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Unit Commitment Problem with Stochastic Demand

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Abstract

The unit commitment problem is a typical scheduling problem in an electric power system. The problem is determining the schedules for power generating units and the generating level of each unit. The decisions concern which units to commit during each time period and at what level to generate power to meet the electricity demand. In this paper we develop a stochastic programming model which incorporates the uncertainties of electric power demand. It is assumed that demand uncertainty can be represented by a scenario tree. We propose a stochastic integer programming model in which the objective is to minimize expected cost. In this model, on/off decisions for each generator are made at the first stage. The approach to solving the problem is based on Lagrangian relaxation and dynamic programming.

Mathematics Subject Classification: 90C11, 90C15, 90C90
Keywords: Unit Commitment, Stochastic Programming, Lagrangian Relaxation

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1 Introduction

Electric power utilities have to maintain sufficient capacity to meet electricity demand during peak load periods. The economic operation and planning of electric power generation occupy an important position in the electric power industry. Wood and Wollenberg [11] offered a brief overview and applied many operations research methods to real electric power problems. The unit commitment problem is determining the schedules for power generating units and the generating level of each unit. The decisions concern which units to commit during each time period, and at what level to generate power to meet the electricity demand. The objective is to minimize the operational cost. This is the sum of the fuel and the start-up costs. The problem is a typical scheduling problem in an electric power system.

Many types of optimization technique have been applied to the unit commitment problem. Among many approaches, the Lagrangian relaxation technique seems to be the most promising because it decomposes the original problem into smaller subproblems. Muckstadt and Koenig [6] used this approach by relaxing the demand constraints. Bard [1] used the Lagrangian relaxation to disaggregate the problem into subproblems that were then solved by dynamic programming.

In these studies, the electricity demand at any period is known in advance. For many actual problems, however, such an assumption is often unjustified. These data contain uncertainty and are represented as random variables since the data represent information about the future. Stochastic programming (Birge [3], Birge and Louveaux [4], Shiina [8]) is a method for an optimization problem under uncertainty. Takriti, Birge and Long [10] is the first paper that deals with the stochastic programming approach. They extended a technique used in the traditional deterministic unit commitment problem. The uncertainty in demand is modeled by introducing a set of scenarios. The problem is decomposed and solved using a Lagrangian relaxation type method, called a progressive hedging algorithm (Rockafellar and Wets [7]). Shiina and Birge [9] proposed another algorithm, based on the Dantzig-Wolfe decomposition (Dantzig-Wolfe [5]) and the column generation approach (Barnhart et al. [2]), to solve the stochastic unit commitment problem.

In this paper we develop a new stochastic programming model in which demand uncertainty is incorporated. In our model, switching decisions for generators are made at the first stage since some types of generators such as coal fired units involve time delay before the generators become available. It is assumed that demand uncertainty can be represented by a scenario tree. We propose a stochastic integer programming model in which the objective is to minimize expected cost. This problem is formulated as a multi-stage stochastic quadratic integer programming problem because the fuel cost function is assumed to be a convex quadratic function. The solution approach is based on Lagrangian relaxation method and dynamic programming. The problem is decomposed into subproblems of single units. The feasible schedule is obtained by solving dynamic programming on a scenario tree. To refine the solution obtained by dynamic programming, we solve an economic dispatch problem in which the equality demand constraint is relaxed to the inequality constraint with upper and lower limit. To solve this problem we develop an algorithm which combines the lambda iteration method and golden section.

2 Uncertainty in Electricity Demand

We assume the duration of the planning horizon in T periods. Since the electricity demand at any point in a period may be uncertain, we have to model the unit commitment problem as a stochastic programming problem.

To model uncertainty, we define the total demand for electricity during period t as a random variable $\tilde{d}_t(\geq 0)$. It is assumed that $\tilde{d}_t$ is defined on a known probability space and has a finite discrete distribution. Let d_t be realization of random variable $\tilde{d}_t$. A sequence of the realization of electricity demand $(d_1, \ldots, d_T)$ is called scenario. It is assumed that we have a set of S scenarios. We use a superscript s to denote an index of scenario s. We associate a probability p_s with each scenario $d^s = (d_1^s, \ldots, d_T^s), s = 1, \ldots, S$. The scenarios are described using a scenario tree as shown in Figure 1. Four scenarios are represented by the scenario tree in Figure 1. The scenario tree divides into branches corresponding to different realizations of random variables $\tilde{d}_t$. Scenario 1 and 2 have the same total demand for $t = 1$. They follow the same first branch. Then they divide separately for $t = 2$, since d_2^1 is not equal to d_2^2.

