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The Gamma-Generalized Inverse Weibull Distribution with Applications to Pricing and Lifetime Data

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Abstract

A new distribution called the gamma-generalized inverse Weibull distribution which includes inverse exponential, inverse Rayleigh, inverse Weibull, Fréchet, generalized inverse Weibull, gamma-exponentiated inverse exponential, exponentiated inverse exponential, Zografos and Balakrishnan-generalized inverse Weibull, Zografos and Balakrishnan-inverse Weibull, Zografos and Balakrishnan-generalized inverse exponential, Zografos and Balakrishnan-inverse exponential, Zografos and Balakrishnan-generalized inverse Rayleigh, Zografos and Balakrishnan-inverse Rayleigh, and Zografos and Balakrishnan-Fréchet distributions as special cases is proposed and studied in detail. Some structural properties of this new distribution including density expansion, moments, Rényi entropy, distribution of the order statistics, moments of order

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statistics and L-moments are presented. Maximum likelihood estimation technique is used to estimate the model parameters and applications to a real datasets to illustrate its usefulness are presented.

Mathematics Subject Classification: 62E15; 60E05
Keywords: Zografos and Balakrishnan gamma generator; Gamma distribution; Inverse Weibull distribution; Maximum likelihood estimation

1 Introduction

The relevance and usefulness of the inverse Weibull (IW) distribution in various areas including reliability, and branching processes can be seen in Oluyede and Yang (2014), Kersey and Oluyede (2012), Calabria and Pulcini (1989, 1990, 1994) and in references therein. The IW model also provides a very good fit to data on times to breakdown of an insulating fluid, subject to constant tension (Badar and Priest (1982)), and references therein for additional results.

There are several new and important generalizations of distributions in the literature including those of Eugene et al. (2002) dealing with the beta-normal distribution and results on weighted inverse Weibull distribution by Sherina and Oluyede (2014). Pararai et al. (2014) developed a new class of generalized inverse Weibull distribution obtained via the use of Ristić and Balakrishnan (2012) alternative-gamma-generator given by equation (6) when $\lambda = 1$. Famoye et al. (2005) discussed and presented results on the beta-Weibull distribution. Nadarajah (2005) presented results on the exponentiated beta distribution. Kong and Sepanski (2007) presented the beta-gamma distribution.

In this note, we present, study and analyze the gamma-exponentiated or generalized inverse Weibull (GEIW or GGIW) distribution. The inverse Weibull (IW) cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta) = exp\{-(\alpha(x-x_0))^{-\beta}\}, \qquad x \geq 0,\, \alpha > 0,\, \beta > 0, \tag{1}$$

where α, x_0 and β are the scale, location and shape parameters respectively. The parameter x_0 is called the minimum life or guarantee time. When $\alpha =$

$\beta = 1$ and $x = x_0 + \alpha$, then $F(\alpha + x_0; 1, \beta) = F(\alpha + x_0; 1, 1) = e^{-1} = 0.3679$. This value is in fact the characteristic life of the distribution. We assume that $x_0 = 0$. The quantile function is $Q_F(y) = \{\frac{-\log(y)}{\alpha}\}^{-1/\beta}$. Note that when $\alpha = 1$, we have the Fréchet distribution function. Also, the IW probability density function (pdf) $f(x; \alpha, \beta)$, satisfies:

$$xf(x; \alpha, \beta) = \beta F(x; \alpha, \beta)(-\ln(F(x; \alpha, \beta)), \qquad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{2}$$

In a recent note, Zografos and Balakrishnan (2009) defined the gamma-generator (when $\lambda = 1$) with pdf $g(x)$ and cdf $G(x)$ (for $\delta > 0$) given by

$$g(x) = \frac{1}{\Gamma(\delta)\lambda^\delta}[-\log(\overline{F}(x))]^{\delta-1}(1 - F(x))^{(1/\lambda)-1} f(x), \tag{3}$$

and

$$G(x) = \frac{1}{\Gamma(\delta)\lambda^\delta} \int_0^{-\log(\overline{F}(x))} t^{\delta-1} e^{-t/\lambda} dt = \frac{\gamma(\delta, -\lambda^{-1}\log(\overline{F}(x)))}{\Gamma(\delta)}, \tag{4}$$

respectively, where $F(x)$ is a baseline cdf, $g(x) = dG(x)/dx$, $\Gamma(\delta) = \int_0^\infty t^{\delta-1}e^{-t}dt$ is the gamma function, and $\gamma(z, \delta) = \int_0^z t^{\delta-1}e^{-t}dt$ is the incomplete gamma function. The corresponding hazard rate function (hrf) is given by

$$h_G(x) = \frac{[-\log(1 - F(x))]^{\delta-1} f(x)(1 - F(x))^{(1/\lambda)-1}}{\lambda^\delta(\Gamma(\delta) - \gamma(-\lambda^{-1}\log(1 - F(x)), \delta))}. \tag{5}$$

When $\lambda = 1$, the distribution which of a special case of the family of distributions given in equation (3) is referred to as the ZB-G family of distributions. Also, when $\lambda = 1$, Ristić and Balakrishnan (2012) proposed an alternative gamma-generator defined by the cdf and pdf

$$G_2(x) = 1 - \frac{1}{\Gamma(\delta)\lambda^\delta} \int_0^{-\log F(x)} t^{\delta-1} e^{-t/\lambda} dt, \quad x \in \mathbf{R},\ \delta > 0, \tag{6}$$

and

$$g_2(x) = \frac{1}{\Gamma(\delta)\lambda^\delta}[-\log(F(x))]^{\delta-1}(F(x))^{(1/\lambda)-1} f(x), \tag{7}$$

respectively.

In this paper, we develop and present a generalization of the IW distribution via the family given in equation (3). Zografos and Balakrishnan (2009) motivated the ZB-G model as follows. Let $X_{(1)}, X_{(2)}, \ldots\ldots, X_{(n)}$ be lower record values from a sequence of independent and identically distributed (i.i.d.) random variables from a population with pdf $f(x)$. Then, the pdf of the n^{th} upper record value is given by equation (3), when $\lambda = 1$. A logarithmic transformation of the parent distribution F transforms the random variable X with density (3) to a gamma distribution. That is, if X has the density (3), then the random variable $Y = -\log[1 - F(X)]$ has a gamma distribution $GAM(\delta; 1)$ with density $k(y; \delta) = \frac{1}{\Gamma(\delta)} y^{\delta-1} e^{-y}$, $y > 0$. The opposite is also true, if Y has a gamma $GAM(\delta; 1)$ distribution, then the random variable $X = G^{-1}(1 - e^{-Y})$ has a ZB-G distribution (Zografos and Balakrishnan (2009)).

