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Iteration methods for two nonexpansive mappings and semigroups on two sets

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Abstract

In this paper, we introduce some new iteration methods based on the hybrid method in mathematical programming, the descent-like iterative method and the Halpern's method for finding a common fixed point of two nonexpansive mappings and nonexpansive semigroups on two closed and convex subsets in Hilbert spaces.

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1 Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle .,. \rangle$ and $\|.\|$, respectively, and let C be a nonempty, closed

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and convex subset of H. Denote by $P_C x$ the metric projection from $x \in H$ onto C. Let T be a nonexpansive mapping on C, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [1].

Let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C, that is,

(1) for each $t > 0, T(t)$ is a nonexpansive mapping on C;

(2) $T(0)x = x$ for all $x \in C$;

(3) $T(t_1 + t_2) = T(t_1) \circ T(t_2)$ for all $t_1, t_2 > 0$; and

(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

Denote by $\mathcal{F} = \cap_{t>0} F(T(t))$ the set of common fixed points for the semigroup $\{T(t) : t > 0\}$. We know that $\mathcal{F}$ is a closed convex subset in H and $\mathcal{F} \neq \emptyset$ if C is compact (see, [2]).

Let $C_i, i = 1, 2$, be two closed and convex subsets in H. Let T_i and $\{T_i(t) : t > 0\}$, $i = 1, 2$, be two nonexpansive mappings and semigroups on C_i, respectively. The problems studied in this paper is to find two elements

$$p \in F := F(T_1) \cap F(T_2) \tag{1.1}$$

and

$$q \in \mathcal{F}_{1,2} := \mathcal{F}_1 \cap \mathcal{F}_2, \tag{1.2}$$

where $\mathcal{F}_i = \cap_{t>0} F(T_i(t))$. Assume that F and $\mathcal{F}_{1,2}$ are not empty. Some particular cases of (1.1) and (1.2) are the following:

(i) when $T_1 = T_2 = I$, the indentity mapping in H, (1.1) is the convex feasibility problem studied in [3].

(ii) when $C_1 = C_2 = C$, problems (1.1) and (1.2) are considered in [4]-[6].

For finding a fixed point of a nonexpansive mapping T on C, in 1953, Mann [7] proposed the following method:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \end{aligned} \tag{1.3}$$

that converges only weakly, in general (see [8] for an example). In 1967, Halpern [9] firstly proposed the following iteration process:

$$x_{n+1} = \beta_n u + (1 - \beta_n) T x_n, \quad n \geq 0, \tag{1.4}$$

where u, x_0 are two fixed elements in C and $\{\beta_n\} \subset (0, 1)$. He pointed out that the conditions $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$ are necessary in the sense that, if the iteration (1.4) converges to a fixed point of T, then these conditions must be satisfied. Further, the iteration method was investigated by Lions [10], Reich [11], Wittmann [12] and Song [13]. Recently, Alber [14] proposed the following descent-like method

$$x_{n+1} = P_C(x_n - \mu_n[x_n - Tx_n]), n \geq 0, \tag{1.5}$$

and proved that if $\{\mu_n\} : \mu_n > 0, \mu_n \to 0$, as $n \to \infty$ and $\{x_n\}$ is bounded, then:

(i) there exists a weak accumulation point $\tilde{x} \in C$ of $\{x_n\}$;
(ii) all weak accumulation points of $\{x_n\}$ belong to $F(T)$; and
(iii) if $F(T)$ is a singleton, i.e., $F(T) = \{\tilde{x}\}$, then $\{x_n\}$ converges weakly to $\tilde{x}$.