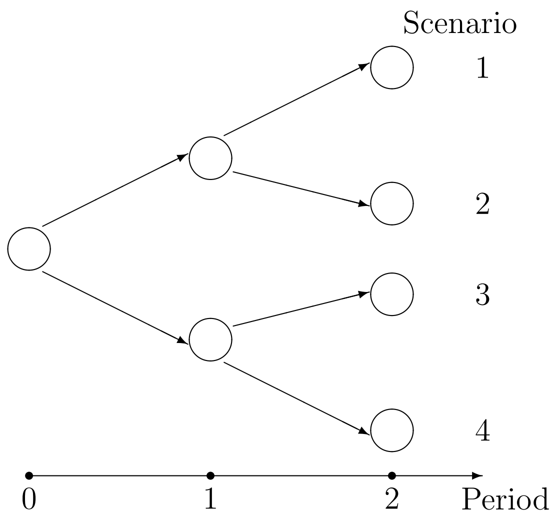

Figure 1: Scenario tree

If two scenarios d^{s_1}, d^{s_2}, $(s_1 \neq s_2)$ satisfy the following conditions $(d_1^{s_1}, \ldots, d_t^{s_1}) = (d_1^{s_2}, \ldots, d_t^{s_2})$, for a period t, they are indistinguishable up to period t. The decisions made for scenario d^{s_1} up to period t must be the same as those made for d^{s_2} up to period t. These two scenarios are said to be included in the same bundle at period t. The set of indices for the scenarios $\{1, \ldots, S\}$ at each period can be partitioned into disjoint subsets which represent scenario bundles. We define $B(s,t)$ to be the bundle in which scenario s is a member at period t.

If $B(s',t) = B(s,t)$ and $B(s',t+1) \neq B(s,t+1), s' < s$, the period $t+1$ is a point when scenario s splits from the other scenario s'. The scenario s' is called a predecessor of scenario s. If there are multiple predecessors for s, we define the scenario with the lowest index as the predecessor of s. The predecessor of scenario s is denoted by $Pred(s)$. The period $\tau(s)$ is defined as the first period in which a scenario s does not share a bundle with another scenario $s' < s$. For scenario 1, we define $\tau(1) = 1$ and $Pred(1) = 1$.

3 Stochastic Unit Commitment Problem

We assume that there are I generating units. In determining an optimal unit commitment schedule, there are two types of decision variables, denoted by u_{it} and x^s_{it}. The status of unit i at period t is represented by the 0-1 variable u_{it}. Unit i is on at time period t, if $u_{it} = 1$, and off if $u_{it} = 0$. The power generating level of the unit i at period t under scenario s is represented by $x^s_{it}(\geq 0)$. The sum of the levels of generation must be greater than the demand.

$$\sum_{i=1}^{I} x^s_{it} \geq d^s_t, t = 1, \ldots, T, s = 1, \ldots, S \tag{1}$$

Rapid changes of temperature are not allowed for thermal units. When unit i is switched on, it must continue to run for at least a certain period L_i. These minimum up-time constraints are described in (2).

$$u_{it} - u_{i,t-1} \leq u_{iv}, v = t+1, \ldots, \min\{t + L_i - 1, T\}, t = 2, \ldots, T \tag{2}$$

Similarly, when unit i is switched off, it must continue to be off at least l_i periods. These constraints are called minimum down-time constraints (3).

$$u_{i,t-1} - u_{it} \leq 1 - u_{iv}, v = t+1, \ldots, \min\{t + l_i - 1, T\}, t = 2, \ldots, T \tag{3}$$

Let $[q_i, Q_i]$ be an operating range of the generating unit i. That is, x^s_{it} has to satisfy the following constraints (4).

$$q_i u_{it} \leq x^s_{it} \leq Q_i u_{it}, i = 1, \ldots, I, t = 1, \ldots, T, s = 1, \ldots, S \tag{4}$$

For two scenarios which are members of the same bundle, the decision variables must be the same.

$$\begin{aligned} x^{s_1}_{it} = x^{s_2}_{it}, \quad & i = 1, \ldots, I, t = 1, \ldots, T, \\ & \forall s_1, s_2 \in \{1, \ldots, S\}, s_1 \neq s_2, B(s_1, t) = B(s_2, t) \end{aligned} \tag{5}$$

This type of constraint is called a nonanticipativity constraint, or a bundle constraint.

The fuel cost function $f_i(x^s_{it})$ is given by a convex quadratic function of x^s_{it}. This function relates to the output of power generated by unit i and depends on the consumption of fuel. The fuel cost function is regarded as convex quadratic, since the incremental fuel cost is a linear increasing function of x^s_{it}. The start up cost function $g_i(u_{i,t-1}, u_{it})$ satisfies the condition $g_i(0, 1) >$

$0, g_i(0,0) = 0, g_i(1,0) = 0, g_i(1,1) = 0$. The mathematical formulation of the stochastic unit commitment problem is described as follows.

(Stochastic Unit Commitment Problem)

$$\begin{aligned} \min \quad & \sum_{s=1}^{S} p_s \sum_{t=1}^{T}\sum_{i=1}^{I}\{f_i(x_{it}^s)u_{it} + g_i(u_{i,t-1}, u_{it})\} \\ \text{subject to} \quad & \text{constraints (1), (2), (3), (4), (5)} \\ & u_{it} \in \{0,1\}, i = 1, \ldots, I, t = 1, \ldots, T \\ & x_{it}^s \geq 0, i = 1, \ldots, I, t = 1, \ldots, T, s = 1, \ldots, S \end{aligned}$$

The problem results in a large scale stochastic mixed integer nonlinear programming problem that combines S deterministic unit commitment problems. The objective is to minimize the expected cost which is given as the sum of the fuel cost and the start up cost.