Ristić and Balakrishnan (2011) gave motivations for the new family of distributions given in equation (7) when $\lambda = 1$, that is, for $n \in N$, equation (7) is the pdf of the n^{th} lower record value of a sequence of independent and identically distributed (i.i.d.) variables. Ristić and Balakrishnan (2011) used the exponentiated exponential (EE) distribution with cdf $F(x) = (1 - e^{-\beta x})^{\alpha}$, where $\alpha > 0$ and $\beta > 0$, and $\lambda = 1$ in equation (7) to obtained and study the gamma-exponentiated exponential (GEE) model. See references therein for additional results on the GEE model. Pinho et al. (2012) presented results on the gamma-exponentiated Weibull distribution. In this note, we obtain a useful and natural extension of the IW distribution, which we refer to as the gamma-generalized inverse Weibull (GGIW) distribution. Note that if $\lambda = 1$ and $\delta = n + 1$, in equation (4), we obtain the cdf and pdf of the upper record values U given by

$$G_U(u) = \frac{1}{n!} \int_0^{-\log(1-F(u))} y^n e^{-y} dy, \quad \text{and} \quad g_U(u) = f(u)[-\log(1 - F(u))]^n / n!,$$

respectively. Similarly, from equation (7), the pdf of the lower record values is given by

$$g_L(t) = f(t)[-\log(F(t))]^n / n!.$$

In addition to the motivations provided by Zografos and Balakrishnan (2009), we are also interested in the generalization of the inverse Weibull distribution via the gamma-generator and establishing the relationship between the distributions in equations (3) and (7), and weighted distributions in general.

Weighted distribution provides a very useful approach to dealing with model specification and data interpretation problems. Fisher (1934) introduced the concept of weighted distribution, in order to study the effect of ascertainment upon estimation of frequencies. Rao (1965) unified concept of weighted distribution and use it to identify various sampling situations. Cox (1962) and Zelen (1974) introduced weighted distribution to present length biased sampling. Patil and Rao (1978) used weighted distribution as stochastic models in the study of harvesting and predation. The use of weighted distribution to model biased samples in various areas including medicine, ecology, reliability, and branching processes can be seen in Nanda and Jain (1999), Gupta and Keating (1985), Oluyede (1999) and in references therein.

Suppose Y is a non-negative random variable with its natural pdf $f(y;\underline{\theta})$, where $\underline{\theta}$ is a vector of parameters, then the pdf of the weighted random variable Y^w is given by

$$f^w(y;\underline{\theta},\underline{\beta}) = \frac{w(y,\underline{\beta})f(y;\underline{\theta})}{\omega}, \tag{10}$$

where the weight function $w(y,\underline{\beta})$ is a non-negative function, that may depend on the vector of parameters $\underline{\beta}$, and $0 < \omega = E(w(Y,\underline{\beta})) < \infty$ is a normalizing constant. A general class of weight function $w(y)$ is defined as follows

$$w(y) = y^k e^{ly} F^i(y) \overline{F}^j(y). \tag{11}$$

Setting $k = 0$; $k = j = i = 0$; $l = i = j = 0$; $k = l = 0$; $i \to j-1$; $j = n-i$; $k = l = i = 0$ and $k = l = j = 0$ in this weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively, where $F(y) = P(Y \leq y)$ and $\overline{F}(y) = 1 - F(y)$. If $w(y) = y$, then $Y^* = Y^w$ is called the size-biased version of Y.

This paper is organized as follows. In section 2, some basic results, the model, series expansion, sub-models, hazard and reverse hazard functions are presented. Moments and moment generating function are given in section 3. Section 4 contains some additional and useful results on Rényi entropy, the distribution of order statistics, moments of the order statistics and L-moments. In section 5, results on the estimation of the parameters of the GGIW distribution via the method of maximum likelihood are presented. Applications are given in section 6, followed by concluding remarks.

2 GGIW Distribution, Series Expansion and Sub-models

In this section, the GGIW distribution, density expansion and some of the sub-models are presented. First, we consider the generalized or exponentiated inverse Weibull (GIW or EIW) distribution given by

$$F_{GIW}(x;\eta,\beta) = (\exp[-(\alpha x)^{-\beta}])^{\theta} = \exp[-\eta x^{-\beta}], \quad x \geq 0,\ \alpha > 0,\ \beta > 0,\ \theta > 0, \tag{12}$$

where $\eta = \theta\alpha^{-\beta}$. By inserting the GIW distribution in equation (3), we obtain the cdf of the GGIW distribution as follows

$$G_{GGIW}(x) = \frac{1}{\Gamma(\delta)\lambda^{\delta}} \int_0^{-\log[1-e^{-\eta x^{-\beta}}]} t^{\delta-1} e^{-t/\lambda} dt = \frac{\gamma(-\lambda^{-1}\log(1-e^{-\eta x^{-\beta}}),\delta)}{\Gamma(\delta)}, \tag{13}$$

where $x > 0$, $\eta > 0$, $\beta > 0$, $\lambda > 0$, $\delta > 0$, and $\gamma(x,\delta) = \int_0^x t^{\delta-1}e^{-t}dt$ is the lower incomplete gamma function. The GGIW quantile function is obtained by solving the equation

$$G(Q_G(y)) = y, \quad 0 < y < 1. \tag{14}$$

The quantile function is

$$Q_G(y) = \eta^{-1/\beta}\left[-\log\left(1-\exp(-\lambda\gamma^{-1}(\Gamma(\delta)y,\delta))\right)\right]^{1/\beta}. \tag{15}$$

The GGIW pdf is given by

$$\begin{aligned} g_{GGIW}(x) &= \frac{\eta\beta x^{-\beta-1}e^{-\eta x^{-\beta}}}{\Gamma(\delta)\lambda^{\delta}} \\ &\times [-\log(1-e^{-\eta x^{-\beta}})]^{\delta-1}[1-e^{-\eta x^{-\beta}}]^{(1/\lambda)-1}. \end{aligned} \tag{16}$$

If a random variable X has the GGIW density, we write $X \sim GGIW(\eta,\beta,\lambda,\delta)$.

2.1 Expansion of GGIW Density Function

In this subsection, a series expansion of the GGIW density function is presented. Let $y = \exp[-\eta x^{-\beta}]$, and $\psi = 1/\lambda$, then using the series representation

$-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\Big[-\log(1-y)\Big]^{\delta-1} = y^{\delta-1}\Big[\sum_{m=1}^{\infty}\binom{\delta-1}{m} y^m \Big(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\Big)^m\Big],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\Big(\sum_{s=0}^{\infty} a_s y^s\Big)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{18}$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn and Ryzhik (2000)), the GGIW pdf can be written as

$$\begin{aligned}
g_{GGIW}(x) &= \frac{\eta\beta x^{-\beta-1}}{\Gamma(\delta)\lambda^{\delta}} y^{\delta}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m} b_{s,m} y^{m+s}\sum_{k=0}^{\infty}\binom{\psi-1}{k}(-1)^k y^k \\
&= \frac{\eta\beta x^{-\beta-1}}{\Gamma(\delta)\lambda^{\delta}}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\delta-1}{m}\binom{\psi-1}{k}(-1)^k b_{s,m} y^{\delta+m+s+k} \\
&= \frac{1}{\Gamma(\delta)\lambda^{\delta}}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\delta-1}{m}\binom{\psi-1}{k}(-1)^k b_{s,m} \\
&\times\ \eta\beta x^{-\beta-1} e^{-\eta(\delta+m+s+k)x^{-\beta}} \\
&= \frac{1}{\Gamma(\delta)\lambda^{\delta}}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\delta-1}{m}\binom{\psi-1}{k}(-1)^k \frac{b_{s,m}}{\delta+m+s+k} \\
&\times\ \eta(\delta+m+s+k)\beta x^{-\beta-1} e^{-\eta(\delta+m+s+k)x^{-\beta}},
\end{aligned}$$

where $f(x;\beta,\eta(\delta+m+s+k))$ is the generalized inverse Weibull pdf with parameters $\eta(\delta+m+s+k)$, and β. Let $C = \{(m,s,k) \in \mathbf{Z}_{+}^{\mathbf{3}}\}$, then the weights in the GGIW pdf above are

$$w_{\nu} = \frac{\psi^{\delta}}{\Gamma(\delta)}(-1)^k\binom{\delta-1}{m}\binom{\psi-1}{k}\frac{b_{m,s}}{\delta+m+s+k},$$

and the GGIW pdf can be written as

$$g_{GGIW}(x) = \sum_{\nu\in C} w_{\nu} f(x;\beta,\eta(\delta+m+s+k)). \tag{19}$$

