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Some Numerical Methods for Options Valuation

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Abstract

Numerical Methods form an important part of options valuation and especially in cases where there is no closed form analytic formula. We discuss three numerical methods for options valuation namely Binomial model, Finite difference methods and Monte Carlo simulation method. Then we compare the convergence of these methods to the analytic Black-Scholes price of the options. Among the methods considered, Crank Nicolson finite difference method is unconditionally stable, more accurate and converges faster than binomial model and Monte Carlo Method when pricing vanilla options, while Monte carlo simulation method is good for pricing path dependent options.

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1 Introduction

Numerical methods are needed for pricing options in cases where analytic solutions are either unavailable or not easily computable. Examples of the former include American style options and most discretely observed path dependent options, while the latter includes the analytic formula for valuing continuously observed Asian options which is very hard to calculate for a wide range of parameter values encountered in practice.

The subject of numerical methods in the area of options valuation and hedging is very broad. A wide range of different types of contracts are available and in many cases there are several candidate models for the stochastic evolution of the underlying state variables [13].

Now, we present an overview of three popular numerical methods available in the context of Black-Scholes-Merton [2, 8] for vanilla and path dependent options valuation which are binomial model for pricing options based on risk-neutral valuation derived by Cox-Ross-Rubinstein [6], finite difference methods for pricing derivative governed by solving the underlying partial differential equations was considered by Brennan and Schwarz [5] and Monte Carlo approach introduced by Boyle [3] for pricing European option and path dependent options. The comparative study of finite difference method and Monte Carlo method for pricing European option was considered by [9]. These procedures provide much of the infrastructure in which many contributions to the field over the past three decades have been centered.

2 Numerical Methods for Options Valuation

This section presents three numerical methods for options valuation namely:

- Binomial Methods.
- Finite Difference Methods.
- Monte Carlo Methods.

2.1 Binomial Model

This is defined as an iterative solution that models the price evolution over the whole option validity period. For some types of options such as the American options, using an iterative model is the only choice since there is no known closed form solution that predicts price over time. Black-Scholes model seems dominated the option pricing, but it is not the only popular model, the Cox-Ross-Rubinstein (CRR) "Binomial" model has a large popularity. The binomial model was first suggested by Cox-Ross-Rubinstein model in paper "Option Pricing: A Simplified Approach" [6] and assumes that stock price movements are composed of a large number of small binomial movements. The Cox-Ross-Rubinstein model [7] contains the Black-Scholes analytical formula as the limiting case as the number of steps tends to infinity.

We know that after one time period, the stock price can move up to Su with probability p or down to Sd with probability $(1-p)$. Therefore the corresponding value of the call option at the first time movement δt is given by

$$f_u = \max(Su - K, 0) \tag{1}$$

$$f_d = \max(Sd - K, 0) \tag{2}$$

where f_u and f_d are the values of the call option after upward and downward movements respectively.

We need to derive a formula to calculate the fair price of the option. The risk neutral call option price at the present time is

$$f = e^{-r\delta t}[pf_u + (1-p)f_d] \tag{3}$$

where the risk neutral probability is given by

$$p = \frac{e^{r\delta t} - d}{u - d} \tag{4}$$

Now, we extend the binomial model to two periods. Let f_{uu} denote the call value at time $2\delta t$ for two consecutive upward stock movement, f_{ud} for one downward and one upward movement and f_{dd} for two consecutive downward movement of the stock price. Then we have

$$f_{uu} = \max(Suu - K, 0) \tag{5}$$

$$f_{ud} = \max(Sud - K, 0) \tag{6}$$

$$f_{dd} = \max(Sdd - K, 0 \tag{7}$$

The values of the call options at time δt are

$$f_u = e^{-r\delta t}[pf_{uu} + (1-p)f_{ud}] \tag{8}$$

$$f_d = e^{-r\delta t}[pf_{ud} + (1-p)f_{dd}] \tag{9}$$

Substituting (8) and (9) into (3), we have

$$f = e^{-r\delta t}[pe^{-r\delta t}(pf_{uu} + (1-p)f_{ud}) + (1-p)e^{-r\delta t}(pf_{ud} + (1-p)f_{dd})]$$

$$f = e^{-2r\delta t}\left[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right] \tag{10}$$

(10) is called the current call value using time $2\delta t$, where the numbers p^2, $2p(1-p)$ and $(1-p)^2$ are the risk neutral probabilities that the underlying asset prices Suu, Sud and Sdd respectively are attained.