4 Solution Algorithm

4.1 Lagrangian Relaxation

First, we consider solving the stochastic unit commitment problem using a Lagrangian relaxation approach. Instead of solving the problem directly, we solve the Lagrangian relaxation which results from relaxing the demand constraints. Let $\lambda_t^s(\geq 0)$ be Lagrange multipliers associated with constraints (1). The Lagrangian relaxation problem is shown as follows.

(Lagrangian Relaxation Problem)

$$\begin{aligned} L(\lambda) = \min \quad & \sum_{s=1}^{S} p_s \sum_{t=1}^{T}\sum_{i=1}^{I}\{f_i(x_{it}^s)u_{it} + g_i(u_{i,t-1}, u_{it})\} - \sum_{s=1}^{S}\sum_{t=1}^{T}\lambda_t^s(\sum_{i=1}^{I} x_{it}^s - d_t^s) \\ \text{subject to} \quad & \text{constraints}(2), (3), (4), (5) \\ & u_{it} \in \{0,1\}, i = 1, \ldots, I, t = 1, \ldots, T \\ & x_{it}^s \geq 0, i = 1, \ldots, I, t = 1, \ldots, T, s = 1, \ldots, S \end{aligned}$$

This relaxation decomposes the problem into smaller single-generator subproblems. The objective function of Lagrangian relaxation problem $L(\lambda)$ can be rewritten as follows.

$$\begin{aligned} L(\lambda) = \min & \sum_{i=1}^{I}\sum_{t=1}^{T}\sum_{s=1}^{S}[p_s\{f_i(x_{it}^s)u_{it}^s\} - \lambda_t^s x_{it}^s] \\ & +\sum_{i=1}^{I}\sum_{t=1}^{T} g_i(u_{i,t-1}, u_{i,t}) + \sum_{s=1}^{S}\sum_{t=1}^{T}\lambda_t^s d_t^s \end{aligned} \tag{6}$$

The last term of objective function (6) is a constant. So the function (6) is separable in each unit.

The Lagrangian relaxation problem can be solved by calculating dynamic programming on the scenario tree. First, we solve the following generation level decision problem to obtain optimal $\hat{x}_{it}^s, t = \tau(s), \ldots, T, s = 1, \ldots, S$. The problem is a convex quadratic programming problem which can be solved easily.

(Generation Level Decision Problem for Unit i at Period t under Scenario s)

$$\min \sum_{s'' \in B(s,t)} p_{s''}\{f_i(x_{it}^s)\} - \lambda_t^s x_{it}^s$$

subject to $q_i \le x_{it}^s \le Q_i$

Let $\hat{x}_{it}^s, t = \tau(s), \ldots, T, s = 1, \ldots, S$ be solutions of the generation level decision problem. By setting $x_{it}^{s''} = \hat{x}_{it}^s, t = 1, \ldots, T, \forall s'' \in B(s,t)$, the scenario bundle constraints (5) can be satisfied. Then the binary variables $u_{it}, t = 1, \ldots, T$ are determined. The calculation of dynamic programming is done by the following recursive equations. A unit i must be in one of $L_i + l_i$ states. The first L_i states mean that the unit i is on, and the last l_i states mean that the unit i is off. Let $C_i(t,k)$ be the optimal cost of unit i under the scenario s from stage t to the end of the horizon, if unit i is in state k at stage t. The recursive equations are defined as shown in (7).

$$
C_i(t,k) = \begin{cases}
C_i(t+1,k+1) + \sum\limits_{s\in\{s':\tau(s')\le t\}} (\sum\limits_{s''\in B(s,t)} p_{s''})\{f_i(\hat{x}^s_{it})\} - \lambda^s_t \hat{x}^s_{it} & \text{if } 1 \le k < L_i \\
\min \left[C_i(t+1,k) + \sum\limits_{s\in\{s':\tau(s')\le t\}} (\sum\limits_{s''\in B(s,t)} p_{s''})\{f_i(\hat{x}^s_{it})\} - \lambda^s_t \hat{x}^s_{it}, \right. & \\
\left. C_i(t+1,k+1) + \sum\limits_{s\in\{s':\tau(s')\le t\}} (\sum\limits_{s''\in B(s,t)} p_{s''})\{f_i(\hat{x}^s_{it})\} - \lambda^s_t \hat{x}^s_{it} \right] & \text{if } k = L_i \\
C_i(t+1,k+1) & \text{if } L_i < k < L_i + l_i \\
\min\{C_i(t+1,k), C_i(t+1,1) + g_i(0,1)\} & \text{if } k = L_i + l_i
\end{cases} \tag{7}
$$

These relations are illustrated in Fig. 2. If a generator i has been on for less than L_i at period t, it must be on at period $t+1$. If a generator has been on for L_i or over, there are two possible choices, keeping it on, or switching it off. In the same way, if a generator has been off for less than l_i, the only decision is to remain it off. If a generator has been off for l_i or more, it is allowed to keep it off or to switch it on.