It follows therefore that the GGIW density is a linear combination of the generalized or exponentiated inverse Weibull densities. The statistical and mathematical properties can be readily obtained from those of the generalized inverse Weibull distribution. For the convergence of equation (19), as well elsewhere in this paper including moments and Rényi entropy, note that for $\delta > 0$,

$$\begin{aligned}[-\log(1-y)]^{\delta-1} &= \left[y\left(1+y\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)\right]^{\delta-1} \\ &= y^{\delta-1}\sum_{k=0}^{\infty}\binom{\delta-1}{k}y^k\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^k,\end{aligned}$$

so that

$$\left[1+y\sum_{k=0}^{\infty}\frac{y^k}{k+2}\right]^{\delta-1}=\sum_{k=0}^{\infty}\binom{\delta-1}{k}y^k\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^k$$

is convergent if and only if $0 < \left(y\sum_{k=0}^{\infty}\frac{y^k}{k+2}\right)^k < 1\ \forall y \in (0,1)$, since $0 < y = e^{-\eta x^{-\beta}} < 1$, $x > 0$, $\eta, \beta > 0$. Now, $y\sum_{k=0}^{\infty}\frac{y^k}{k+2} = \frac{-\log(1-y)}{y} - 1$, so we must have $0 < \frac{-\log(1-y)}{y} - 1 < 1$. This leads to $1-y > \exp(-2y)$, and on the other hand $\exp(-y) = \sum_{k=0}^{\infty}\frac{(-1)^k y^k}{k!} > 1-y$. Thus, we have the system of inequalities $1-y > \exp(-2y)$ and $\exp(-y) > 1-y$, which is satisfied $\forall y \in (0, 0.7968)$.

Note that $g_{GGIW}(x)$ is a weighted pdf with weight function

$$w(x) = [-\log(1-F_{GIW}(x))]^{\delta-1}[1-F_{GIW}(x)]^{\frac{1}{\lambda}-1},$$

that is,

$$\begin{aligned}g_{GGIW}(x) &= \frac{[-\log(1-F_{GIW}(x))]^{\delta-1}[1-F_{GIW}(x)]^{\frac{1}{\lambda}-1}}{\lambda^{\delta}\Gamma(\delta)}f_{GIW}(x) \\ &= \frac{w(x)f_{GIW}(x)}{E_{F_{GIW}}(w(X))},\end{aligned}$$

where $0 < E_{F_{GIW}}\{[-\log(1-F_{GIW}(x))]^{\delta-1}[1-F_{GIW}(x)]^{\frac{1}{\lambda}-1}\} = \lambda^{\delta}\Gamma(\delta) < \infty$, is the normalizing constant. Similarly,

$$g_2(x) = \frac{[-\log(F_{GIW}(X))]^{\delta-1}[F_{GIW}(X)]^{\frac{1}{\lambda}-1}}{\lambda^{\delta}\Gamma(\delta)}f_{GIW}(x) = \frac{w(x)f_{GIW}(x)}{E_{F_{GIW}}(w(X))},$$

where $0 < E_{F_{GIW}}(w(X)) = E_{F_{GIW}}([-\log(F_{GIW}(X))]^{\delta-1}[F_{GIW}(X)]^{\frac{1}{\lambda}-1}) = \lambda^{\delta}\Gamma(\delta) < \infty$.

The graphs in Figure 1 are asymmetric and right skewed. For some combinations of the GGIW model parameter values the graph of the pdf can be L-shaped.

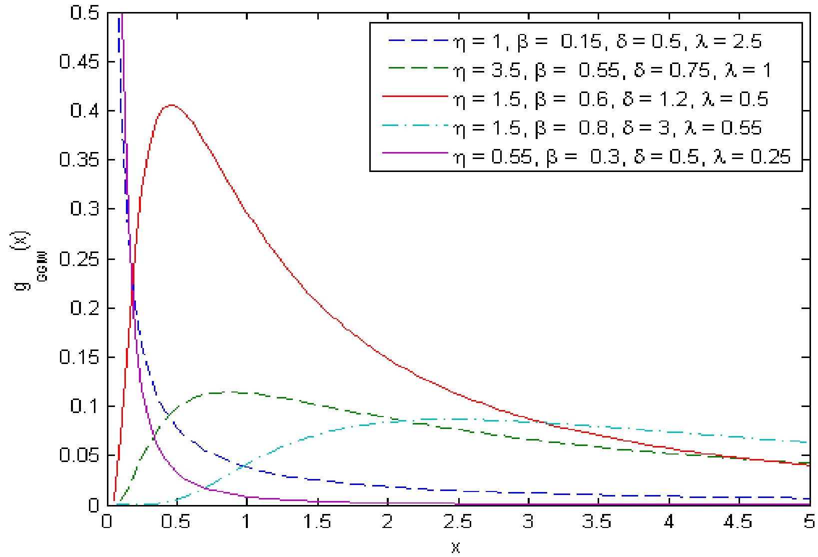

Figure 1: Plots of GGIW pdf for selected values of the parameters

2.2 Some Sub-models of the GGIW Distribution

Some of the sub-models of the GGIW distribution are listed below.

- If $\lambda = 1$, we obtain the gamma-generalized inverse Weibull distribution via the ZB-G (ZBIW) distribution. Also, with $\lambda = \beta = 1$, we have the ZB-inverse exponential (ZBIE) distribution. Similarly, if $\lambda = 1$, and $\beta = 2$, we obtain the ZB-inverse Rayleigh (ZBIR) distribution.

- If $\eta = 1$, we get the gamma-generalized Fréchet (GGF) distribution.

- When $\beta = 1$, we have the gamma-generalized inverse exponential (GGIE) distribution.

- If $\beta = 2$, we obtain the gamma-generalized inverse Rayleigh (GGIR) distribution.

- When $\delta = \lambda = 1$, we have the inverse Weibull (IW) distribution.

- If $\beta = 2$, and $\delta = \lambda = 1$, we obtain the inverse Rayleigh (IR) distribution.

- When $\delta = \beta = \lambda = 1$, we get the Inverse exponential (IE) distribution.

- When $\eta = \delta = \lambda = 1$, we obtain Fréchet (F) distribution.