We generalize the result in (10) to value an option at $T = N\delta t$ as

$$f = e^{-Nr\delta t}\sum_{j=0}^{N}\tbinom{N}{j}p^j(1-p)^{N-j}f_{u^j d^{N-j}}$$

$$f = e^{-Nr\delta t}\sum_{j=0}^{N}\binom{N}{j}p^j(1-p)^{N-j}\max(Su^j d^{N-j} - K, 0) \tag{11}$$

where $f_{u^j d^{N-j}} = \max(Su^j d^{N-j} - K, 0)$ and $\binom{N}{j} = \frac{N!}{(N-j)!j!}$ is the binomial coefficient . We assume that m is the smallest integer for which the option's intrinsic value in (11) is greater than zero. This implies that $Su^m d^{N-m} \geq K$. Then (11) is written as

$$f = Se^{-Nr\delta t}\sum_{j=0}^{N}\binom{N}{j}p^j(1-p)^{N-j}u^j d^{N-j}$$

$$-Ke^{-Nr\delta t}\sum_{j=0}^{N}\binom{N}{j}p^j(1-p)^{N-j} \tag{12}$$

which gives us the present value of the call option.

The term $e^{-Nr\delta t}$ is the discounting factor that reduces f to its present value. The first term $\binom{N}{j}p^j(1-p)^{N-j}$ is the binomial probability of j upward movements to occur after the first N trading periods and $Su^j d^{N-j}$ is the corresponding value of the asset after j upward move of the stock price.

The second term is the present value of the option‘s strike price. Let $R = e^{r\delta t}$, we substitute R in the first term in (11) to yield

$$f = SR^{-N} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} u^j d^{N-j} - Ke^{-Nr\delta t} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j}$$

$$= S \sum_{j=0} \binom{N}{j} \left[R^{-1}pu\right]^j \left[R^{-1}(1-p)d\right]^{N-j}$$

$$-Ke^{-Nr\delta t} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} \tag{13}$$

Now, let $\Phi(m; N, p)$ be the binomial distribution function. That is

$$\Phi(m; N, p) = \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} \tag{14}$$

(14) is the probability of at least m success in N independent trials, each resulting in a success with probability p and in a failure with probability $(1-p)$. Then, letting $\hat{p} = R^{-1}pu$, we easily see that $R^{-1}(1-p)d = 1-\hat{p}$. Consequently it follows from (13) that

$$f = S\Phi(m; N, \hat{p}) - Ke^{-rT}\Phi(m; N, p) \tag{15}$$

where $\delta t = \frac{T}{N}$.

The model in (15) was developed by Cox-Ross-Rubinstein [3] and we will refer to it as CRR model. The corresponding value of the European put option can be obtained as

$$f = Ke^{-rT}\Phi(m; N, p) - S\Phi(m; N, \hat{p}) \tag{16}$$

using call put relationship given by

$$C_E + Ke^{-rt} = P_E + S \tag{17}$$

where r denotes the risk free interest rate, C_E is the European Call, P_E is European put and S is the initial stock price. European options can only be exercised at expiration, while for an American options, we check at each node to see whether early exercise is preferable to holding the option for a further time period δt. When early exercise is taken into account, this value of f must be compared with the option's intrinsic value [5].

2.1.1 Variations of Binomial Model

The variations of binomial models is of two forms namely:

- **Underlying stock paying a dividend or known dividend yield**

The value of a share reflects the value of the company. After a dividend is paid, the value of the company is reduced so the value of the share.

Let us assume a single dividend λ during the life of an option and the the dividend which is given as a proportional to the share price at the ex-dividend by $i\delta t$, then the share prices at nodes are

$$\left.\begin{array}{r} Su^j d^{N-j}, \;\; j = 0, 1, ..., N, \quad N = 0, 1, ..., i-1 \\ \\ S(1-\lambda)u^j d^{N-j}, \;\; j = 0, 1, ..., N, \;\; N = i, i+1, ... \end{array}\right\} \tag{18}$$

- **Underlying stock with continuous dividend yield**

A stock index is composed of several hundred different shares. Each share gives dividend away a different time so the stock index can be assumed to provide a dividend continuously.

We explored Merton's model, the adjustment for the Black-Scholes model to carter for European options on stocks that pay dividend. Referring to Black-Scholes analytic pricing formula on a dividend paying stock for European option, we saw that the risk-free interest rate is modified from r to $(r-\lambda)$ where λ is the continuous dividend yield [14]. We apply the same principle in our binomial model for the valuation of the options. The risk neutral probability in (4) is modified but the other parameters remains the same.

$$p = \frac{e^{(r-\lambda)\delta t} - d}{u - d} \tag{19}$$

where $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}}$.

These parameters apply when generating the binomial tree of stock prices for both the American and European options on stocks paying a continuous dividend and the tree will be identical in both cases. The probability of a stock price increase varies inversely with the level of the continuous dividend rate λ.