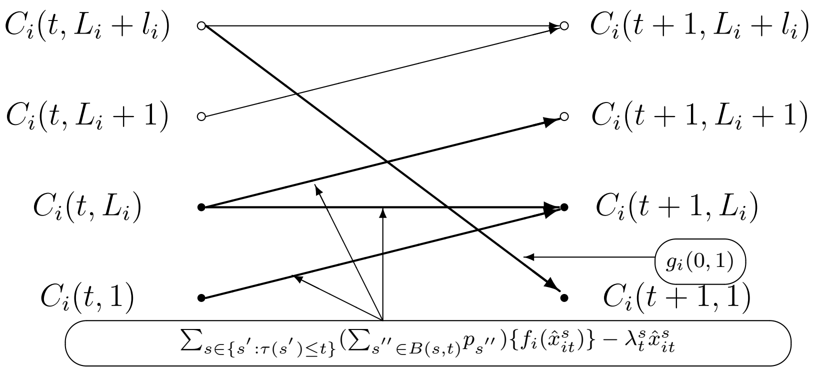

Figure 2: Recursive equation of dynamic programming($L_i = l_i = 2$)

In the recursive equation of dynamic programming (7) at period t, the probability of scenario s ($\tau(s) \le t$) is replaced by $\sum_{s''\in B(s,t)} p_{s''}$. This procedure makes possible an exact calculation of dynamic programming. We define the optimal $u_{it}, i = 1,\ldots,I, t = 1,\ldots,T$ obtained from (7) by $\hat{u}_{it}, i = 1,\ldots,I, t = 1,\ldots,T$. Then we maximize the Lagrangian dual function because the function is concave.

(Lagrangian Dual Problem)

$$\max\ L(\lambda) = \sum_{s=1}^{S} p_s \sum_{t=1}^{T}\sum_{i=1}^{I}\{f_i(\hat{x}_{it}^s)\hat{u}_{it} + g_i(\hat{u}_{i,t-1}, \hat{u}_{it})\} - \sum_{s=1}^{S}\sum_{t=1}^{T}\lambda_t^s(\sum_{i=1}^{I}\hat{x}_{it}^s - d_t^s)$$

subject to $\lambda \geq 0$

The optimal objective value of the Lagrangian relaxation problem provides the lower bound of the original objective value if the solution satisfies the constraints (1). Then we wish to obtain the largest possible lower bound. However, the function is not differentiable, so we use the subgradient optimization technique. Let λ^0 be any initial multiplier. The Lagrangian multipliers are refined for $l = 0, 1, 2, \ldots,$ as shown in (8), where α_l denotes a step size at iteration l.

$$\lambda^{l+1} = \lambda^l + \alpha_l \xi^l \tag{8}$$

A vector ξ^l is called subgradient of $L(\lambda)$ at (λ^l) if inequality (9) holds.

$$L(\lambda) \leq L(\lambda^l) + (\lambda - \lambda^l)^\top \xi^l \tag{9}$$

In the Lagrangian dual problem, ξ^l is given as follows.

$$\xi^l = -(\sum_{i=1}^{I}\hat{x}_{it}^s - d_t^s) \tag{10}$$

The nessesary and sufficient condition for the convergence of the sequence (λ^l) to obtain the optimal solution is shown as follows.

$$\alpha_l\|\xi^l\| \to 0 \ \text{ and } \ \sum_l \alpha_l\|\xi^l\| \to 0 \tag{11}$$

A geometric convergence rate can be achieved if we set α_l as (12), where L^* denotes an optimal value of $L(\lambda)$.

$$\alpha_l = \frac{L^* - L(\lambda^l)}{\|\xi^l\|^2} \tag{12}$$

But we cannot know the value of L^* in advance. Instead, we adopt a heuristic for selecting the step length.

$$\alpha_l = \frac{\theta^l(UB - L(\lambda^l))}{\|\xi^l\|^2} \tag{13}$$

In this expression, UB denotes an upper bound of $L(\lambda)$ and θ^l is chosen between 0 and 2.

4.2 Economic dispatch problem

The optimal solution obtained from solving the Lagrangian relaxation problem may not give a primal feasible solution. After solving the Lagrangian relaxation problem by dynamic programming, the set of solutions $\hat{x}_{it}^s, \hat{u}_{it}, i = 1, \ldots, I, s = 1, \ldots, S, t = \tau(s), \ldots, T$ is obtained. These solutions may violate constraints (1). To modify the level of power generation $\hat{x}_{it}^s$, we solve the following economic dispatch problem for $s = 1, \ldots, S, t = \tau(s), \ldots, T$.

(Economic Dispatch Problem at Period t under Scenario s)

$$\begin{aligned} \min \quad & \sum_{i=1}^{I} f_i(x_{it}^s)\hat{u}_{it} \\ \text{subject to} \quad & \sum_{i=1}^{I} x_{it}^s \geq d_t^s \\ & q_i\hat{u}_{it} \leq x_{it}^s \leq Q_i\hat{u}_{it}, i = 1, \ldots, I \end{aligned}$$

If the constraints $q_i\hat{u}_{it} \leq x_{it}^s \leq Q_i\hat{u}_{it}, i = 1, \ldots, I$ do not exist, the problem can be solved by the lambda iteration method (Wood and Wollenberg [11]). The lambda iteration method, which is based on the method of indeterminate coefficients, seeks the optimal value of undetermined multipliers using binary search. We develop a solution method which combines the lambda iteration method and the golden section. In our approach, the economic power dispatch problem is reformulated as a parametric optimization problem.