To obtain strong convergence for (1.3), Nakajo and Takahashi [15] introduced the hybrid Mann's iteration method:

$$\begin{aligned} x_0 &\in C, \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. They showed that $\{x_n\}$ defined by (1.6) converges strongly to $P_{F(T)}x_0$ as $n \to \infty$. Recently, Yanes and Xu [16] adapted the iteration process (1.4) as follows:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_n &= \beta_n x_0 + (1 - \beta_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\qquad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0. \end{aligned} \tag{1.7}$$

They proved that if T is a nonexpansive mapping on a closed convex subset C with $F(T) \neq \emptyset$ and the sequence $\{\beta_n\} \subset (0, 1)$ is chosen such that

$$\lim_{n\to\infty} \beta_n = 0,$$

then the sequence $\{x_n\}$ defined by (1.7) converges strongly to $P_{F(T)}x_0$ as $n \to \infty$.

For finding an element $p \in \mathcal{F}$, Nakajo and Takahashi [15] also introduced an iteration procedure as follows:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_n &= \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, n \ge 0, \end{aligned} \tag{1.8}$$

where $\alpha_n \in$ [0,a] for some $a \in$ [0,1) and $\{t_n\}$ is a positive real number divergent sequence. Under the conditions on $\{\alpha_n\}$ and $\{t_n\}$, the sequence $\{x_n\}$ defined by (1.8) converges strongly to $P_{\mathcal{F}}x_0$.

If $C \equiv H$, then C_n and Q_n in (1.6)-(1.8) are two halfspaces. So, the projection x_{n+1} onto $C_n \cap Q_n$ in these methods can be found by an explicit formula [17]. Clearly, if C is a proper subset of H, then C_n and Q_n in (1.6)-(1.8) are not two halfspaces. Then, the following problem is posed: how to construct the closed convex subsets C_n and Q_n and if we can express x_{n+1} of (1.6)-(1.8) in a similar form as in [17]? This problem is solved very recently in [18]-[20]. In this works, C_n and Q_n are replaced by two halfspaces and y_n is the right hand side of (1.5) with a modification. In this paper, motivated by (1.5), (1.7) and [14], [15], to solve problems (1.1) and (1.2) we introduce the following new iteration processes:

$$\begin{aligned} x_0 &\in H \quad \text{any element}, \\ z_n &= x_n - \mu_n(x_n - T_1 P_{C_1} x_n), \\ y_n &= \beta_n x_0 + (1-\beta_n) T_2 P_{C_2} z_n, \\ H_n &= \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\qquad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{H_n \cap W_n} x_0, n \ge 0; \end{aligned} \tag{1.9}$$

and

$$\begin{aligned}
&x_0 \in H \quad \text{any element},\\
&z_n = x_n - \mu_n\Big(x_n - \frac{1}{t_n}\int_0^{t_n} T_1(s)P_{C_1}x_n ds\Big),\\
&y_n = \beta_n x_0 + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T_2(s)P_{C_2}z_n ds,\\
&H_n = \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2\\
&\qquad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\},\\
&W_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{H_n\cap W_n}x_0, \quad n \ge 0.
\end{aligned} \tag{1.10}$$

We shall prove the strong convergence of the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.9) and (1.10) to some elements p and q in Sections 2 and 3, respectively.

Below, the symbols $\rightharpoonup$ and $\to$ denote weak and strong convergences, respectively.

2 Strong convergence to a common fixed point of two nonexpansive mappings

We formulate the following facts needed in the proof of our results.

Lemma 2.1. [21] *Let H be a real Hilbert. There holds the following identity:* $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$.

Lemma 2.2. [16] *Let C be a nonempty, closed and convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that $\|z - x\| \le \|y - x\|$ for all $y \in C$, and $z = P_C x$ if and only if $\langle z - x, y - z\rangle \ge 0$ for all $y \in C$.*

Lemma 2.3. (Demiclosedness principle) [21] *If C is a nonempty, closed and convex subset of a real Hilbert space H, T is a nonexpansive mapping on C, $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$.*

Lemma 2.4. [22] *Every Hilbert space H has Randon-Riesz property or Kadec-Klee property, that is, for a sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there hodls $x_n \to x$.*

Now, we are in a position to prove the following result.

