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Application on Inner Product Space with Fixed Point Theorem in Probabilistic

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Abstract

In this present paper we obtain a fixed point theorem in complete probabilistic - Inner product space. To study the existence and uniqueness of solution for linear valterra integral equation.

Mathematics Subject Classification : 46S50
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1 Introduction

This paper is to obtain a new fixed point theorem in probabilistic Δ-inner product space, where Δ is a t-norm of h-type, to study the existence and

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uniqueness of solution for linear Valterra integral equation in complete Δ-PIP-space. Throughout this paper, let $R = (-\infty, +\infty)$, $R+ = (0, +\infty)$, D denotes the set of all distribution functions.

2 Preliminaries

Definition 2.1. *A mapping $F : R \to R^+$ is called a distribution function if it is nondecreasing and left-continuous with $\inf_{t\in R} F(t) = 0, \sup_{t\in R} F(t) = 1$.*

Definition 2.2. *A probabilistic Δ-inner product space (briefly, a Δ-PIP-space) is a triplet (X, F, Δ), where X is a real linear space, Δ is a t-norm and F is a mapping of $X \times X \to D(F_{x,y}$ will denote the distribution function $F(x, y)$ and $F_{x,y}(t)$ will represent the value of $F_{x,y}$ at $t \in R)$ satisfying the following conditions:*
$(PI - 1 - \Delta) f_{x,x}(0) = 0$
$(PI - 2 - \Delta) f_{x,y} = F_{y,x}$
$(PI - 3 - \Delta) f_{x,x} = H(t) \forall t > 0 \Leftrightarrow x = 0$
where

$$H(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

$(PI - 4 - \Delta)$

$$F_{ax,y} = \begin{cases} f_{x,y}(\frac{t}{a}), & a > 0 \\ H(t), & a = 0 \\ 1 - F_{x,y}(\frac{t}{a}+), & a < 0 \end{cases}$$

$(PI - 5 - \Delta)$
$F_{x+y,z}(t) = sup\Delta(F_{x,z}(s), F_{y,z}(s), F_{y,z}(r), t \in R, S + r = t, S \in R, r \in R,$

Definition 2.3. *A t-norm Δ is h-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equi-continuous at $t = 1$, where*

$$\Delta^1(t) = \Delta(t, t), \Delta^m(t) = \Delta(t, \Delta^{m-1}(t)), t \in [0, 1], m = 2, 3, \ldots$$

Definition 2.4. *Let (X, F, Δ) be a Δ-PIP-space.*

1. *A sequence $x_n \subset X$ is said to converge to $x \in X$ if $\forall \epsilon > 0, \forall \alpha \in (0, 1], \exists N, when n \geq N, F_{x_n - x, x_n - x}(\varepsilon) > 1 - \alpha$.*

2. *A sequence $x_n \subset X$ is called a Cauchy sequence if $\forall \epsilon > 0, \forall \alpha \in (0, 1], \exists N$, when $m, n \geq N$, $F_{x_n - x_m, x_n - x_m}(\varepsilon) > 1 - \alpha$.*

3 Main Results

Theorem 3.1. *Let (X, F, Δ) be a complete Δ PIP space and Δ be a t-norm of h-type. Let $T : (X, F, \Delta) \to (X, F, \Delta)$ be a linear mapping satisfying the following condition.*

$$F_{T_{x,y,z}}(t) \geq \frac{F_{xy}\left(\dfrac{t}{k(\alpha, \beta)}\right) + F_{y.z}\left(\dfrac{t}{k(\beta.\gamma)}\right)}{F_{x,y}\left(\dfrac{t}{k(\alpha.\beta)}\right)} \tag{1}$$

For all $x, y, z \in X, t \geq 0, \alpha, \gamma \in (0, +\infty)$ and $k(\alpha.\gamma) : (0, +\infty) \times (0, +\infty) \to (0.1)$ is a function. Then T has exactly one fixed point $x* \in X$. Further more for any $x_0 \in X$ The iterative sequence $\{T^n x_0\}$ T converges to X_*.