(Parametric Optimization Problem at Period t under Scenario s: P(α))

$$\begin{aligned} z(\alpha) = \min \quad & \sum_{i=1}^{I} f_i(x_{it}^s)\hat{u}_{it} \\ \text{subject to} \quad & \sum_{i=1}^{I} x_{it}^s = \alpha \\ & q_i\hat{u}_{it} \leq x_{it}^s \leq Q_i\hat{u}_{it}, i = 1, \ldots, I \end{aligned}$$

We solve the parametric optimization problem in the following range of parameter α.

$$\max\{\sum_{i=1}^{I} q_i\hat{u}_{it}, d_t^s\} \le \alpha \le \min\{\sum_{i=1}^{I} Q_i\hat{u}_{it}, d_t^s\} \tag{14}$$

It follows that solving economic dispatch problem is equivalent to find optimal parameter α in the range of (14). It is necessary to examine in more detail the property of $z(\alpha)$. For two parameters α_1, α_2 which satisfies (14), we define $\hat{x}_{it}^{s1}$, $\hat{x}_{it}^{s2}$ be the optimal solution of the problem $\mathrm{P}(\alpha_1)$, $\mathrm{P}(\alpha_2)$, respectively. It can be shown that function $z(\alpha)$ is a convex function of α. Since $\hat{x}_{it}^{s1}$ and $\hat{x}_{it}^{s2}$ satisfy the constraints of problem $\mathrm{P}(\alpha_1)$ and $\mathrm{P}(\alpha_2)$, the following relations hold for $0 \le \gamma \le 1$.

$$\sum_{i=1}^{I}\{\gamma\hat{x}_{it}^{s1} + (1-\gamma)\hat{x}_{it}^{s2}\} = \gamma\alpha_1 + (1-\gamma)\alpha_2 \tag{15}$$

$$q_i\hat{u}_{it} \le \gamma\hat{x}_{it}^{s1} + (1-\gamma)\hat{x}_{it}^{s2} \le Q_i\hat{u}_{it}, i = 1, \ldots, I \tag{16}$$

Since the convex combination of $\hat{x}_{it}^{s1}$ and $\hat{x}_{it}^{s2}$, that is $\gamma\hat{x}_{it}^{s1} + (1-\gamma)\hat{x}_{it}^{s2}$, is a feasible solution for problem $\mathrm{P}(\gamma\alpha_1 + (1-\gamma)\alpha_2)$, we have the following inequality which proves the convexity of $z(\alpha)$.

$$\begin{aligned}
& z(\gamma\alpha_1 + (1-\gamma)\alpha_2) \\
\le\ & \sum_{i=1}^{I} f_i(\gamma\hat{x}_{it}^{s1} + (1-\gamma)\hat{x}_{it}^{s2})\hat{u}_{it} - e_t^s\{\sum_{i=1}^{I}(\gamma\hat{x}_{it}^{s1} + (1-\gamma)\hat{x}_{it}^{s2}) - d_t^s\} && (17) \\
\le\ & \gamma\{\sum_{i=1}^{I} f_i(\hat{x}_{it}^{s1})\hat{u}_{it} - e_t^s(\sum_{i=1}^{I}\hat{x}_{it}^{s1} - d_t^s)\} \\
& +(1-\gamma)\{\sum_{i=1}^{I} f_i(\hat{x}_{it}^{s2})\hat{u}_{it} - e_t^s(\sum_{i=1}^{I}\hat{x}_{it}^{s2} - d_t^s)\} && (18) \\
=\ & \gamma z(\alpha_1) + (1-\gamma)z(\alpha_2) && (19)
\end{aligned}$$

The second inequality (18) uses the convexity of fuel cost function f. Therefore, an optimal parameter α can be obtained using the golden section.

The algorithm to solve economic dispatch problem is shown in Figure 3.

Figure 4 illustrates how the golden section works.

In step 3 of the algorithm shown in Figure 3, parametric optimization problem $\mathrm{P}(\alpha)$ for fixed α is solved. The lambda iteration method is well-known for solving the economic dispatch problem when the problem has no

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Algorithm to solve Economic Dispatch Problem by Golden Section