Theorem 2.5. *Let C_1 and C_2 be two nonempty, closed and convex subsets in a real Hilbert space H and let T_1 and T_2 be two nonexpansive mappings on C_1 and C_2, respectively, such that $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\mu_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\mu_n \in (a, b)$ for some $a, b \in (0, 1)$ and $\beta_n \to 0$. Then, the sequences $\{x_n\}, \{z_n\}$ and $\{y_n\}$, defined by (1.9), converge strongly to the same point $u_0 = P_F x_0$, as $n \to \infty$.*

Proof. First, note that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)$$

is equivalent to

$$\langle (1-\beta_n)x_n + \beta_n x_0 - y_n, z\rangle \leq \langle x_n - y_n, x_n\rangle - \frac{1}{2}\|y_n - x_n\|^2 + \frac{\beta_n}{2}\|x_0\|^2.$$

Thus, H_n is a halfspace. It is clear that

$$F(T) = F(TP_C) := \{p \in H : TP_C p = p\}$$

for any mapping T from C into C. So, we have that $F = F(\tilde{T}_1) \cap F(\tilde{T}_2)$ where $\tilde{T}_i = T_i P_{C_i}, i = 1, 2$, and $\tilde{T}_i, i = 1, 2$, are also two nonexpansive mappings on H. Hence, by (1.9) and Lemma 2.1, we obtain for any $p \in F$ that

$$\begin{aligned}\|z_n - p\|^2 &= \|(1-\mu_n)(x_n - p) + \mu_n(\tilde{T}_1 x_n - p)\|^2 \\ &= (1-\mu_n)\|x_n - p\|^2 + \mu_n\|\tilde{T}_1 x_n - p\|^2 \\ &\quad - (1-\mu_n)\mu_n\|x_n - \tilde{T}_1 x_n\|^2 \\ &\leq (1-\mu_n)\|x_n - p\|^2 + \mu_n\|x_n - p\|^2 \\ &\quad - (1-\mu_n)\mu_n\|x_n - \tilde{T}_1 x_n\|^2 \\ &\leq \|x_n - p\|^2 - (1-\mu_n)\mu_n\|x_n - \tilde{T}_1 x_n\|^2 \leq \|x_n - p\|^2.\end{aligned} \tag{2.1}$$

By the similar argument and the convexity of $\|.\|^2$, we also obtain

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_n x_0 + (1-\beta_n)\tilde{T}_2 z_n - p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|\tilde{T}_2 z_n - \tilde{T}_2 p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|z_n - p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|x_n - p\|^2\\ &= \|x_n - p\|^2 + \beta_n(\|x_0 - p\|^2 - \|x_n - p\|^2)\\ &= \|x_n - p\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle).\end{aligned}$$

Therefore, $p \in H_n$ for all $n \ge 0$. It means that $F(T) \subset H_n$ for all $n \ge 0$.

Next, we show by mathematical induction that $F(T) \subset H_n \cap W_n$ for each $n \ge 0$. For $n = 0$, we have $W_0 = H$, and hence $F(T) \subset H_0 \cap W_0$. Suppose that x_i is given and $F(T) \subset H_i \cap W_i$ for some $i > 0$. There exists a unique element $x_{i+1} \in H_i \cap W_i$ such that $x_{i+1} = P_{H_i \cap W_i} x_0$. Therefore, by Lemma 2.2,

$$\langle x_{i+1} - x_0, p - x_{i+1}\rangle \ge 0$$

for each $p \in H_i \cap W_i$. Since $F(T) \subset H_i \cap W_i$, we get $F(T) \subset W_{i+1}$. So, we have $F(T) \subset H_{i+1} \cap W_{i+1}$.