2.2 Finite Difference Methods

Schwartz [5] first applied the finite difference techniques to solve option valuation problems for which closed form solutions are unavailable. He considered the valuation of an American option on stock which pays discrete dividends. The finite difference methods attempt to solve the Black-Scholes partial differential equation by approximating the differential equation over the area of integration by a system of algebraic equations [7]. They are a means of obtaining numerical solutions to partial differential equations [12]. They provide a general numerical solution to the valuation problems, as well as an optimal early exercise strategy and other physical sciences. They are also referred to as grid methods. The most common finite difference methods for solving the Black-Scholes partial differential equation are:

- Implicit method
- Crank Nicolson method

These are closely related but differ in stability, accuracy and execution speed. In the formulation of a partial differential equation problem, there are three components to be considered:

- The partial differential equation.
- The region of space-time on which the partial differential is required to be satisfied.
- The ancillary boundary and initial conditions to be met.

2.2.1 Discretization of the Equation

The finite difference method consists of discretizing the partial differential equation and the boundary conditions using a forward or a backward difference approximation. The Black-Scholes partial differential equation can be written as

$$\frac{\partial f(t,S_t)}{\partial t} + rS_t\frac{\partial f(t,S_t)}{\partial S_t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f(t,S_t)}{\partial {S_t}^2} = rf(t,S_t) \tag{20}$$

We discretize the equation with respect to time and to the underlying price of the asset. Divide the (t, S) plane into a sufficiently dense grid or mesh and approximate the infinitesimal steps δt and δS by some small fixed finite steps. Further, define an array of $N+1$ equally spaced grid points $t_0, t_1, ..., t_N$ to discretize the time derivative with $t_{n+1} - t_n = \delta t$ and $\delta t = \frac{T}{N}$.

We know that the stock price cannot go below zero and we have assumed that $S_{\max} = 2S_0$. We have $M+1$ equally spaced grid points $S_0, S_1, ..., S_M$ to discretize the stock price derivative with $S_{M+1} - S_M = \delta S$ and $\delta S = \frac{S_{\max}}{M}$. This gives us a rectangular region on the (t, S) plane with sides $(0, S_{\max})$ and $(0, T)$. The grid coordinates (n, m) enables us to compute the solution at discrete points.

The time and stock price points define a grid consisting of a total of $(M+1) \times (N+1)$ points. The (n, m) point on the grid is the point that corresponds to time $n\delta t$ for $n = 0, 1, 2, ..., N$ and stock price $m\delta S$ for $m = 0, 1, 2, ..., M$

We will denote the value of the derivative at time step t_n when the underlying asset has value S_m as

$$f_{m,n} = f(n\delta t, m\delta S) = f(t_n, S_m) = f(t, S) \tag{21}$$

Where n and m are the number of discrete increments in the time to maturity and stock price respectively. The discrete increments in the time to maturity and stock price are given by δt and δS respectively [9].

2.2.2 Finite Difference Approximations

In finite difference methods we replace the partial derivative occurring in the partial differential equations by approximations based on Taylor series expansions of function near the point or points of interest. The derivative we seek is expressed with many desired order of accuracy.

Assuming that $f(t, S)$ is represented in the grid by $f_{n,m}$, the respective expansions of $f(t, \delta S + S)$ and $f(t, \delta S - S)$ in Taylors series are

$$f(t, \delta S + S) = f(t, S) + \frac{\partial f}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\delta S^2 + \frac{1}{6}\frac{\partial^3 f}{\partial S^3}\delta S^3 + O(\delta S^4) \tag{22}$$

$$f(t, S - \delta S) = f(t, S) - \frac{\partial f}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\delta S^2 - \frac{1}{6}\frac{\partial^3 f}{\partial S^3}\delta S^3 + O(\delta S^4) \tag{23}$$

Using (22), the forward difference is given by

$$\frac{\partial f(t, S)}{\partial S} = \frac{f(t, \delta S + S) - f(t, S)}{\partial S} + O(\delta S)$$

$$\approx \frac{f_{n,m+1} - f_{n,m}}{\delta S} \tag{24}$$

Where $f_{n,m+1} = f(t, \delta S + S)$ and $f_{n,m} = f(t, S)$. (23) gives the corresponding backward difference as

$$\frac{\partial f(t,S)}{\partial S} = \frac{f(t,S) - f(t,S-\delta S)}{\partial S} + O(\delta S)$$

$$\approx \frac{f_{n,m} - f_{n,m-1}}{\delta S} \tag{25}$$

Subtracting (23) from (22) and taking the first order partial derivative results in the central difference given by