- **Step 0.** Set $\underline{\alpha} = \max\{\sum_{i=1}^{I} q_i \hat{u}_{it}, d_t^s\}$, $\overline{\alpha} = \min\{\sum_{i=1}^{I} Q_i \hat{u}_{it}, d_t^s\}$, $\epsilon > 0$.
- **Step 1.** Set $\alpha_1 = \underline{\alpha} + F_1 \cdot (\overline{\alpha} - \underline{\alpha}), \alpha_2 = \underline{\alpha} + F_2 \cdot (\overline{\alpha} - \underline{\alpha})$, where $F_1 = \frac{3-\sqrt{5}}{2}, F_2 = \frac{\sqrt{5}-1}{2}$.
- **Step 2.** If $\overline{\alpha} - \underline{\alpha} < \epsilon$, then stop.
- **Step 3.** If $z(\alpha_1) < z(\alpha_2)$, then set $\overline{\alpha} = \alpha_2$, $\alpha_2 = \alpha_1$ and $\alpha_1 = \underline{\alpha} + F_1 \cdot (\overline{\alpha} - \underline{\alpha})$. If $z(\alpha_1) \geq z(\alpha_2)$, then set $\underline{\alpha} = \alpha_1$, $\alpha_1 = \alpha_2$ and $\alpha_2 = \underline{\alpha} + F_2 \cdot (\overline{\alpha} - \underline{\alpha})$. Go to step 2.

Figure 3: Algorithm to solve economic dispatch problem

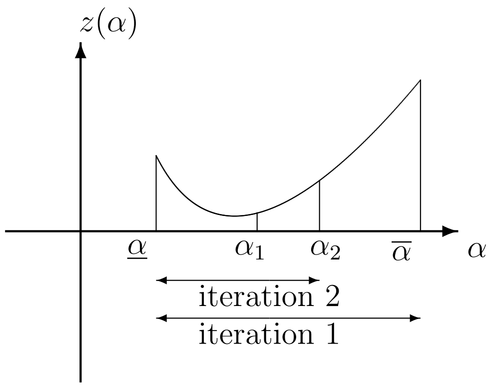

Figure 4: Economic dispatch problem by golden section

minimum and maximum power constraints $q_i \hat{u}_{it} \leq x_{it}^s \leq Q_i \hat{u}_{it}, i = 1, \ldots, I$. Traditional lambda iteration method cannot be applied to solve problem P(α) due to the minimum and maximum power constraints. Thus, it is required to modify the lambda iteration method. We briefly summarize the outline of our method. First, we consider the following economic dispatch problem in which the constraints $q_i \hat{u}_{it} \leq x_{it}^s \leq Q_i \hat{u}_{it}, i = 1, \ldots, I$ are relaxed.

(Relaxed Economic Dispatch Problem)

$$\min \quad \sum_{i=1}^{I} f_i(x_{it}^s)\hat{u}_{it}$$

$$\text{subject to} \quad \phi = \sum_{i=1}^{I} x_{it}^s - \alpha = 0$$

To solve the relaxed economic dispatch problem, the indeterminate coefficients method is applied. The constraint function ϕ is multiplied by an undetermined multiplier π and added to the objective function. This function is known as the Lagrange function denoted by L.

$$L = \sum_{i=1}^{I} f_i(x_{it}^s)\hat{u}_{it} - \pi(\sum_{i=1}^{I} x_{it}^s - \alpha) \tag{20}$$

The nessesary condition for an extreme value of the objective function is shown as follows.

$$\frac{\partial L}{\partial x_{it}^s} = \frac{df_i(x_{it}^s)}{dx_{it}^s}\hat{u}_{it} - \pi = 0, i = 1, \ldots, I \tag{21}$$

$$\frac{\partial L}{\partial \pi} = \phi = \sum_{i=1}^{I} x_{it}^s - \alpha = 0 \tag{22}$$

Equation (21) means that the incremental fuel cost, the derivative of fuel cost function $f_i(x_{it}^s)$ with respect to x_{it}^s, is equal for all committed generators with $\hat{u}_{it} = 1$. The lambda iteration method seeks an optimal power output level x_{it}^s by obtaining the optimal π using the binary search. In our algorithm, the lambda iteration method is modified so as to satisfy the constraints $q_i\hat{u}_{it} \leq x_{it}^s \leq Q_i\hat{u}_{it}, i = 1, \ldots, I$. The algorithm of modified lambda iteration is shown in Figure 5.

If the solution x_{it}^s of the equation $f_i^{'}(x_{it}^s)\hat{u}_{it} - \frac{\pi + \overline{\pi}}{2} = 0$ does not satisfy the minimum and maximum power constraints $q_i\hat{u}_{it} \leq x_{it}^s \leq Q_i\hat{u}_{it}, i = 1, \ldots, I$ in step 2, the power output level x_{it}^s is set to mimimum or maximum value so as to satisfy the minimum and maximum power constraints. Figure 6 illustrates how the algorithm works.

The whole algorithm of Lagrangian relaxation method is shown in Figure 7. It should be noted that the golden section and the modified lambda iteration are applied in step 4.

