

$(e.3)$ $(x \odot_u y) \oplus_u (x \odot_u z) \oplus_u (\ominus_u x)) = ((x \odot_u y) \oplus_u (x \odot_u z)) \oplus_u (2u - x) =$
$= (x \odot_u y + x \odot_u z - u) \oplus_u (2u - x) = x \odot_u y + x \odot_u z - x =$
$= (x-u) \odot (y-u) + (x-u) \odot (z-u) + 2u - x$.

Using $(e.2)$ and $(e.3)$ we obtain the equality (e). Hence, the relation (3.1.4)(1) from Definition 3.1 holds. In the same manner we can prove that the relations (3.1.4)(2) – (3.1.4)(4) from Definition 3.1 are verified.

(iii) Taking $u = 0$ in the relations $(2) - (4)$ we obtains that $(V(0), \oplus_0, \otimes_0)$ is a vector space and $(V(0), \alpha, \beta, \odot_0, i, \{0\})$ is a vector groupoid with a single unit, where $x \oplus_0 y = x + y,\ k \otimes_0 x = kx,\ x \odot_0 y = x \cdot y,\ = forall x, y \in V(0), k \in K$.

The map $\varphi : V(0) \to V(u)$ is bijective. For each $x \in V(0)$ we have $(\alpha \circ \varphi)(x) = \alpha(x + u) = \alpha(x) + \alpha(u) = u = \varphi_0(0) = \varphi_0(\alpha(x)) = (\varphi_0 \circ \alpha)(x)$, that is $\alpha \circ \varphi = \varphi_0 \circ \alpha$. Similarly, we have $\beta \circ \varphi = \varphi_0 \circ \beta$.

Also, for all $x, y \in V(0)$ we have

$$\varphi(x \odot_0 y) = (x \odot_0 y) + u = x \cdot y + u = (x + u) \odot_u (y + u) = \varphi(x) \odot_u \varphi(y).$$

Hence, the conditions (i) and (ii) from Definition 2.6 are satisfied. It is easy to verify that φ is linear. Then, φ is a vector groupoid isomorphism. $\square$

We call $(V(u), \alpha, \beta, \odot_u, i, V_0(u) = \{u\})$ the *isotropy vector groupoid* at $u \in V_0$ associated to vector space $(V(u), \oplus_u, \otimes_u)$.

Example 3.12. *The anchor map $(\alpha, \beta) : V \to V_0 \times V_0$ is a vector groupoid homomorphism between the vector groupoid $(V, \alpha, \beta, m, i, V_0)$ and the pair vector groupoid $(V_0 \times V_0, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{m}, \widetilde{i}, \Delta_{V_0})$.*

Indeed, if we denote $(\alpha, \beta) := f$ and consider $x, y \in G$ such that $(f(x), f(y)) \in (V_0 \times V_0)_{(2)}$, then $\widetilde{\beta}(f(x)) = \widetilde{\alpha}(f(y))$ and we have

$$\widetilde{\beta}(\alpha(x), \beta(x)) = \widetilde{\alpha}(\alpha(y), \beta(y)) \ \Rightarrow \ (\beta(x), \beta(x)) = (\alpha(y), \alpha(y)) \ \Rightarrow$$

$$\beta(x)) = \alpha(y), \text{ i.e. } (x, y) \in V_{(2)}.$$

For $(x, y) \in V_{(2)}$ we have $f(m(x, y)) = f(xy) = (\alpha(xy), \beta(xy)) = (\alpha(x), \beta(y))$ and $\widetilde{m}(f(x), f(y)) = \widetilde{m}((\alpha(x), \beta(x)), (\alpha(y), \beta(y))) = (\alpha(x), \beta(y))$. Hence, the conditions (i) and (ii) from Definition 2.6 are verified.

Let now $x, y \in V$ such that $(f(x), f(y)) \in (V_0 \times V_0)_{(2)}$. Then $\widetilde{\beta}(f(x)) = \widetilde{\alpha}(f(y))$. Since $f(x) = (\alpha(x), \beta(x))$ and $f(y) = (\alpha(y), \beta(y))$ we deduce that $(\beta(x), \beta(x)) = (\alpha(y), \alpha(y))$. Hence $\beta(x) = \alpha(y)$ and $(x, y) \in V_{(2)}$. Therefore the condition (4) is satisfied and f is a groupoid homomorphism. Using the linearity of α and β, it is easy to verify that f is linear. Hence, f is a vector groupoid homomorphism.

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