Further, since $F(T)$ is a nonempty, closed and convex subset of H, there exists a unique element $u_0 \in F(T)$ such that $u_0 = P_{F(T)} x_0$. From $x_{n+1} = P_{H_n \cap W_n}(x_0)$, we obtain

$$\|x_{n+1} - x_0\| \le \|z - x_0\|$$

for every $z \in H_n \cap W_n$. As $u_0 \in F(T) \subset W_n$, we get

$$\|x_{n+1} - x_0\| \le \|u_0 - x_0\| \quad \forall\, n \ge 0. \tag{2.2}$$

This implies that $\{x_n\}$ is bounded. Now, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.3}$$

From the definition of W_n and Lemma 2.2, we have $x_n = P_{W_n} x_0$. As $x_{n+1} \in H_n \cap W_n$, we obtain

$$\|x_{n+1} - x_0\| \ge \|x_n - x_0\| \quad \forall\, n \ge 0.$$

Therefore, $\{\|x_n - x_0\|\}$ is a nondecreasing and bounded sequence. So, there exists $\lim_{n\to\infty} \|x_n - x_0\| = c$. On the other hand, from $x_{n+1} \in W_n$, it follows that

$$\langle x_n - x_0, x_{n+1} - x_n\rangle \ge 0,$$

and hence

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_0 - (x_{n+1} - x_0)\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_{n+1} - x_0\rangle + \|x_{n+1} - x_0\|^2 \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \quad \forall n \ge 0.\end{aligned}$$

Thus, (2.3) is proved by using the last inequality and $\lim_{n\to\infty} \|x_n - x_0\| = c$.

Next, since $x_{n+1} \in H_n$ we have that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \beta_n(\|x_0\| + 2\langle x_n - x_0, z\rangle)\}.$$

Therefore, from (2.3), the boundedness of $\{x_n\}$, $\beta_n \to 0$ and the last inequality, it follows that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{2.4}$$

This together with (2.3) implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.5}$$

Noticing that $\tilde{T}_2 z_n = y_n - \beta_n(x_n - \tilde{T}_2 z_n) + \beta_n(x_n - x_0)$, we have

$$\|x_n - \tilde{T}_2 z_n\| \le \|x_n - y_n\| + \beta_n\|x_n - \tilde{T}_2 z_n\| + \beta_n\|x_n - x_0\|.$$

From (2.2) and the last inequality, it follows that

$$\|x_n - \tilde{T}_2 z_n\| \le \frac{1}{1-\beta_n}\Bigg(\|x_n - y_n\| + \beta_n\|u_0 - x_0\|\Bigg).$$

By $\beta_n \to 0$ ($\beta_n \le 1 - \beta$ for some $\beta \in (0,1)$), (2.5) and the last inequality, we obtain

$$\lim_{n\to\infty} \|x_n - \tilde{T}_2 z_n\| = 0. \tag{2.6}$$

Now, we shall prove that $\|x_n - \tilde{T}_1 x_n\| \to 0$ and $\|x_n - \tilde{T}_2 x_n\| \to 0$, as $n \to \infty$. Indeed, since $\{x_n\}$ is bounded, for any $p \in F$ and any subsequence $\{\tilde{T}_1 x_{n_k} - x_{n_k}\}$ of $\{\tilde{T}_1 x_n - x_n\}$ there exists a subsequence $\{x_{n_j}\} \subset \{x_{n_k}\}$ such that

$$\lim_{j\to\infty} \|x_{n_j} - p\| = \limsup_{k\to\infty} \|x_{n_k} - p\| = a.$$

By (2.6), (2.1) and the following inequalities

$$\begin{aligned}\|x_{n_j} - p\| &\le \|x_{n_j} - \tilde{T}_2 z_{n_j}\| + \|\tilde{T}_2 z_{n_j} - p\| \\ &\le \|x_{n_j} - \tilde{T}_2 z_{n_j}\| + \|z_{n_j} - p\| \\ &\le \|x_{n_j} - \tilde{T}_2 z_{n_j}\| + \|x_{n_j} - p\|,\end{aligned}$$