Modified Algorithm of Lambda Iteration

- **Step 0.** Suppose $\sum_{i=1}^{I} q_i \hat{u}_{it} - \alpha < 0$ and $\sum_{i=1}^{I} Q_i \hat{u}_{it} - \alpha > 0$. Set $\underline{\pi} = f_i^{'}(q_i)\hat{u}_{it}$ and $\overline{\pi} = f_i^{'}(Q_i)\hat{u}_{it}$.
- **Step 1.** Let $\hat{x}_{it}^s$ be the solution of the equation $f_i^{'}(x_{it}^s)\hat{u}_{it} - \frac{\underline{\pi}+\overline{\pi}}{2} = 0$. If $\hat{x}_{it}^s < q_i$, then $\hat{x}_{it}^s = q_i$. Else if $\hat{x}_{it}^s > Q_i$, then $\hat{x}_{it}^s = Q_i$.
- **Step 2.** If $\sum_{i=1}^{I} \hat{x}_{it}^s - \alpha < 0$, then $\overline{\pi} = \underline{\pi} + \frac{\underline{\pi}+\overline{\pi}}{2}$. Otherwise, if $\sum_{i=1}^{I} \hat{x}_{it}^s - \alpha > 0$, then $\underline{\pi} = \underline{\pi} + \frac{\underline{\pi}+\overline{\pi}}{2}$. Go to step 1.

Figure 5: Modified algorithm of lambda iteration

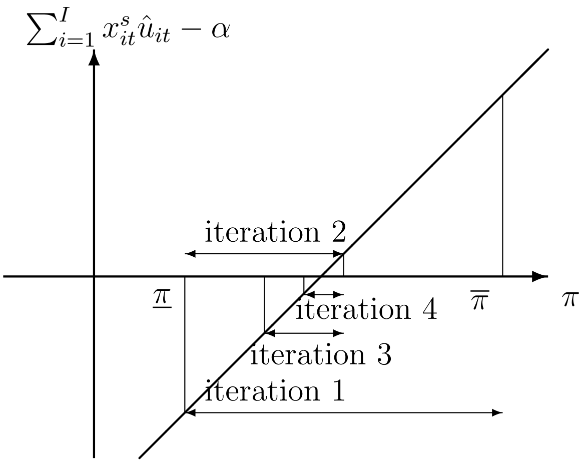

Figure 6: Binary search in lambda iteration

5 Numerical Experiments

This section demonstrates how our algorithm works. The Lagrangian relaxation method for the stochastic unit commitment problem was implemented using C language. The whole framework of the algorithm was coded in C. The test problems considered in this section consist of $I = 10$ units, $T = 168$ periods and $S = 16$ scenarios.

The base demand $d_t^{base}, t = 1, \ldots, 24$ were already given as shown in Table 1.

Lagrangian relaxation

- **Step 0.** Given Lagrangian multipliers $\lambda^0(>0)$. Set iteration number $l = 0$.
- **Step 1.** If the difference between the upper bound and lower bound is relatively small, then stop.
- **Step 2.** Decompose Lagrangian dual into smaller single-generator subproblems by relaxing constraints (1).
- **Step 3.** Solve generation level decision problem to obtain $\hat{x}^s_{it}, s = 1, \ldots, S, t = \tau(s), \ldots, T$. Then solve dynamic programming problem by recursive equations (7) to obtain $\hat{u}_{it}, i = 1, \ldots, I, t = \tau(s), \ldots, T$.
- **Step 4.** Solve economic dispatcing problem using the golden section algorithm and the modified lambda iteration as shown in Figure 3 and Figure 5.
- **Step 5.** Update Lagrange multiplier by subgradient optimization (8). Go to step 1. Set $l = l + 1$.

Figure 7: Algorithm of Lagrangian relaxation method

Table 1: Base power demand

t	1	2	3	4	5	6	7	8	9	10	11	12
d_t^{base}	850	1025	1400	1970	2400	2850	3150	3300	3400	3275	2950	2700
	13	14	15	16	17	18	19	20	21	22	23	24
	2550	2725	3200	3300	2900	2125	1650	1300	1150	1025	1000	900

The predicted demand $d_t^{predicted}$ for $t = 1, \ldots, T$ were caluculated from d_t^{base}

as shown in (23).

$$d_t^{predicted} = \begin{cases} 1.00 d_t^{base} & \text{if} \quad 1 \le t \le 24 \\ 1.05 d_{t-24}^{base} & \text{if} \quad 25 \le t \le 48 \\ 1.10 d_{t-48}^{base} & \text{if} \quad 49 \le t \le 72 \\ 1.10 d_{t-72}^{base} & \text{if} \quad 73 \le t \le 96 \\ 1.05 d_{t-96}^{base} & \text{if} \quad 97 \le t \le 120 \\ 1.00 d_{t-120}^{base} & \text{if} \quad 121 \le t \le 144 \\ 0.95 d_{t-144}^{base} & \text{if} \quad 145 \le t \le 168 \end{cases} \tag{23}$$

The scenarios are generated from the predicted power demand by increasing predicted demand $d_t^{predicted}$, as shown in Table 2.

Table 2: Demand increases and decreases for different scenarios

Scenario	Probability	Period t			
		Mon.	Tue.	Wed.	Thu.
s	p_s	25-48	49-72	73-96	97-120
1	0.0625	0	0	0	0
2	0.0625	0	0	0	+10%
3	0.0625	0	0	+20%	0
4	0.0625	0	0	+20%	+10%
5	0.0625	0	+20%	0	0
6	0.0625	0	+20%	0	+10%
7	0.0625	0	+20%	+20%	0
8	0.0625	0	+20%	+20%	+10%
9	0.0625	+10%	0	0	0
10	0.0625	+10%	0	0	+10%
11	0.0625	+10%	0	+20%	0
12	0.0625	+10%	0	+20%	+10%
13	0.0625	+10%	+20%	0	0
14	0.0625	+10%	+20%	0	+10%
15	0.0625	+10%	+20%	+20%	0
16	0.0625	+10%	+20%	+20%	+10%

We applied our solution approach to the following generating system. The data on the units are shown in Table 3.