2.3 Hazard and Reverse Hazard Functions

Let X be a continuous random variable with cdf F, and pdf f, then the hazard function, reverse hazard function and mean residual life functions are given by $h_F(x) = f(x)/\overline{F}(x)$, $\tau_F(x) = f(x)/F(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$, respectively. The functions $h_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent (Shaked and Shanthikumar (1994)). The hazard and reverse hazard functions of the GGIW distribution are given by

$$h_G(x) = \frac{\eta\beta x^{-\beta-1}e^{-\eta x^{-\beta}}(-\log(1-e^{-\eta x^{-\beta}}))^{\delta-1}[1-e^{-\eta x^{-\beta}}]^{\lambda^{-1}-1}}{\lambda^{\delta}(\Gamma(\delta)-\gamma(-\lambda^{-1}\log(1-e^{-\eta x^{-\beta}}),\delta))},$$

and

$$\tau_G(x) = \frac{\eta\beta x^{-\beta-1}e^{-\eta x^{-\beta}}(-\log(1-e^{-\eta x^{-\beta}}))^{\delta-1}[1-e^{-\eta x^{-\beta}}]^{\lambda^{-1}-1}}{\lambda^{\delta}(\gamma(-\lambda^{-1}\log(1-e^{-\eta x^{-\beta}}),\delta))},$$

for $x \geq 0$, $\eta > 0$, $\beta > 0$, $\lambda > 0$, $\delta > 0$, respectively. Plots of the GGIW hazard rate function for selected values of the parameters are give in Figure 2.

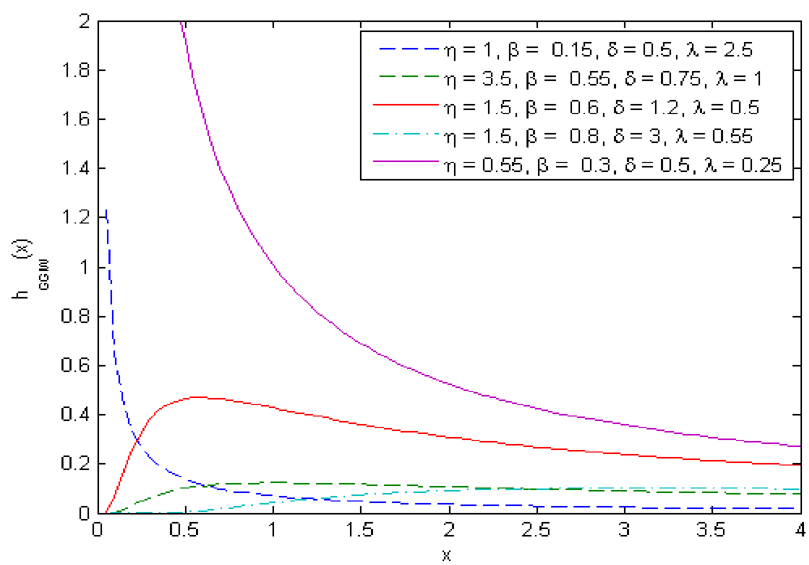

Figure 2: Plots of GGIW hazard function for selected values of the parameters

The graphs of the hazard rate function given in Figure 2 for five combinations of the parameter values are unimodal and upside down bathtub shaped.

3 Moments and Moment Generating Function

In this section, we obtain moments and moment generating function of the GGIW distribution. Let $\eta^* = \eta(\delta + m + s + k)$, and $Y \sim GIW(\beta, \eta^*)$. Note that from $Y \sim GIW(\beta, \eta^*)$, the j^{th} moment of the random variable Y is

$$E(Y^j) = (\eta^*)^{j/\beta}\Gamma(1 - j\beta^{-1}), \tag{23}$$

so that the j^{th} raw moment of GGIW distribution is given by

$$E(X^j) = \sum_{\nu \in C} w_\nu E(Y^j).$$

The moment generating function (MGF), for $|t| < 1$, is given by

$$M_X(t) = \sum_{\nu \in C} w_\nu M_Y(t) = \sum_{\nu \in C} \sum_{i=0}^{\infty} w_\nu \frac{t^i}{i!}(\eta^*)^{i/\beta}\Gamma(1 - i\beta^{-1}).$$

Theorem 3.1.

$$E\{[-\log(1 - F_{GIW}(X))]^r[(1 - F_{GIW}(X))^s]\} = \frac{\lambda^r\Gamma(r+\delta)}{(s\lambda+1)^\delta\Gamma(\delta)}.$$

If $s = 0$,

$$E[-\log(1 - F_{GIW}(X))^r] = \frac{\lambda^r\Gamma(r+\delta)}{\Gamma(\delta)},$$

and similarly, if $r = 0$,

$$E[(1 - F_{GIW}(X))^s] = [s\lambda + 1]^{-\delta}.$$

Proof:

$$\begin{aligned} E\{[-\log(1 - F_{GIW}(X))]^r[(1 - F_{GIW}(X))^s]\} &= \int_0^\infty \frac{[-\log(1 - F_{GIW}(x))]^{r+\delta-1}}{\lambda^\delta\Gamma(\delta)} \\ &\times [1 - F_{GIW}(x)]^{s+(1/\lambda)-1} f_{GIW}(x)dx \\ &= \frac{\lambda^r\Gamma(r+\delta)}{(s\lambda+1)^\delta\Gamma(\delta)}. \end{aligned}$$

If $s = 0$, we have

$$\begin{aligned} E[-\log(1 - F_{GIW}(X))^r] &= \int_0^\infty \frac{1}{\lambda^\delta\Gamma(\delta)}[-\log(1 - F_{GIW}(x))]^{r+\delta-1} \\ &\times [1 - F_{GIW}(x)]^{(1/\lambda)-1} f_{GIW}(x)dx \\ &= \frac{\lambda^{r+\delta}\Gamma(r+\delta)}{\lambda^\delta\Gamma(\delta)} \int_0^\infty \frac{f_{GIW}(x)}{\lambda^{r+\delta}\Gamma(r+\delta)} \\ &\times [-\log(1 - F_{GIW}(x))]^{r+\delta-1}[1 - F_{GIW}(x)]^{(1/\lambda)-1}dx \\ &= \frac{\lambda^{r+\delta}\Gamma(r+\delta)}{\lambda^\delta\Gamma(\delta)}. \end{aligned}$$

Let $\lambda^* = s + \frac{1}{\lambda}$, then with $r = 0$, we obtain

$$\begin{aligned} E[(1-F_{GIW}(X))^s] &= \int_0^\infty \frac{1}{\lambda^\delta \Gamma(\delta)} [-\log(1-F_{GIW}(x))]^{\delta-1} \\ &\times [1-F_{GIW}(x)]^{s+(1/\lambda)-1} f_{GIW}(x)dx \\ &= \int_0^\infty \frac{(\lambda^*)^\delta}{\Gamma(\delta)} [-\log(1-F_{GIW}(x))]^{\delta-1} \\ &\times \left(\frac{1}{\lambda\lambda^*}\right)^\delta [1-F_{GIW}(x)]^{\lambda^*-1} f_{GIW}(x)dx \\ &= [s\lambda+1]^{-\delta}. \end{aligned}$$

4 Rényi Entropy and Order Statistics

Order Statistics play an important role in probability and statistics. The concept of entropy plays a vital role in information theory. Entropy of a random variable is defined in terms of its probability distribution and is a good measure of randomness or uncertainty. In this section, we present Rényi entropy, the distribution of the order statistics and L-moments for the GGIW distribution.