we get that

$$\lim_{j\to\infty}\|x_{n_j}-p\| = \lim_{j\to\infty}\|z_{n_j}-p\| = a.$$

Again from (2.1) and the condition on μ_n, it implies that

$$a(1-b)\|\tilde{T}_1x_{n_j}-x_{n_j}\| \le \|x_{n_j}-p\| - \|z_{n_j}-p\|.$$

So, $\|\tilde{T}_1x_{n_j}-x_{n_j}\| \to 0$ and hence $\|\tilde{T}_1x_n-x_n\| \to 0$, as $n\to\infty$. Further, since

$$\begin{aligned}\|\tilde{T}_2x_n-x_n\| &\le \|\tilde{T}_2x_n-\tilde{T}_2z_n\| + \|\tilde{T}_2z_n-x_n\|\\ &\le \|x_n-z_n\| + \|\tilde{T}_2z_n-x_n\|,\end{aligned}$$

$$\lim_{n\to\infty}\|z_n-x_n\| = \lim_{n\to\infty}\mu_n\|\tilde{T}_1x_n-x_n\| = 0, \tag{2.7}$$

by (2.6) and $\|\tilde{T}_1x_n-x_n\|\to 0$, we also obtain that $\|\tilde{T}_2x_n-x_n\|\to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that convegers weakly to some element $p\in H$ as $i\to\infty$. By Lemmas 2.3 and $\|\tilde{T}_1x_n-x_n\|, \|\tilde{T}_2x_n-x_n\|\to 0$, we have that $p\in F$.

Now, from (2.2) and the weak lower semicontinuity of the norm it implies that

$$\|x_0-u_0\| \le \|x_0-p\| \le \liminf_{j\to\infty}\|x_0-x_{n_j}\| \le \limsup_{j\to\infty}\|x_0-x_{n_j}\| \le \|x_0-u_0\|.$$

Thus, we obtain $\lim_{j\to\infty}\|x_0-x_{n_j}\| = \|x_0-u_0\| = \|x_0-p\|$. This implies $x_{k_j}\to p=u_0$ by Lemma 2.4. By the uniqueness of the projection $u_0=P_Fx_0$, we have that $x_n\to u_0$. Consequently, from (2.7) it follows that $z_n\to u_0$. From (2.5), we also get that $y_n\to u_0$. This completes the proof. □

We have the following corollaries.

Corollary 2.6. *Let $C_i, i=1,2$, be two nonempty, closed and convex subsets in a real Hilbert space H. Let $T_i, i=1,2$, be two nonexpansive mappings on C_i such that $F(T_1)\cap F(T_2)\ne\emptyset$. Assume that $\{\mu_n\}$ is a sequence such that $0<a\le\mu_n\le b<1$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned} x_0 &\in H \quad \textit{any element},\\ y_n &= T_2P_{C_2}(x_n-\mu_n(x_n-T_1P_{C_1}x_n)),\\ H_n &= \{z\in H : \|y_n-z\|\le\|x_n-z\|\},\\ W_n &= \{z\in H : \langle x_n-z, x_0-x_n\rangle \ge 0\},\\ x_{n+1} &= P_{H_n\cap W_n}(x_0), n\ge 0,\end{aligned}$$

converge strongly to the same point $u_0 = P_{F(T)}x_0$, *as* $n \to \infty$.

Proof. By putting $\beta_n \equiv 0$ in Theorem 2.5, we obtain the conclusion. □

Corollary 2.7. *Let* $C_i, i = 1, 2,$ *be two nonempty, closed and convex subsets in a real Hilbert space* H *such that* $C := C_1 \cap C_2 \neq \emptyset$. *Assume that* $\{\mu_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,\ 1]$ *such that* $\mu_n \in (a, b)$ *for some* $a, b \in (0, 1)$ *and* $\beta_n \to 0$. *Then, the sequences* $\{x_n\}, \{z_n\}$ *and* $\{y_n\}$, *defined by*

$$\begin{aligned}
x_0 &\in H \quad \text{any element,}\\
z_n &= x_n - \mu_n(x_n - P_{C_1}x_n),\\
y_n &= \beta_n x_0 + (1-\beta_n)P_{C_2}z_n,\\
H_n &= \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2\\
&\quad + \beta_n(\|x_0\| + 2\langle x_n - x_0, z\rangle)\},\\
W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\
x_{n+1} &= P_{H_n \cap W_n}x_0, n \geq 0,
\end{aligned}$$

converge strongly to the same point $u_0 = P_C x_0$, *as* $n \to \infty$.