Proof. Firstly we prove that any $x_0 \in X$. The sequence $\{x_m\}_{m=0}^{\infty}$ is a τ - cauchy sequence where

$$\{x_m\}_{m=0}^{\infty} = \{x_0, x_1 = Tx_0 .... x_m = T^m x_0 ....\}$$

□

Let us consider

$$F_{T_{x,y,z}}(t) \geq \frac{F_{xy}\left(\dfrac{t}{k(\alpha.\beta)}\right) + F_{y.z}\left(\dfrac{t}{k(\beta.\gamma)}\right)}{F_{xy}\left(\dfrac{t}{k(\alpha.\beta)}\right)}$$

$$F_{xyz}\left(t\right) \geq \frac{F_{x,z}\begin{pmatrix} t \\ k\left(\alpha.\gamma\right) \end{pmatrix}}{F_{xy}\begin{pmatrix} t \\ k\left(\alpha.\beta\right) \end{pmatrix}}$$

or

$$F_{x,z}\left(t\right) \geq \frac{F_{x,z}\begin{pmatrix} t \\ k\left(\alpha.\gamma\right) \end{pmatrix}}{F_{xy}\begin{pmatrix} t \\ k\left(\alpha.\beta\right) \end{pmatrix}}$$

By (Δ - PI – 5) we have

$$F_{x_0-T^m x_0.z}\begin{pmatrix} t \\ k\left(\alpha.\gamma\right) \end{pmatrix} = F_{x_0-Tx_0+T^m x_0,z}\begin{pmatrix} t \\ k\left(\alpha.\gamma\right) \end{pmatrix}$$

$$\geq \Delta(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}, F_{Tx_0-T^m x_0,z}\begin{pmatrix} tk\left(\alpha,\gamma\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix})$$

$$\geq \Delta(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}, F_{Tx_0-T^{m-1} x_0,z}\begin{pmatrix} t \\ k\left(\alpha,\gamma\right) \end{pmatrix})$$

$$= \Delta(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}, F_{Tx_0-Tx_0+Tx_0-T^{m-1} x_0,z}\begin{pmatrix} t \\ k\left(\alpha,\gamma\right) \end{pmatrix})$$

$$\geq \Delta(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}, \Delta\left(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}\right.,$$

$$F_{Tx_0-T^{m-1} x_0,z}\begin{pmatrix} tk\alpha,\beta \\ k\left(\alpha,\gamma\right) \end{pmatrix})$$

$$\geq \Delta(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\beta\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}, \Delta\left(F_{x_0-Tx_0,y}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}\right.,$$

$$F_{Tx_0-T^{m-2} x_0,z}\begin{pmatrix} t \\ k\left(\alpha,\gamma\right) \end{pmatrix})$$

$$\geq \ldots\ldots \geq \Delta\left(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}, \Delta\left(F_{x_0-Tx_0,y}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}\right.\right.,$$

$$\left.\Delta\left(\ldots\ldots, \Delta\left(F_{x_0-Tx_0,z}\begin{pmatrix} t\left(1-k\left(\alpha,\gamma\right)\right) \\ k\left(\alpha,\gamma\right) \end{pmatrix}\right., \left(F_{x_0-Tx_0,y}\begin{pmatrix} t \\ k\left(\alpha,\gamma\right) \end{pmatrix}\ldots\ldots\right)\right.$$

Because of k $(\alpha,\gamma)\in(0,1)$, therefore we get

$$\frac{t\,(1-k\,(\alpha,\gamma))}{k\,(\alpha,\gamma)}\leq\frac{t}{k\,(\alpha,\gamma)}$$

By the property of a distribution function, we have

$$F_{x_0-Tx_0,z}\left(\frac{t}{k\,(\alpha,\gamma)}\right)\geq F_{x_0-Tx_0,z}\left(\frac{t\,(1-k\,(\alpha,\gamma))}{k\,(\alpha,\gamma)}\right)$$