4.1 Rényi Entropy

Rényi entropy is an extension of Shannon entropy. Rényi entropy of the GGIW distribution is defined to be

$$I_R(v) = \frac{1}{1-v} \log\left(\int_0^\infty [g_{GGIW}(x;\eta,\beta,\lambda,\delta)]^v dx\right), v \neq 1, v > 0.$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Note that

$$\begin{aligned} \int_0^\infty g^v_{GGIW}(x)dx &= \left(\frac{\eta\beta}{\lambda^\delta \Gamma(\delta)}\right)^v \int_0^\infty x^{-v\beta-v} e^{-v\eta x^{-\beta}} [1-e^{-\eta x^{-\beta}}]^{\frac{v}{\lambda}-1} \\ &\times [-\log(1-e^{-\eta x^{-\beta}})]^{v\delta-v} dx. \end{aligned}$$

Let $y = e^{-\eta x^{-\beta}}$, then using the same results as in section 2, we have for $\delta > 1$, and v/λ a natural number,

$$\begin{aligned}\int_0^\infty g^v_{_{GGIW}}(x)dx &= \left(\frac{\eta\beta}{\lambda^\delta\Gamma(\delta)}\right)^v \sum_{m=1}^\infty \sum_{s,k=0}^\infty (-1)^k \binom{v\delta - v}{m}\binom{(v/\lambda)-1}{k} b_{s,m} \\ &\times \int_0^\infty x^{-v\beta-v} e^{-\eta(v\delta+m+s+k)x^{-\beta}} dx \\ &= \frac{\eta^v\beta^{v-1}\Gamma(v+\frac{1}{\beta}(v-1))}{(\lambda^\delta\Gamma(\delta))^v} \cdot \sum_{m=0}^\infty \sum_{s,k=0}^\infty (-1)^k \binom{v\delta - v}{m}\binom{(\frac{v}{\lambda})-1}{k} \\ &\times b_{s,m}[\eta(v\delta+m+s+k)]^{\frac{1}{\beta}(1-v)-v}.\end{aligned}$$

Consequently, Rényi entropy of the GGIW distribution is given by

$$\begin{aligned}I_R(v) &= \left(\frac{1}{1-v}\right)\log\left[\frac{\eta^v\beta^{v-1}\Gamma(v+\frac{1}{\beta}(v-1))}{(\lambda^\delta\Gamma(\delta))^v}\right. \\ &\times \sum_{m=0}^\infty \sum_{s,k=0}^\infty (-1)^k \binom{v\delta - v}{m}\binom{(\frac{v}{\lambda})-1}{k} \\ &\times \left. b_{s,m}[\eta(v\delta+m+s+k)]^{\frac{1}{\beta}(1-v)-v}\right],\end{aligned}$$

for $v > 0$, $v \neq 1$.

4.2 Order Statistics

The distribution of the i^{th} order statistic and the j^{th} moment of the distribution of the i^{th} order statistic from the GGIW distribution are presented in this subsection. Moments of order statistics are often used in several areas including reliability, engineering, biometry, insurance and quality control for the prediction of future failures times from a set of past or previous failures. $L-$moments (Hoskings (1990)) are expectations of some linear combinations of order statistics and they exist whenever the mean of the distribution exits, even when some higher moments may not exist are particularly important in probability and statistics.

Let $X_1, X_2, ..., X_n$ be independent and identically distributed GGIW random variables. We apply the general binomial series expansion, that is,

$$[1-G(x)]^{n-i} = \sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[G(x)]^{i+j-1}$$

and the result on power series raised to a positive inter used in section 2 to obtain the pdf of the i^{th} order statistic from the GGIW distribution. The pdf of the i^{th} order statistic from the GGIW pdf $g_{_{GGIW}}(x) = g(x)$ is given by

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[G(x)]^{i+j-1} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j} \\
&\times \left[\frac{\gamma(-\lambda^{-1}\log(1-\overline{F}(x),\delta))}{\Gamma(\delta)}\right]^{i+j-1}.
\end{aligned}$$

Using the fact that $\gamma(x,\delta) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{m+\delta}}{(m+\delta)m!}$, and setting $c_m = (-1)^m/((m+\delta)m!)$, the pdf of the i^{th} order statistic from the GGIW distribution can be written as follows

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}} \\
&\times [-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)} \\
&\times \left[\sum_{m=0}^{\infty}\frac{(-1)^m(-\lambda^{-1}\log(\overline{F}(x)))^m}{(m+\delta)m!}\right]^{i+j-1} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j} \\
&\times \frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}}[-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)} \\
&\times \sum_{m=0}^{\infty}d_{m,i+j-1}(-\lambda^{-1}\log(\overline{F}(x)))^m,
\end{aligned}$$

where $d_0 = c_0^{(i+j-1)}$, $d_{m,i+j-1} = (mc_0)^{-1}\sum_{l=1}^{m}[(i+j-1)l - m + l]c_l d_{m-l,i+j-1}$.

It follows therefore that

$$\begin{aligned} g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\ &\times\ [-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)+m} \\ &= \frac{n![-\log(\overline{F}(x))]^{\delta-1}[\overline{F}(x)]^{\psi-1}f(x)}{(i-1)!(n-i)!\Gamma(\delta)\lambda^{\delta}}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\ &\times\ [-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)+m} \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j}} \\ &\times\ \frac{[-\log(\overline{F}(x))]^{\delta(i+j-1)+m+\delta-1}[\overline{F}(x)]^{\psi-1}f(x)}{\lambda^{i+j}}. \\ &= \frac{n!}{(i-1)!(n-1)!\Gamma(\delta)\lambda}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\binom{i-1}{j}\frac{(-1)^j d_{m,n-i+j}}{[\Gamma(\delta)]^{n-i+j+1}} \\ &\times\ \frac{\Gamma(\delta(n-i+j)+m+\delta)}{\Gamma(\delta(n-i+j)+m+\delta)}[-\lambda^{-1}\log(\overline{F}(x))]^{\delta(n-i+j)+m+\delta-1}[\overline{F}(x)]^{\psi-1}f(x). \end{aligned}$$

That is, the pdf of the i^{th} order statistic from the GGIW distribution is given by

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,n-i+j}}{[\Gamma(\delta)]^{i+j}}\frac{1}{\lambda^{\delta(i+j)+m}} \\ &\times\ [-\log(\overline{F}(x))]^{\delta(i+j)+m-1}[\overline{F}(x)]^{\psi-1}f(x) \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}\Gamma(\delta(i+j)+m)}{[\Gamma(\delta)]^{i+j}} \\ &\times\ \frac{[-\log(\overline{F}(x))]^{\delta(i+j)+m-1}[\overline{F}(x)]^{\psi-1}f(x)}{\Gamma(\delta(i+j)+m)\lambda^{\delta(i+j)+m}} \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}\Gamma(\delta(i+j)+m)}{[\Gamma(\delta)]^{i+j}} \\ &\times\ g(x;\eta,\beta,\lambda,\delta^*), \end{aligned}$$

where $g(x;\eta,\beta,\lambda,\delta^*)$ is the GGIW pdf with parameters $\eta,\beta,\ \lambda$, and shape parameter $\delta^*=\delta(i+j)+m$. It follows therefore that the j^{th} moment of the distribution of the i^{th} order statistic from the GGIW distribution is given by

$$\begin{aligned} E(X_{i:n}^{j}) &= \frac{n!}{(i-1)!(n-i)!\Gamma(\delta)} \sum_{\nu \in C} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^{j} w_{\nu} d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j}} \\ &\times \Gamma(\delta(i+j)+m)(\eta^{*})^{j/\beta}\Gamma(1-j\beta^{-1}), \end{aligned} \quad (29)$$

for $j < \beta$. These moments are often used in several areas including reliability, engineering, biometry, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

4.3 L-moments

$L-$moments (Hoskings (1990)) are relatively robust to the effects of outliers and are given by

$$\lambda_{k+1} = \frac{1}{k+1} \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} E(X_{k+1-j:k+1}), \quad k = 0, 1, 2, ... \quad (30)$$

The $L-$moments of the GGIW distribution can be readily obtained from equation (29). The first four $L-$moments are given by $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$, respectively.