Proof. By putting $T_1 = T_2 = I$ in Theorem 2.5, we obtain the conclusion. □

3 Strong convergence to a common fixed point of two nonexpansive semigroups

We need the following Lemma in the proof of our result.

Lemma 3.1. [23] *Let* C *be a nonempty bounded closed convex subset in a real Hilbert space* H *and let* $\{T(t) : t > 0\}$ *be a nonexpansive semigroup on* C. *Then, for any* $h > 0$

$$\limsup_{t\to\infty}\sup_{y\in C}\left\|T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right) - \frac{1}{t}\int_0^t T(s)yds\right\| = 0.$$

Now, we prove the following result.

Theorem 3.2. *Let C_1 and C_2 be two nonempty closed convex subsets in a real Hilbert space H and let $\{T_1(t) : t > 0\}$ and $\{T_2(t) : t > 0\}$ be two nonexpansive semigroups on C_1 and C_2, respectively, such that $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$ where $\mathcal{F}_i = \cap_{t>0} F(T_i(t)), i = 1, 2$. Assume that $\{\mu_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\mu_n \in (a, b)$ for some $a, b \in (0, 1)$ and $\beta_n \to 0$ and $\{t_n\}$ is a positive real divergent sequence. Then, the sequences $\{x_n\}, \{z_n\}$ and $\{y_n\}$, defined by (1.10), converge strongly to the same point $u_0 = P_{\mathcal{F}} x_0$, as $n \to \infty$.*

Proof. For each $p \in \mathcal{F}$, we have for each $s > 0$ that

$$p = P_{C_i} p = \tilde{T}_i(s)p, \quad i = 1, 2,$$

where $\tilde{T}_i(s) = T_i(s)P_{C_i}$, and hence from (1.10) and Lemma 2.1, we obtain that

$$\begin{aligned}
\|z_n - p\|^2 &= \left\| (1-\mu_n)(x_n - p) + \mu_n \left(\frac{1}{t_n} \int_0^{t_n} \tilde{T}_1(s)x_n)ds - p \right) \right\|^2 \\
&= \left\| (1-\mu_n)(x_n - p) + \mu_n \left(\frac{1}{t_n} \int_0^{t_n} [\tilde{T}_1(s)x_n - \tilde{T}_1(s)p]ds \right) \right\|^2 \\
&= (1-\mu_n)\|x_n - p\|^2 + \mu_n \left\| \frac{1}{t_n} \int_0^{t_n} \tilde{T}_1(s)x_n - \tilde{T}_1(s)pds \right\|^2 \\
&\quad - (1-\mu_n)\mu_n \left\| x_n - \frac{1}{t_n} \int_0^{t_n} \tilde{T}_1(s)x_n ds \right\|^2 \qquad (3.1) \\
&\leq \|x_n - p\|^2 - (1-\mu_n)\mu_n \left\| x_n - \frac{1}{t_n} \int_0^{t_n} \tilde{T}_1(s)x_n ds \right\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned}$$