By the property of t-norm, we obtain

$$\begin{aligned}
&F_{x_0-T^mx_0,z}\left(\frac{t}{k\,(\alpha,\gamma)}\right)\\
&\geq\left(F_{x_0-Tx_0,z}\left(\frac{t\,(1-k\,(\alpha,\gamma))}{k\,(\alpha,\gamma)}\right),\Delta\left(F_{x_0-Tx_0,z}\left(\frac{t\,(1-k\,(\alpha,\gamma))}{k\,(\alpha,\gamma)}\right)\right)\right.\\
&\Delta\left(......,\Delta\left(F_{x_0-Tx_0,z}\left(\frac{t\,(1-k\,(\alpha,\gamma))}{k\,(\alpha,\gamma)}\right),F_{x_0-Tx_0,z}\left(\frac{t\,(1-k\,(\alpha,\gamma))}{k\,(\alpha,\gamma)}\right)\right)\right).....\Bigg)\\
&=\Delta^{m-1}\left(F_{x_0-Tx_0,z}\left(\frac{t\,(1-k\,(\alpha,\gamma))}{k\,(\alpha,\gamma)}\right)\right)
\end{aligned}$$

So for any positive integer m, n we have

$$\begin{aligned}
&F_{T^nx_0-T^{m+n}x_0,T^nx_0-T^{m+n}x_0}\left(\frac{t}{k\,(\alpha,\gamma)}\right)\\
&\geq F_{x_0-T^mx_0,T^nx_0-T^{m+n}x_0}\left(\frac{t}{k^{n+1}\,(\alpha,\gamma)}\right)\\
&\geq F_{x_0-T^mx_0,x_0-T^mx_0}\left(\frac{t}{k^{2n+1}\,(\alpha,\gamma)}\right)\\
&\geq\Delta^{m-1}\left(F_{x_0-T^mx_0,x_0-T^mx_0}\left(\frac{t\,(1-k\,(\alpha,\gamma))}{k^{2n+1}\,(\alpha,\gamma)}\right)\right)\\
&\geq\Delta^{m-1}\left(\Delta^{m-1}\left(F_{x_0-Tx_0,x_0-Tx_0}\left(\frac{t\,(1-k\,(\alpha,\gamma))^2}{k^{2n+1}\,(\alpha,\gamma)}\right)\right)\right)\\
&\geq\Delta^{2m-2}\left(F_{x_0-Tx_0,x_0-Tx_0}\left(\frac{t\,(1-k\,(\alpha,\gamma))^2}{k^{2n+1}\,(\alpha,\gamma)}\right)\right).
\end{aligned}$$

Note that Δ is a t − norm of h-type, the family of functions $\Delta^m(t)_{m=1}^{\infty}$ is equicontinuous at t = 1, and the distribution function F is nondecreaseing with supt∈R F(t) = 1, then we have

$$\begin{aligned}&\lim_{1\to\infty} F_{T^n x_0 - T^{m+n}x_0, T^n x_0 - T^{m+n}x_0}\left(\begin{array}{c} t \\ k(\alpha,\gamma)\end{array}\right) \\ &\qquad \geq \lim_{n\to\infty} \Delta^{2m-2}\left(F_{x_0 - Tx_0, x_0 - Tx_0}\left(\begin{array}{c} t(1-k(\alpha,\gamma))^2 \\ k^{2n+1}(\alpha,\gamma)\end{array}\right)\right)\end{aligned}$$

By (Δ - P1-3), we have $\lim\limits_{n\to\infty}(T^n x_0 - T^{n+m}x_0) = 0$. so $\{T^m x_0\}_{m=0}^{\infty}$ is a Cauchy sequence in X. By the completeness of X, let $x_m \to x_* \in X(m \to \infty)$.