5 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample from the GGIW distribution and $\mathbf{\Theta} = (\eta, \beta, \lambda, \delta)$ the vector of model parameters. The likelihood function is given by

$$\begin{aligned} L(\eta, \beta, \lambda, \delta) &= \frac{(\eta\beta)^{n}}{[\lambda^{\delta}\Gamma(\delta)]^{n}} e^{-\eta \sum_{i=1}^{n} x_i^{-\beta}} \prod_{i=1}^{n} \Bigg\{ x_i^{-\beta-1} \\ &\times \left[-\log\left(1 - e^{-\eta x_i^{-\beta}}\right)\right]^{\delta-1} \left[1 - e^{-\eta x_i^{-\beta}}\right]^{(1/\lambda)-1} \Bigg\}. \end{aligned} \quad (31)$$

Now, the log-likelihood function denoted by ℓ is given by

$$\begin{aligned}\ell &= \log[L(\eta,\beta,\lambda,\delta)] \\ &= n\log(\eta)+n\log(\beta)-n\log(\Gamma(\delta))-n\delta\log(\lambda)+(-\beta-1)\sum_{i=1}^{n}\log(x_i) \\ &- \eta\sum_{i=1}^{n}x_i^{-\beta}+(\delta-1)\sum_{i=1}^{n}\log\left[-\log\left(1-e^{-\eta x_i^{-\beta}}\right)\right] \\ &+ \left(\frac{1}{\lambda}-1\right)\sum_{i=1}^{n}\log\left(1-e^{-\eta x_i^{-\beta}}\right). \end{aligned} \tag{32}$$

The entries of the score function are given by

$$\begin{aligned}\frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta}-\sum_{i=1}^{n}\log(x_i)+\eta\sum_{i=1}^{n}x_i^{-\beta}\log(x_i) \\ &- (\delta-1)\sum_{i=1}^{n}\frac{\eta x_i^{-\beta}e^{-\eta x_i^{-\beta}}\log(x_i)}{(1-e^{-\eta x_i^{-\beta}})\log(1-e^{-\eta x_i^{-\beta}})} \\ &- \left(\frac{1}{\lambda}-1\right)\sum_{i=1}^{n}\frac{\eta x_i^{-\beta}e^{-\eta x_i^{-\beta}}\log(x_i)}{(1-e^{-\eta x_i^{-\beta}})},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\eta} &= \frac{n}{\eta}-\sum_{i=1}^{n}x_i^{-\beta}+(\delta-1)\sum_{i=1}^{n}\frac{x_i^{-\beta}e^{-\eta x_i^{-\beta}}}{(1-e^{-\eta x_i^{-\beta}})\log(1-e^{-\eta x_i^{-\beta}})} \\ &+ \left(\frac{1}{\lambda}-1\right)\sum_{i=1}^{n}\frac{x_i^{-\beta}e^{-\eta x_i^{-\beta}}}{(1-e^{-\eta x_i^{-\beta}})},\end{aligned}$$

$$\frac{\partial\ell}{\partial\delta}=-\frac{n\Gamma'(\delta)}{\Gamma(\delta)}-n\log(\lambda)+\sum_{i=1}^{n}\log\left(-\log\left(1-e^{-\eta x_i^{-\beta}}\right)\right),$$

and

$$\frac{\partial\ell}{\partial\lambda}=-\frac{n\delta}{\lambda}-\frac{1}{\lambda^2}\sum_{i=1}^{n}\log\left(1-e^{-\eta x_i^{-\beta}}\right),$$

respectively. The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters η,β,λ and δ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Theta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell}{\partial\eta},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\delta})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix (FIM) given by $\mathbf{I}(\mathbf{\Theta})=[\mathbf{I}_{\theta_i,\theta_j}]_{4X4}=$

$E(-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j})$, $i,j=1,2,3,4$, can be numerically obtained by MATHLAB or R software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Theta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Theta}}) \approx \left[-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\bigg|_{\mathbf{\Theta}=\hat{\mathbf{\Theta}}}\right]_{4X4}, \quad i,j=1,2,3,4. \tag{35}$$

For a given set of observations, the matrix given in equation (35) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software. Elements of the observed information matrix are given in the Appendix.

The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Theta}}=(\hat{\eta},\hat{\beta},\hat{\lambda},\hat{\delta})$ be the maximum likelihood estimate of $\mathbf{\Theta}=(\eta,\beta,\lambda,\delta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}}-\mathbf{\Theta}) \xrightarrow{d} N_4(\underline{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. The multivariate normal distribution $N_4(\underline{\mathbf{0}}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\underline{\mathbf{0}}=(0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. A large sample $100(1-\alpha)\%$ confidence intervals for η,β,λ, and δ are:

$$\widehat{\eta}\pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\eta\eta}(\widehat{\mathbf{\Theta}})},\ \widehat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta\beta}(\widehat{\mathbf{\Theta}})},\ \widehat{\lambda}\pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\lambda\lambda}(\widehat{\mathbf{\Theta}})},\ \widehat{\delta}\pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\delta\delta}(\widehat{\mathbf{\Theta}})},$$

respectively, where $I^{-1}_{\eta\eta}(\hat{\mathbf{\Theta}})$, $I^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})$, $I^{-1}_{\lambda\lambda}(\hat{\mathbf{\Theta}})$, and $I^{-1}_{\delta\delta}(\hat{\mathbf{\Theta}})$ are the diagonal elements of $I_n^{-1}(\hat{\mathbf{\Theta}})$, and $Z_{\frac{\alpha}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

The maximum likelihood estimates (MLEs) of the GGIW parameters η, β, λ, and δ are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC=2p-2\ln(L)$, Bayesian Information Criterion, $BIC=p\ln(n)-2\ln(L)$, and Consistent Akaike Information Criterion, $AICC=AIC+2\frac{p(p+1)}{n-p-1}$, where $L=L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Tables 1, 2, and 3. The values of the Kolmogorov-Smirnov statistic, $KS=\max_{1\le i\le n}\{G(x_i)-\frac{i-1}{n},\frac{i}{n}-G(x_i)\}$ are also presented in Tables 1, 2, and 3. The GGIW distribution is fitted to the datasets and compared to the fits for the GGIE, GIW, IW and ZBIE distributions.