$$\frac{\partial f(t,S)}{\partial S} = \frac{f(t,S+\delta S) - f(t,S-\delta S)}{2\delta S} + O(\delta S^2)$$

$$\approx \frac{f_{n,m+1} - f_{n,m-1}}{2\delta S} \tag{26}$$

The second order partial derivatives can be estimated by the symmetric central difference approximation. We sum (22) and (23) and take the second order partial derivative to have

$$\frac{\partial^2 f(t,S)}{\partial S^2} = \frac{f(t,S+\delta S) - 2f(t,S) + f(t,S-\delta S)}{\delta S^2} + O(\delta S^2)$$

$$\approx \frac{f_{n,m+1} - 2f_{n,m} + f_{n,m-1}}{\delta S^2} \tag{27}$$

Although there are other approximations, this approximation to $\frac{\partial^2 f(t,S)}{\partial S^2}$ is preferred, as its symmetry preserves the reflectional symmetry of the second order partial derivative. It is also invariant and more accurate than the similar approximations [1].

We expand $f(t + \delta t, S)$ in Taylors series as

$$f(t+\delta t, S) = f(t,S) + \frac{\partial f}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\delta t^2 + \frac{1}{6}\frac{\partial^3 f}{\partial t^3}\delta t^3 + O(\delta t^4) \tag{28}$$

The forward difference for the time is given by

$$\frac{\partial f(t,S)}{\partial t} = \frac{f(t+\delta t,S) - f(t,S)}{\delta t}$$

$$\approx \frac{f_{n+1,m} - f_{n,m}}{\delta t} \tag{29}$$

Substituting equations (26), (27) and (29) into the equation (20) and noting that $S = m\delta S$ gives

$$\frac{f_{n+1,m} - f_{n,m}}{\delta t} + rm\delta S \frac{f_{n,m+1} - f_{n,m-1}}{2\delta S} + \frac{1}{2}\sigma^2 m^2 S^2 \frac{f_{n,m+1} - 2f_{n,m} + f_{n,m-1}}{\delta S^2} = rf_{n,m}$$

for $m = 1, 2, 3, ..., M-1$ and $n = 0, 1, 2, ..., N-1$. Rearranging terms, we obtain

$$a_m f_{n,m-1} + b_m f_{n,m} + c_m f_{n,m+1} = f_{n+1,m} \tag{30}$$

where

$$\begin{aligned} a_m &= \frac{1}{2} rm\delta t - \frac{1}{2}\sigma^2 m^2 \delta t \\ b_m &= 1 + \sigma^2 m^2 \delta t + r\delta t \\ c_m &= -\frac{1}{2} rm\delta t - \frac{1}{2}\sigma^2 m^2 \delta t \end{aligned}$$

(30) is a finite difference equation which gives equation that we use to approximate the solution $f(t, S)$ [4]. Similarly, we obtained the following finite difference methods using above approach as follows.

For implicit case, we have

$$f_{n+1,m} = \frac{1}{1 - r\delta t} \left[\alpha_{1m} f_{n,m-1} + \alpha_{2m} f_{n,m} + \alpha_{3m} f_{n,m+1}\right] \tag{31}$$

for $n = 0, 1, 2, ..., N-1$ and $m = 1, 2, ..., M-1$. The implicit method is accurate to $O(\delta t, \delta S^2)$. The weights $\alpha_{km}'s$ for $k = 1, 2, 3$ are given by

$$\alpha_{1m} = \frac{1}{2} rm\delta t - \frac{1}{2}\sigma^2 m^2 \delta t,$$

$$\alpha_{2m} = 1 + \sigma^2 m^2 \delta t,$$

$$\alpha_{3m} = -\frac{1}{2} rm\delta t - \frac{1}{2}\sigma^2 m^2 \delta t$$

For Crank Nicolson finite difference method, we have

$$\gamma_{1m} f_{n,m-1} + \gamma_{2m} f_{n,m} + \gamma_{3m} f_{n,m+1}$$

$$= \rho_{1m} f_{n+1,m+1} + \rho_{2m} f_{n+1,m} + \rho_{3m} f_{n+1,m+1} \tag{32}$$

for $n = 0, 1, ..., N-1$ and $m = 1, 2, ..., M-1$. Then the weights γ_{km} and ρ_{km} for $k = 1, 2, 3$ are given by

$$\gamma_{1m} = \frac{rm\delta t}{4} - \frac{\sigma^2 m^2 \delta t}{4},$$

$$\gamma_{2m} = 1 + \frac{r\delta t}{2} + \frac{\sigma^2 m^2 \delta t}{2},$$
$$\gamma_{3m} = -\frac{\sigma^2 m^2 \delta t}{4} - \frac{r m \delta t}{4},$$
$$\rho_{1m} = \frac{\sigma^2 m^2 \delta t}{4} - \frac{r m \delta t}{4},$$
$$\rho_{2m} = 1 - \frac{r\delta t}{2} - \frac{\sigma^2 m^2 \delta t}{2},$$
$$\rho_{3m} = \frac{r m \delta t}{4} + \frac{\sigma^2 m^2 \delta t}{4}$$