By the similar argument and the convexity of $\|.\|^2$, we also obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \left\| \beta_n(x_0 - p) + (1-\beta_n) \left(\frac{1}{t_n} \int_0^{t_n} \tilde{T}_2(s)z_n ds - p \right) \right\|^2 \\
&\leq \beta_n \|x_0 - p\|^2 + (1-\beta_n) \left\| \frac{1}{t_n} \int_0^{t_n} [\tilde{T}_2(s)z_n - \tilde{T}_2(s)p]ds \right\|^2 \\
&\leq \beta_n \|x_0 - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\leq \beta_n \|x_0 - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\
&= \|x_n - p\|^2 + \beta_n(\|x_0 - p\|^2 - \|x_n - p\|^2) \\
&= \|x_n - p\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle).
\end{aligned}$$

Therefore, $p \in H_n$ for $n \geq 0$. It means that $\mathcal{F} \subset H_n$ for $n \geq 0$. As in the proof of Theorem 2.5, we can obtain the following properties:

(i) $\mathcal{F} \subset H_n \cap W_n$,

$$\|x_{n+1} - x_0\| \leq \|u_0 - x_0\|, u_0 = P_{\mathcal{F}} x_0 \tag{3.2}$$

for $n \geq 0$. This implies that $\{x_n\}$ is bounded.

(ii)

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.4}$$

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.5}$$

Noticing that

$$\frac{1}{t_n}\int_0^{t_n} \tilde{T}_2(s) z_n ds = y_n - \beta_n \Big(x_n - \frac{1}{t_n}\int_0^{t_n} \tilde{T}_2(s) z_n ds \Big) + \beta_n (x_n - x_0),$$

we have

$$\begin{aligned}\|x_n - \frac{1}{t_n}\int_0^{t_n} \tilde{T}_2(s) z_n ds\| &\leq \|x_n - y_n\| \\ &+ \beta_n \Big\| x_n - \frac{1}{t_n}\int_0^{t_n} \tilde{T}_2(s) z_n ds \Big\| + \beta_n \|x_n - x_0\|.\end{aligned}$$

From (3.2) and the last inequality, it follows that

$$\Big\| x_n - \frac{1}{t_n}\int_0^{t_n} \tilde{T}_2(s) z_n ds \Big\| \leq \frac{1}{1-\beta_n}\Big(\|x_n - y_n\| + \beta_n \|u_0 - x_0\| \Big).$$

By $\beta_n \to 0$ ($\beta_n \leq 1 - \beta$ for some $\beta \in (0,1)$), (3.5) and the last inequality, we obtain

$$\lim_{n\to\infty} \Big\| x_n - \frac{1}{t_n}\int_0^{t_n} \tilde{T}_2(s) z_n ds \Big\| = 0. \tag{3.6}$$

As in the proof of Theorem 2.5, by using (3.6) we can obtain that

$$\lim_{n\to\infty} \Big\| x_n - \frac{1}{t_n}\int_0^{t_n} \tilde{T}_i(s) x_n ds \Big\| = 0, i = 1, 2, \tag{3.7}$$

and

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.8}$$

Since

$$\frac{1}{t_n}\int_0^{t_n}\tilde{T}_i(s)x_n ds \in C_i, i = 1,2,$$

we have that

$$\begin{aligned}\left\|P_{C_i}x_n - \frac{1}{t_n}\int_0^{t_n}\tilde{T}_i(s)x_n ds\right\| &= \left\|P_{C_i}x_n - P_{C_i}\frac{1}{t_n}\int_0^{t_n}\tilde{T}_i(s)x_n ds\right\| \\ &\le \left\|x_n - \frac{1}{t_n}\int_0^{t_n}\tilde{T}_i(s)x_n ds\right\|,\end{aligned}$$

and hence from (3.7) it implies that

$$\lim_{n\to\infty}\left\|P_{C_i}x_n - \frac{1}{t_n}\int_0^{t_n}\tilde{T}_i(s)x_n ds\right\| = 0, i = 1,2. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to some element $q \in H$ as $j \to \infty$. From (3.7) and (3.9), we also obtain that $u^i_{n_j} := P_{C_i}x_{n_j} \to q$ as $j \to \infty$. It means that $q \in C_1 \cap C_2$. Then, for each $h > 0$, we have that