We can use the likelihood ratio (LR) test to compare the fit of the GGIW distribution with its sub-models for a given dataset. For example, to test $\lambda = \delta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\eta}, \hat{\beta}, \hat{\lambda}, \hat{\delta})) - \ln(L(\tilde{\eta}, \tilde{\beta}, 1, 1))]$, where $\hat{\eta}$, $\hat{\beta}$, $\hat{\lambda}$, and $\hat{\delta}$, are the unrestricted estimates, and $\tilde{\eta}$, and $\tilde{\beta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

6 Applications

In this section, we present examples to illustrate the flexibility of the GGIW distribution and its sub-models for data modeling. Estimates of the parameters of GGIW distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov statistic (KS) are given in Tables 1, 2, and 3. Plots of the fitted densities and the histogram of the data are given in Figures 3, 4 and 5. Probability plots (Chambers et al. (1983)) are also presented in Figures 3, 4 and 5. For the probability plot, we plotted $G_{GGIW}(x_{(j)}; \hat{\eta}, \hat{\beta}, \hat{\lambda}, \hat{\delta})$ against $\dfrac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. We also computed a measure of closeness of each plot to the diagonal line. This measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[G_{GGIW}(x_{(j)}; \hat{\eta}, \hat{\beta}, \hat{\lambda}, \hat{\delta}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2 .$$

6.1 Guinea Pig Survival Times Data

The first dataset from Bjerkedal (1960) represents the survival time, in days, of guinea pigs injected with different doses of tubercle bacilli. It is known that guinea pigs have high susceptibility of human tuberculosis. The dataset consists of 72 observations.

Table 1: Estimates of Models for Bjerkedal Data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW($\eta,\beta,\lambda,\delta$)	6.7266	0.3096	0.03433	5.8272	780.5	788.5	789.1	797.6	0.1944	0.7453
	(32.6026)	(0.7888)	(0.1637)	(41.1586)						
GGIE($\eta,1,\lambda,\delta$)	0.05157	1	0.06965	104.94	780.6	786.6	787.0	793.5	0.0972	0.1771
	(0.3388)		(0.06418)	(190.19)						
GIW($\eta,\beta,1,1$)	283.84	1.4148	1	1	791.3	795.3	795.5	799.9	0.3333	3.0557
	(125.63)	(0.1173)								
IE($\eta,1,1,1$)	60.0975	1	1	1	805.3	807.3	807.4	809.6	0.4444	6.2891
	(7.0826)									
ZBIE($\eta,1,1,\delta$)	230.68	1	1	0.279	797	801	801.2	805.6	0.625	13.0313
	(130.53)			(0.1622)						

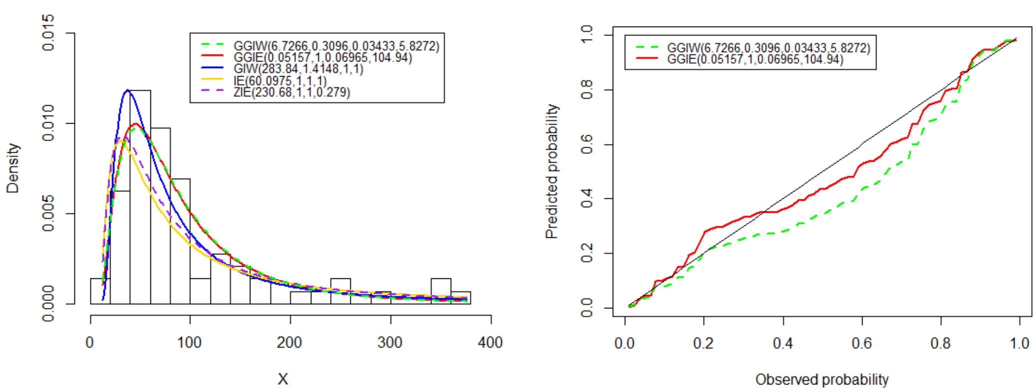

Figure 3: Fitted Densities and Probability Plots for Bjerkedal (pigs) Data

For the Bjerkedal data, the likelihood ratio (LR) test statistic indicates that there is no significant difference between the GGIE and GGIW distributions. There are significant differences between the $GGIW$ and the sub-models GIW, IE, and $ZBIE$, respectively, based on the LR tests. The value of the statistics AIC, AICC, BIC and KS are smaller for GGIE model. The value of SS is also smaller for this model, so we conclude that the GGIE distribution is a "superior" fit for this data.

6.2 Price of Cars Data

This example consists of price of 428 new vehicles for the 2004 year. The data was published in the Kiplinger's Personal Finance magazine, December 2003. See Huang and Oluyede (2016) for additional details.

Table 2: Estimates of Models for Car Prices Data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW$(\eta, \beta, \lambda, \delta)$	0.001651	6.7706	0.8001	16.9713	1488	1496	1496.1	1512.3	0.0701	0.7962
	(0.1277)	(1.0087)	(0.6555)	(22.3023)						
GGIE$(\eta, 1, \lambda, \delta)$	1.5848	1	0.1511	5.8679	1488	1494.9	1494.9	1507	0.1215	2.6045
	(2.0889)		(0.5504)	(8.4821)						
GIW$(\eta, \beta, 1, 1)$	6.7735	2.3166	1	1	1506.5	1510.5	1510.5	1518.6	0.2477	14.2982
	(0.4850)	(0.08417)								
IE$(\eta, 1, 1, 1)$	2.5838	1	1	1	1856.8	1858.8	1858.9	1862.9	0.5584	55.7895
	(0.1249)									
ZBIE$(\eta, 1, 1, \delta)$	8.6363	1	1	0.3176	1789.1	1793.1	1793.2	1801.3	0.715	96.3835
	(1.419)			(0.05316)						

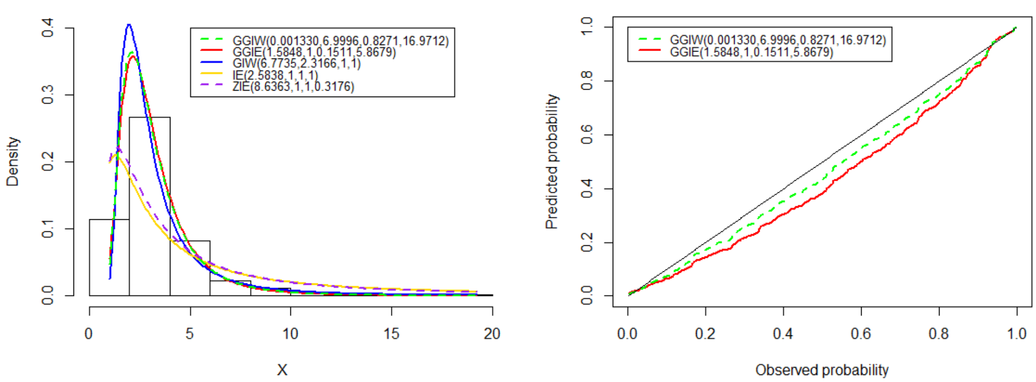

Figure 4: Fitted Densities and Probability Plots for Car Prices Data

The LR test of $H_0 : GGIE$ against $H_a : GGIW$ shows that there is no significant difference between these two models. However, there are significant differences between the $GGIW$ and the sub-models GIW, IE, and $ZBIE$, respectively, based on the LR tests. However, the values of KS statistic and SS from Table 2 supports the $GGIW$ distribution as a "better" or "superior" fit for the car prices data when compared to the nested models.