2.2.3 Boundary and Initial Conditions

A partial differential equation without the ancillary boundary or initial conditions will either has an infinitely many solutions or has no solution. We need to specify the boundary and initial conditions for the European put option whose payoff is given by $\max(K - S_T, 0)$, where S_T is the stock price at time T. When the stock is worth nothing, a put is worth its strike price K. That is

$$f_{n,0} = K, \quad for \ \ n = 0, 1, 2, ..., N \tag{33}$$

As the price of the underlying asset increases, the value of the put option approaches zero or worths nothing. Accordingly, we choose $S_{\max} = S_M$ and from this we get

$$f_{n,M} = 0, \quad for \ \ n = 0, 1, 2, ..., N \tag{34}$$

We know the value of the put option at time T and can impose the initial condition

$$f_{N,m} = \max(K - m\delta S, 0), \quad for \ \ m = 0, 1, 2, ..., M \tag{35}$$

The initial condition gives us the value of f at the end of the time period and not at the beginning. This means that we move backward from the maturity date to time zero. The price of the put option is given by $f_{0,\frac{M+1}{2}}$ and $f_{0,\frac{M}{2}}$ when M is odd and even respectively.

This method is suited for European put options where early exercise in not permitted. The call-put parity in (17) is used to obtain the corresponding value of the European call option. To value an American put option, where early exercised is permitted, we need to make only one simple modification [9]. After each linear system solution, we need to consider whether early exercise is optimal or not. We compare $f_{n,m}$ with the intrinsic value of the option, $(K - m\delta S)$. If the intrinsic value is greater, then set $f_{n,m}$ to the intrinsic value.

The American call and put options are handled in almost exactly the same way. We have for call and put respectively.

$$\left.\begin{aligned} f_{N,m} &= \max(m\delta S - K, 0), \quad \text{for } m = 0,1,2,...,M \\ f_{N,m} &= \max(K - m\delta S, 0), \quad \text{for } m = 0,1,2,...,M \end{aligned}\right\} \tag{36}$$

2.2.4 Stability Analysis

The two fundamental sources of error, the truncation error in the stock price discretization and in the time discretization. The importance of truncation error is that the numerical schemes solves a problem that is not exactly the same as the problem we are trying to solve.

The three fundamental factors that characterize a numerical scheme are consistency, stability and convergence [9].

- **Consistency**: A finite difference of a partial differential equation is consistent, if the difference between the partial differential equation and finite difference equation vanishes as the interval and time step size approach zero. That is , the truncation error vanishes so that

$$\lim_{\delta t \to 0}(PDE - FDE) = 0 \tag{37}$$

  Consistency deals with how well the FDE approximates the PDE and it is the necessary condition for convergence.

- **Stability**: For a stable numerical scheme, the errors from any source will not grow unboundedly with time.

- **Convergence**: It means that the solution to a FDE approaches the true solution to the PDE as both grid interval and time step sizes are reduced.

These three factors that characterize a numerical scheme are linked together by Lax equivalence theorem [8] which states as follows;

Theorem 2.1. *Given a well posed linear initial value problem and a consistent finite difference scheme, stability is the necessary and sufficient condition for convergence.*
In general, a problem is said to be well posed if:

- *a solution to the problem exists.*
- *the solution is unique when it exists.*
- *the solution depends continuously on the problem data.*

2.2.5 A Necessary and Sufficient Condition for Stability

Let $f_{n+1} = Af_n$ be a system of equations. Matrix A and the column vectors f_{n+1} and f_n are as represented in (30). We have

$$\begin{aligned} f_n &= Af_{n-1} \\ &= A^2 f_{n-2} \\ &\vdots \\ &= A^n f_0 \end{aligned}$$

$$\text{for} \quad n = 1, 2, ..., N \tag{38}$$

where f_0 is the vector of initial values. we are concerned with stability and we investigate the propagation of a perturbation. Perturb the vector of initial values f_0 to $\hat{f}_0$. The exact solution at the n^{th} time-row will then be

$$\hat{f}_n = A^n \hat{f}_0 \tag{39}$$

Let the perturbation or 'error' vector e be denoted by

$$e = \hat{f} - f$$

and using the perturbation vector (38) and (39) we have

$$\begin{aligned} e_n &= \hat{f}_n - f_n \\ &= A^n \hat{f}_0 - A^n f_0 \\ &= A^n(\hat{f}_0 - f_0) \end{aligned}$$