$$\begin{aligned}\|T_i(h)u^i_n - u^i_n\| \le &\left\|T_i(h)u^i_n - T_i(h)\left(\frac{1}{t_n}\int_0^{t_n}T_i(s)u^i_n ds\right)\right\| \\ &+ \left\|T_i(h)\left(\frac{1}{t_n}\int_0^{t_n}T_i(s)u^i_n ds\right) - \frac{1}{t_n}\int_0^{t_n}T_i(s)u^i_n ds\right\| \\ &+ \left\|\frac{1}{t_n}\int_0^{t_n}T_i(s)u^i_n ds - u^i_n\right\| \\ \le &\, 2\left\|\frac{1}{t_n}\int_0^{t_n}T_i(s)u^i_n ds - u^i_n\right\| \\ &+ \left\|T(h)\left(\frac{1}{t_n}\int_0^{t_n}T_i(s)u^i_n ds\right) - \frac{1}{t_n}\int_0^{t_n}T_i(s)u^i_n ds\right\|.\end{aligned} \tag{3.10}$$

Let $C^i_0 = \{z \in C_i : \|z - u_0\| \le 2\|x_0 - u_0\|\}$. Since $u_0 = P_{\mathcal{F}}x_0 \in C_i$, we have that

$$\|u^i_{n_j} - u_0\| = \|P_{C_i}x_{n_j} - P_{C_i}u_0\| \le \|x_{n_j} - u_0\| \le 2\|x_0 - u_o\|.$$

So, C^i_0 is a nonempty bounded closed convex subset. It is easy to verify that $\{T_i(t) : t > 0\}$ is a nonexpansive semigroup on C^i_0. By Lemma 3.1, we get

$$\lim_{n\to\infty}\left\|T_i(h)\left(\frac{1}{t_n}\int_0^{t_n}T(s)u^i_n ds\right) - \frac{1}{t_n}\int_0^{t_n}T(s)u^i_n ds\right\| = 0$$

for every fixed $h > 0$ and hence by (3.9)-(3.10) we obtain that

$$\lim_{j\to\infty}\|T_i(h)u^i_{n_j} - u^i_{n_j}\| = 0$$

for each $h > 0$. By Lemma 2.3, $q \in F(T_i(h))$ for all $h > 0$. It means that $q \in \mathcal{F}$. As in the proof of Theorem 2.5, by using (3.2), (3.5) and (3.8), we also obtain that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$, defined by (1.10), converge strongly to u_0 as $n \to \infty$. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in $[0, 1]$ such that $\beta_n \to 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned}
x_0 &\in H \quad \textit{any element},\\
y_n &= \beta_n x_0 + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds,\\
H_n &= \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2\\
&\qquad + \beta_n(\|x_0\| + 2\langle x_n - x_0, z\rangle)\},\\
W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{H_n\cap W_n}(x_0), n \ge 0,
\end{aligned}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}x_0$, as $n \to \infty$.

Proof. By putting $T_1(s) = I$ for all $s > 0, C_1 = H, C_2 = C$ and $T_2(s) = T(s)$ in Theorem 3.2, we obtain the conclusion. □

Corollary 3.4. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\alpha_n \to 1$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned}
x_0 &\in H \quad \textit{any element},\\
y_n &= \frac{1}{t_n}\int_0^{t_n} T(s)P_C\left(x_n - \mu_n\left[x_n - \frac{1}{t_n}\int_0^{t_n} T(s)P_C x_n ds\right]ds\right),\\
H_n &= \{z \in H : \|y_n - z\| \le \|x_n - z\|\},\\
W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{H_n\cap W_n}(x_0), n \ge 0,
\end{aligned}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}x_0$, as $n \to \infty$.

Proof. By putting $\beta_n \equiv 0, C_2 = H, C_1 = C, T_2(s) = I$ and $T_1(s) = T(s)$ for all $s > 0$ in Theorem 3.2, we obtain the conclusion. □

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