6.3 Fatigue Failure Times of Ball Bearing Data

In this example, we consider a real life dataset given by Lawless (1982). The data represents the fatigue failure times of ball bearings: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

Table 3: Estimates of Models for Lawless (1982) Ball Bearing Data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW$(\eta,\beta,\lambda,\delta)$	49.0531	7.8745	0.6250	46.0324	227.1	235.1	237.3	239.6	0.1304	0.0261
	(140.28)	(0.8175)	(0.1723)	(12.6340)						
GGIE$(\eta,1,\lambda,\delta)$	0.2745	1	0.05187	104.98	226.8	232.8	234.0	236.2	0.087	0.0247
	(1.6121)		(0.05727)	(226.71)						
GIW$(\eta,\beta,1,1)$	1240.49	1.8344	1	1	231.6	235.6	236.2	237.8	0.3478	0.8565
	(1231.6)	(0.2692)								
IE$(\eta,1,1,1)$	55.0595	1	1	1	243.5	245.5	245.6	246.6	0.5652	2.7320
	(11.4807)									
ZBIE$(\eta,1,1,\delta)$	194.37	1	1	0.3013	239.9	243.9	244.5	246.2	0.7391	5.0725
	(144.05)			(0.2288)						

The LR test statistic for the hypothesis H_0: $GGIE$ against H_a: $GGIW$, shows that we do not have enough evidence to reject H_0 in favor of H_a. There are significant differences between the $GGIW$ and the sub-models GIW, IE, and $ZBIE$, respectively, based on the LR tests. The values of the SS and of the KS statistic also support the GGIE distribution as a "better" or "superior" fit for this data.

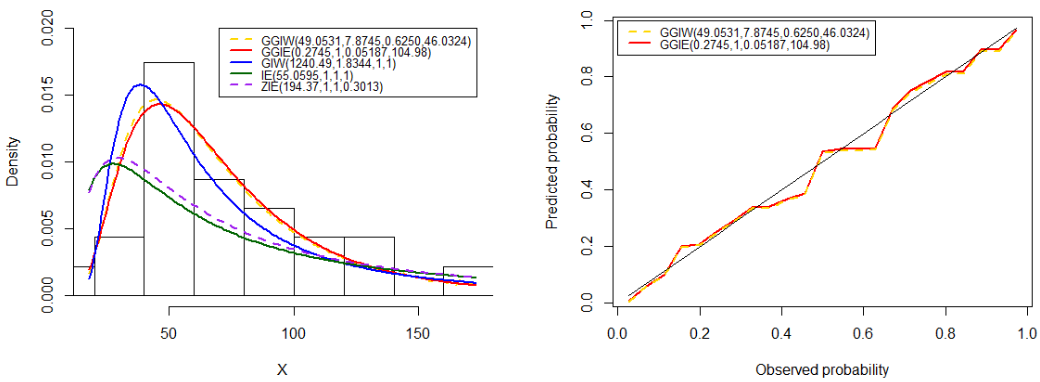

Figure 5: Fitted Density and Probability Plots for Lawless Ball Bearing Data

7 Concluding Remarks

A new class of generalized inverse Weibull distribution called the gamma-generalized inverse Weibull distribution is proposed and studied in details. The GGIW distribution has the GGIE, GIR, IW, IE, IR, ZBGIW, ZBGIE, ZBGIR and Fréchet distributions as special cases. The density of this new class of distributions can be expressed as a linear combination of GIW density functions. The GGIW distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, distribution of order statistics and Rényi entropy. Maximum likelihood estimation technique was used to estimate the model parameters. Finally, the GGIW distribution and some of its sub-models are fitted to real datasets in order to illustrate the applicability and usefulness of this new distribution.

Appendix

Elements of the observed information matrix of the GGIW distribution can be readily obtained from the second and mixed partial derivatives of $\ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)$ given by:

$$\begin{aligned}
\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\eta^2} &= -\frac{e^{-2\eta x^{-\beta}}\left(\frac{1}{\lambda}-1\right)x^{-2\beta}}{\left(1-e^{-\eta x^{-\beta}}\right)^2}-\frac{e^{-\eta x^{-\beta}}\left(\frac{1}{\lambda}-1\right)x^{-2\beta}}{1-e^{-\eta x^{-\beta}}}\\
&-\frac{1}{\eta^2}-\frac{(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln^2\left(1-e^{-\eta x^{-\beta}}\right)}\\
&-\frac{(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln\left(1-e^{-\eta x^{-\beta}}\right)}\\
&-\frac{(\delta-1)e^{-\eta x^{-\beta}}x^{-2\beta}}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\eta\partial\beta} &= x^{-\beta}\ln(x)-\frac{\eta\left(\frac{1}{\lambda}-1\right)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2}\\
&+\frac{\eta\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-2\beta}\ln(x)}{1-e^{-\eta x^{-\beta}}}\\
&-\frac{\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-\beta}\ln(x)}{1-e^{-\eta x^{-\beta}}}\\
&+\frac{\eta(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln^2\left(1-e^{-\eta x^{-\beta}}\right)}\\
&-\frac{\eta(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln\left(1-e^{-\eta x^{-\beta}}\right)}\\
&+\frac{\eta(\delta-1)e^{-\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)}\\
&-\frac{(\delta-1)e^{-\eta x^{-\beta}}x^{-\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},
\end{aligned}$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\eta\partial\lambda} = \frac{e^{-\eta x^{-\beta}}x^{-\beta}}{\lambda^2\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\eta\partial\delta} = \frac{e^{-\eta x^{-\beta}}x^{-\beta}}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\begin{aligned}\frac{\partial^2 \ln g_{_{GGIW}}(x;\eta,\beta,\lambda,\delta)}{\partial\beta^2} &= -\frac{1}{\beta^2}-\frac{\eta^2\left(\frac{1}{\lambda}-1\right)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2}\\ &\quad -\frac{\eta^2\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{1-e^{-\eta x^{-\beta}}}\\ &\quad -\eta x^{-\beta}\ln^2(x)+\frac{\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-\beta}\ln^2(x)}{1-e^{-\eta x^{-\beta}}}\\ &\quad -\frac{\eta^2(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln^2\left(1-e^{-\eta x^{-\beta}}\right)}\\ &\quad -\frac{\eta^2(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln\left(1-e^{-\eta x^{-\beta}}\right)}\\ &\quad -\frac{\eta^2(\delta-1)e^{-\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)}\\ &\quad +\frac{(\delta-1)e^{-\eta x^{-\beta}}x^{-\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},\end{aligned}$$

$$\frac{\partial^2 \ln g_{_{GGIW}}(x;\eta,\beta,\lambda,\delta)}{\partial\beta\partial\lambda} = \frac{\eta e^{-\eta x^{-\beta}}x^{-\beta}\ln(x)}{\lambda^2\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\frac{\partial^2 \ln g_{_{GGIW}}(x;\eta,\beta,\lambda,\delta)}{\partial\beta\partial\delta} = -\frac{\eta e^{-\eta x^{-\beta}}x^{-\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\frac{\partial^2 \ln g_{_{GGIW}}(x;\eta,\beta,\lambda,\delta)}{\partial\lambda^2} = \frac{\delta}{\lambda^2}+\frac{2\ln\left(1-e^{-\eta x^{-\beta}}\right)}{\lambda^3},$$

$$\frac{\partial^2 \ln g_{_{GGIW}}(x;\eta,\beta,\lambda,\delta)}{\partial\lambda\partial\delta} = -\frac{1}{\lambda},$$

and

$$\frac{\partial^2 \ln g_{_{GGIW}}(x;\eta,\beta,\lambda,\delta)}{\partial\delta^2} = -\Psi'(\delta).$$

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