Therefore,

$$e_n = A^n e_0, \quad \text{for} \quad n = 1, 2, ..., N \tag{40}$$

Hence for compatible matrix and vector norms [8]

$$\|e_n\| \leq \|A^n\| \|e_0\|$$

Lax and Richmyer defined the difference scheme to be stable when there exists a positive number L, independent of n, δt and δS such that

$$\|A^n\| \leq L, \quad \text{for} \quad n = 1, 2, ..., N$$

This limits the amplification of any initial perturbation and therefore of any arbitrary initial rounding errors because it implies that

$$\|e_n\| \leq L\|e_0\|$$

since $\|A^n\| = \|A^{n-1}A\| \leq \|A\|\|A^{n-1}\| \leq \cdots \leq \|A\|^n$, then the Lax-Richmyer definition of stability is satisfied when

$$\|A\| \leq 1 \tag{41}$$

condition (41) is the necessary and sufficient condition for the difference equations to be stable [8]. Since the spectral radius $\rho(A)$ satisfies

$$\rho(A) \leq \|A\|$$

it follows automatically from (41) that

$$\rho(A) \leq 1$$

We note that if matrix A is real and symmetric, then by definition [6], we have

$$\|A\|_\infty = \text{moduli of the maximum row of matrix } A$$

$$\|A\|_2 = \rho(A) = \max |\lambda_n| \tag{42}$$

where λ_n is an eigenvalue of matrix A which is given by $\lambda_n = y + 2(\sqrt{xz}) \cos \frac{n\pi}{N}$, for $n = 1, 2, ..., N$ where x, y and z are referred to as weights which may be real or complex.

By Lax equivalence theorem, the two finite difference methods are consistent, unconditionally stable and convergent.

2.3 Monte Carlo Simulation Method

Boyle [3] was the first researcher to introduce Monte Carlo simulation into finance. Monte Carlo method is a numerical method that is useful in many situations when no closed form solution is available. This method is good for pricing path dependent options.

The basis of Monte Carlo simulation is the strong law of large numbers, stating that the arithmetic mean of independent, identically distributed random variables, converges towards their mean almost surely. Monte Carlo simulation method uses the risk valuation result. The expected payoff in a risk neutral world is calculated using a sampling procedure The main procedures are followed when using Monte Carlo simulation.

- Simulate a path of the underlying asset under the risk neutral condition within the desired time horizon
- Discount the payoff corresponding to the path at the risk-free interest rate.
- Repeat the procedure for a high number of simulated sample path
- Average the discounted cash flows over sample paths to obtain the option's value.

A Monte Carlo simulation can be used as a procedure for sampling random outcomes of a process followed by the stock price [9]

$$dS = \mu S dt + \sigma S dW(t) \tag{43}$$

where dW_t is a Wiener process and S is the stock price. If δS is the increase in the stock price in the next small interval of time δt then

$$\frac{\delta S}{S} = \mu \delta t + \sigma Z \sqrt{\sigma t} \tag{44}$$

where $Z \sim N(0,1)$, σ is the volatility of the stock price and μ is its expected return in a risk neutral world (44) is expressed as

$$S(t+\delta t) - S(t) = \mu S(t)\delta t + \sigma S(t) Z \sqrt{\delta t} \tag{45}$$

We can calculate the value of S at time $t+\delta t$ from the initial value of S, then the value of S at time $t+2\delta t$, from the value at $t+\delta t$ and so on. We use N

random samples from a normal distribution to simulate a trial for a complete path followed by S. It is more accurate to simulate $\ln S$ than S, we transform the asset price process using Ito's lemma

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW(t)$$

so that

$$\ln S(t + \delta t) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right) \delta t + \sigma Z \sqrt{\delta t}$$

or

$$S(t + \delta t) = S(t) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) \delta t + \sigma Z \sqrt{\delta t}\right] \tag{46}$$

Monte Carlo simulation is particularly relevant when the financial derivatives payoff depends on the path followed by the underlying asset during the life of the option, that is, for path dependent options. For example, we consider an Asian options whose Stock price process at maturity time T is given by

$$S_T^j = S \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma Z \sqrt{T}\right] \tag{47}$$

where $j = 1, 2, ..., M$ and M denotes the number of trials or the different states of the world. These M simulations are the possible paths that a stock price can have at maturity date T. The estimated Asian call option value is

$$C = \frac{1}{M} \sum_{j=1}^{M} e^{-rT} \max[S_T^j - S_t, 0] \tag{48}$$

where S_t is the strike price determined by either arithmetic or geometric mean [4].