The number of iteration of Lagrangian relaxation is set to 1000, and the number of iteration of golden section (Figure 4) and lambda iteration (Figure 5) is limited to 20 and 30, respectively.

We compare the conventional deterministic unit commitment model, with our model which we developed by incorporating demand uncertainty. In the

Table 3: Unit characteristics

Unit	Max	Min	Fuel Cost Function	Minimum Times Up(h)	Minimum Times Down(h)	Start Up Cost
1	1000	300	$0.00113x^2 + 9.023x + 820$	5	4	2875
2	400	130	$0.00160x^2 + 7.654x + 400$	3	2	2110
3	600	165	$0.00147x^2 + 8.752x + 600$	2	4	3050
4	420	130	$0.00150x^2 + 8.431x + 420$	1	3	2130
5	700	225	$0.00234x^2 + 9.223x + 540$	4	5	3000
6	200	50	$0.00515x^2 + 7.054x + 175$	2	2	2110
7	750	250	$0.00131x^2 + 9.121x + 600$	3	4	3250
8	375	110	$0.00171x^2 + 7.762x + 400$	1	3	1920
9	850	275	$0.00128x^2 + 8.162x + 725$	4	3	3150
10	250	75	$0.00452x^2 + 8.149x + 200$	2	1	1805

conventional model, the supply reserve rate r_t is set to the demand constraints on Monday, Tuesday, Wednesday, and Thursday.

Then, we consider the value of the stochastic solution. Let $\overline{x}_{it}^s, \overline{u}_{it}$ be the solution of the deterministic unit commitment problem which is obtained by replacing all random demands $\tilde{d}_t, t = 1, \ldots, T$ with their base values plus reserved margin as $d_t^{base}(1 + r_t)$. This solution is referred to deterministic solution. We apply this deterministic solution to all available scenarios and the exact value of the expected objective function at $(\overline{u})$ can be computed as the optimal objective value of the following expected supply cost problem.

(Expected supply cost problem):

$$
\begin{aligned}
\min \quad & \sum_{s=1}^{S} p_s \sum_{i=1}^{I} \sum_{t=1}^{T} f_i(x_{it}^s)\bar{u}_{it} + \sum_{i=1}^{I} \sum_{t=1}^{T} g_i(\bar{u}_{i,t-1}, \bar{u}_{i,t}) \\
\text{subject to} \quad & \sum_{i=1}^{I} x_{it}^s \geq d_t^s, t = 1, \ldots, T, s = 1, \ldots, S \\
& q_i\bar{u}_{it} \leq x_{it}^s \leq Q_i\bar{u}_{it}, i = 1, \ldots, I, t = 1, \ldots, T, s = 1, \ldots, S \\
& x_{it}^{s_1} = x_{it}^{s_2}, i = 1, \ldots, I, t = 1, \ldots, T, \\
& \forall s_1, s_2 \in \{1, \ldots, S\}, s_1 \neq s_2, B(s_1, t) = B(s_2, t)
\end{aligned}
$$

The difference between the expected objective value of the deterministic solution and the optimal objective value of the original stochastic programming problem is called the value of the stochastic solution, and is denoted as VSS. VSS is calculated as $188594 = 3858235 - 3669641$ in Table 4. The results indicate that using the stochastic programming model can reduce the expected cost by $4.89\% = 100 \times (1 - 3669641/3858235)$.

Table 4: Results of experiments

Model	Objective	Lower Bound	GAP (%)	Expected Cost
Stochastic Programming	3669641	3574354	2.60	3669641
Deterministic				
supply reserve rate r_t				
(Monday, Tuesday, Wednesday, Thursday)				
(5%,10%,10%,5%)	3661903	3565734	2.63	infeasible
(6%,12%,12%,6%)	3690471	3599839	2.46	infeasible
(7%,14%,14%,7%)	3723632	3634030	2.40	infeasible
(8%,16%,16%,8%)	3752530	3668190	2.25	infeasible
(9%,18%,18%,9%)	3788842	3702357	2.28	3858235
(10%,20%,20%,10%)	3829294	3735353	2.45	3858235

6 Concluding Remarks

We have considered a stochastic integer programming model for the unit commitment in which the objective is to minimize expected cost. This problem is formulated as a multi-stage stochastic quadratic integer programming problem because the fuel cost function is defined to be a convex quadratic function. The approach to solving the problem is based on Lagrangian relaxation method and dynamic programming. The feasible schedule is obtained by solving dynamic programming on a scenario tree. To refine the solution obtained by dynamic programming, we solve an economic dispatch problem in which the equality demand constraint is relaxed to the inequality constraint with upper and lower limit. To solve this problem we develop an algorithm which combines the lambda iteration method and golden section. More research is necessary to make a scenario set which reflects real demand.

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