Secondly, we prove that X_* is a fixed point of T. Because of

$$\begin{aligned} F_{x_i - Tx_i, x_* - Tx_*}(t) &\geq F_{xi-1-Tx_{i-1}, x_* - Tx_*}\left(\begin{array}{c} t \\ k(\alpha,\gamma)\end{array}\right) \\ &\geq \; ..... \geq F_{x_0 - Tx_0, x_* - Tx_*}\left(\begin{array}{c} t \\ k^i(\alpha,\gamma)\end{array}\right)\end{aligned}$$

Then we have

$$\lim_{i\to\infty} F_{x_i - Tx_i, x_* - Tx_*}(t) \geq \lim_{i\to\infty} F_{x_0 - Tx_0, x_* - Tx_*}\left(\begin{array}{c} t \\ k^i(\alpha,\gamma)\end{array}\right) = 1$$

$$F_{x_* - Tx_*, x_* - Tx_i}\left(\begin{array}{c} t \\ k(\alpha,\gamma)\end{array}\right) \geq \Delta\left(F_{x_* - x_i, x_* - Tx_*}\left(\begin{array}{c} t(1-k(\alpha,\gamma)) \\ k(\alpha,\gamma)\end{array}\right), F_{x_i - Tx_i, x_* - Tx_*}(t)\right)$$

Because $x_i \to x_*$ (when $i \to \infty$), $\Delta(\dots)$ is equi-continous at (3.1) and $F_{\theta, x_* - Tx_*}(t) = 1$, we have

$$\lim_{i\to\infty} F_{x_* - Tx_i, x_* - Tx_*}\left(\begin{array}{c} t \\ k(\alpha,\gamma)\end{array}\right) = 1, \forall t > 0.$$

Hence

$$\begin{aligned}&F_{x_* - Tx_*, x_* - Tx_*}\left(\begin{array}{c} t \\ k(\alpha,\gamma)\end{array}\right) \\ &\quad \geq \Delta\left(F_{x_* - Tx_i, x_* - Tx_*}\left(\begin{array}{c} t(1-k(\alpha,\gamma)) \\ k(\alpha,\gamma)\end{array}\right), F_{Tx_i - Tx_*, x_* - Tx_*}(t)\right) \\ &\quad \geq \Delta\left(F_{x_* - Tx_i, x_* - Tx_*}\left(\begin{array}{c} t(1-k(\alpha,\gamma)) \\ k(\alpha,\gamma)\end{array}\right), F_{Tx_i - x_*, x_* - Tx_*}\left(\begin{array}{c} t \\ k(\alpha,\gamma)\end{array}\right)\right)\end{aligned}$$

So $F_{x_*-Tx_*,x_*-Tx_*}\left(\frac{t}{k(\alpha,\gamma)}\right) \to 1\,(i\to\infty, \forall t>0)$. By $(\Delta - PI - 3)$, we have $x_* = Tx_*$.

If there exists a point $y^* \in X$ such that $y_* = Ty_*$, then

$$F_{x_*-y_*,x_*-y_*}(t) = F_{Tx_*-Ty_*,Tx_*-Ty_*}(t) \geq F_{x_*-y_*,x_*-y_*}\left(\frac{t}{k^2(\alpha,\gamma)}\right)$$

In the same way, we obtain

$$F_{x_*-y_*,x_*-y_*}(t) \geq F_{x_*-y_*,x_*-y_*}\left(\frac{t}{k^2(\alpha,\gamma)}\right) \geq \ldots\ldots \geq f_{x_*-y_*,x_*-y_*}\left(\frac{t}{k^{2n}(\alpha,\gamma)}\right)$$

So $F_{x_*-y_*,x_*-y_*}(t) \to 1\,(n\to\infty, \forall t>0)$. By (Δ - PI - 3), we have $x_* = y_*$. Therefore x* is the unique fixed point in X.