3 Numerical Examples

Now, we present some numerical examples as follows:

Example 3.1. We compute the values of both European and American style options. The results in Table 1 and Table 2 for the American and European options are compared to those obtained using Black-Scholes analytic pricing formula. The rate of convergence for multi-period method may be assessed by repeatedly doubling the number of time steps N. Tables 1 and 2 use the parameters:

$$S = 45, K = 40, T = 0.5, r = 0.1, \sigma = 0.25$$

in computing the options prices as we increase the number of steps. The Black-Scholes price for call option and put option are 7.6200 and 0.6692 respectively.

Example 3.2. Consider pricing a vanilla option on a stock paying a known dividend yield with the following parameters:

$$S = 50, r = 0.1, T = 0.5, \sigma = 0.25, \tau = \frac{1}{6}, \lambda = 0.05$$

The results obtained are shown in the Table 3 below.

Example 3.3. We consider the convergence of the multi-period binomial model, fully implicit and the Crank Nicolson method with relation to the Black-Scholes value of the option. We price the European call option on a non dividend paying stock with the following parameters:

$$S = 50, K = 60, r = 0.05, \sigma = 0.2, T = 1$$

The Black-Scholes price for the call option is 1.6237. The results obtained are shown in the Tables 4 and 5.

Example 3.4. We consider the performance of the three numerical methods against the 'true' Black-Scholes price for a European put with

$$K = 50, r = 0.05, \sigma = 0.25, T = 3$$

The results obtained are shown in the Table 6 below.

Next, we will now attempt to price an Asian options for which there is no closed form solution available.

Example 3.5. Consider pricing of an Asian call and put options using Monte Carlo method with the following parameters:

$$S_0 = 50, K = 50, \mu = 0.04, \sigma = 0.1, r = 0.03, dt = \frac{1}{365}, N = 50, n = 1000$$

where $S_0, K, \mu, \sigma, r, dt, N, T$ and n are called price of underlying today, strike price, expected return, volatility rate, risk free rate, time steps, number of days to expiry, expiration date and number of simulation path respectively.

The following shows the results of executing the Asian put and call script:
Put price = 0.3502
Call price = 0.4991

Monte Carlo method is useful in pricing path dependent options and is becoming more appealing and gaining popularity in derivative pricing. We obtained the above results using Matlab codes.

4 Discussion of Results

From Table 1, we can see that Black-Scholes formula for the European call option can be used to value American call option for it is never optimal to exercise an American call option before expiration. As we increase the value of N, the value of the American put option is higher than the corresponding European put option as we can see from this Table, because of the early exercise premium. Sometimes the early exercise of the American put option can be optimal.

Table 2 shows that as the value of N is doubled, Multi-Period Binomial model converges faster and closer to the Black-Scholes value. This method is very flexible in pricing options.

Table 3 shows that the American option on the dividend paying stock is always worth more than its European counterpart. A very deep in the money, American option has a high early exercise premium. The premium of both the put and call option decreases as the option goes out of the money.

The American and European call options are not worth the same as it is optimal to exercise the American call early on a dividend paying stock. A deep out of the money, American and European call and put options are worth the same. This is due to the fact that they might not be exercised early as they are worthless.

Table 4 shows that the Crank Nicolson method in (32) converges faster than fully implicit method in (31) as $N \to \infty$, $\delta t \to 0$ and as $M \to \infty$, $\delta S \to 0$. The multi-period binomial is closer to the solution for small values of N than the two finite difference methods.

Table 5 shows that when N and M are different, the finite difference methods converges faster than when N and M the same. For the implicit and Crank Nicolson schemes, the number of time steps N initially set at 10 and doubled with each grid M refinement. We conclude that the finite difference methods, just as the binomial model are very powerful in pricing of vanilla options. The Crank Nicolson method has a higher accuracy than the implicit method and therefore it converges faster. The results obtained highlight that the two finite difference methods are unconditionally stable.

Table 6 shows the variation of the option price with the underlying price S. The results demonstrate that the three numerical methods perform well, are mutually consistent and agree with the Black-Scholes value. However, finite difference method (F.D.M) is the most accurate and converges faster than multi period binomial model (M.P.B) and Monte Carlo method (M.C.M).

5 Conclusion

We have at our disposal three numerical methods for options valuation. In general, each of the three numerical methods has its advantages and disadvantages of use. Binomial models are good for pricing options with early exercise opportunities and they are relatively easy to implement but can be quite hard to adapt to more complex situations. Finite difference methods converge faster and more accurate, they are fairly robust and good for pricing vanilla options where there is possibilities of early exercise. They can also require sophisticated algorithms for solving large sparse linear systems of equations and are relatively difficult to code. Finally, Monte Carlo method works very well for pricing European options, approximates every arbitrary exotic options, it is flexible in handling varying and even high dimensional financial problems, moreover, despite significant progress, early exercise remain problematic for Monte Carlo method.

Among the methods considered in this work, we conclude that Crank Nicolson finite difference method is unconditionally stable, more accurate and converges faster than binomial model and Monte Carlo Method when pricing vanilla options as shown in Table 4 and Table 6, while Monte carlo simulation method is good for pricing path dependent options.

When pricing options the important thing is to choose the correct numerical method from the wide array of methods available.

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Table of Results

Table 1: Comparison of the Multi-period Binomial to Black-Scholes Value of the Option as we increase N.

N	European call	American call	European put	American put
10	7.6184	7.6184	0.6676	0.7124
20	7.6305	7.6305	0.6797	0.7235
30	7.6042	7.6042	0.6534	0.7027
40	7.6241	7.6251	0.6742	0.7228
50	7.6070	7.6070	0.6562	0.7101
60	7.6219	7.6219	0.6710	0.7199
70	7.6209	7.6209	0.6701	0.7207
80	7.6124	7.6124	0.6616	0.7134
90	7.6210	7.6210	0.6702	0.7201
100	7.6216	7.6216	0.6707	0.7214

Table 2: The Comparison of the Convergence of the Multi Period-Binomial model and Black-Scholes Value of the Option as we double the value of N.

N	European Call	American Call	European Put	American Put
20	7.6305	7.6305	0.6797	0.7235
40	7.6251	7.6251	0.6742	0.7228
60	7.6219	7.6219	0.6710	0.7199
80	7.6124	7.6124	0.6616	0.7134
100	7.6216	7.6216	0.6707	0.7214
120	7.6181	7.6181	0.6673	0.7182
140	7.6209	7.6209	0.6700	0.7211
160	7.6178	7.6178	0.6670	0.7184
180	7.6211	7.6211	0.6703	0.7213
200	7.6171	7.6171	0.6663	0.7185

Table 3: Out of the Money, at the Money and in the Money Vanilla Options on a Stock Paying a Known Dividend Yield.

K	E.Call	A.Call	E.E.Premium	E.Put	A.Put	E.E.Premium
30	18.97	20.50	1.53	0.004	0.004	0.00
45	6.06	6.47	0.41	1.37	1.49	0.12
50	3.32	3.42	0.10	3.38	3.78	0.40
55	1.62	1.63	0.01	6.40	7.31	0.91
70	0.11	0.11	0.00	19.19	21.35	2.16

Table 4: The Comparison of the Convergence of the Implicit Method, the Crank Nicolson Method and the Multi-period Binomial Model as we increase N and M.

N = M	Multi-Period Binomial	Fully Implicit	Crank Nicolson
10	1.6804	1.3113	1.4782
20	1.5900	1.4957	1.5739
30	1.6373	1.5423	1.6010
40	1.6442	1.5603	1.6110
50	1.6386	1.5692	1.6156
60	1.6289	1.5743	1.6181
70	1.6179	1.5776	1.6196
80	1.6178	1.5798	1.6205
90	1.6254	1.5814	1.6212
100	1.6293	1.5826	1.6216

Table 5: Illustrative Results for the Performance of the Implicit Method and Crank Nicolson Method when N and M are different.

N	M	Fully Implicit Method	Crank Nicolson Method
10	20	1.4781	1.5731
20	40	1.5505	1.6108
30	60	1.5677	1.6180
40	80	1.5748	1.6205
50	100	1.5786	1.6216
60	120	1.5808	1.6222
70	140	1.5824	1.6225
80	160	1.5835	1.6227
90	180	1.5844	1.6229
100	200	1.5849	1.6230

Table 6: A Comparison with the Black-Scholes Price for a European put.

S	Black-Scholes	M.P.B	Finite D. Method	M.C. Method
45	6.6021	6.6025	6.6019	6.6014
50	4.9564	4.9556	4.9563	4.9559
55	3.7046	3.7073	3.7042	3.7076
60	2.7621	2.7640	2.7613	2.7602
65	2.0574	2.0592	2.0572	2.0581
70	1.5328	1.5346	1.5326	1.5324
75	1.1430	1.1427	1.1427	1.1407
80	0.8538	0.8549	0.8537	0.8543
85	0.6392	0.6401	0.6391	0.6405
90	0.4797	0.4803	0.4795	0.4790