Finally, we prove that the sequence $\{T_n x_0\}$ T-converges to x_* for any $x_0 \in X$. Because of

$$\begin{aligned}
F_{x_*-T^nx_0,x_*-T^nx_0}(l) &= F_{Tx_*-T^nx_0,x_*-T^{n-1}x_0}(t) \\
&\geq F_{x_*-T^{n-1}x_0,x_*-T^{n-1}x_0}\left(\frac{t}{k^2(\alpha,\gamma)}\right) \\
&= F_{Tx_*-T^{n-1}x_0,Tx_*-T^{n-1}x_0}\left(\frac{t}{k^2(\alpha,\gamma)}\right) \\
&\geq F_{x_*-T^{n-2}x_0,x_*-T^{n-2}x_0}\left(\frac{t}{k^4(\alpha,\gamma)}\right) \\
&\geq \ldots\ldots \geq F_{x_*-x_0,x_*-x_0}\left(\frac{t}{k^{2n}(\alpha,\gamma)}\right)
\end{aligned}$$

We have

$$\lim_{n\to\infty} F_{x_*-T^nx_0,x_*-T^nx_0}(t) \geq \lim_{n\to\infty} F_{x_*-x_0,x_*-x_0}\left(\frac{t}{k^{2n}(\alpha,\gamma)}\right) = 1$$

Similarly $\lim_{n\to\infty} F_{x_*-T^n x_0 - T^n x_0}(t) \geq \lim_{n\to\infty} F_{x_*-x_0, x_*-x_0}\left(\begin{array}{c} t \\ k^{2n}(\alpha,\gamma) \end{array}\right) = 1$

$$\therefore \lim_{n\to\infty} F_{x_*-T^n x_0 x_0 - T^n x_0}(t) \geq \frac{\lim_{n\to\infty} F_{x_*-x_0, x_*-x_0}\left(\begin{array}{c} t \\ k^{2n}(\alpha,\gamma) \end{array}\right)}{\lim_{n\to\infty} F_{x_*-x_0, x_*-x_0}\left(\begin{array}{c} t \\ k^{2n}(\alpha,\gamma) \end{array}\right)} = \frac{1}{1} = 1$$

$\therefore$ So $T^n x_0 \to x_0\,(n \to \infty)$ This completes the proof.

4 Application

Theorem 3.1, we utilize this theorem to study the existence and uniqueness of solution of linear Valterra integral equation in complete Δ - PIP – space. Let [a, b] be a fixed real interval. We define linear operation in $L^2[a,b]$,

$$(x+y)(t) = x(t) + y(t)), (\alpha x(t) = \alpha x(t)$$

Then $L^2\,[a,b]$ is a linear space. We lead inner product into

$$L^2[a,b], (x,y) = \int_a^c x\,(t)\,y\,(t)\,dt.$$

Hence (x,y) is a finite number, $(.,.)$ satisfies all conditions of inner product and $L^2[a,b]$ is a inner product space by $(.,.)$ Because $L^2[a,b]$ is infinite dimension and completeness. Define a space (L^2 [a, b,], F, Δ), where

$$F : L3[a,b,c] \times L3[a,b,c] \;\; \to D, Fx,\, y,\, z(t) = H(t - -(x-y-z)).$$

Then (L^2 [a, b], F, Δ) is a Δ - PIP – space. In fact, let {xn} be a Cauchy sequence in (L^2 [a, b], F, Δ). Then for any $\epsilon > 0$, $\lambda \in$ (0, 1], ∃N, when m, n ≥ N, we have $F_{x_m-x_n, x_m-x_n}(\varepsilon) > 1-\lambda$ because of

$$F_{x_m-x_n, x_m-x_n}(\varepsilon) = H\left(\varepsilon - (x_m - x_n, x_m - x_n)\,\phi(\mu)\right)$$

$$\begin{aligned} x_m - x_n &= H\left(\varepsilon - \int_a^c (x_m - x_n)(t)\,(x_m - x_n)(t)\,dt\phi(\mu)\right) \\ &= H\left(\varepsilon - \int_a^b \left[(x_m - x_n)(t)\right]^2 dt\phi(\mu)\right) > 1 - \lambda, \end{aligned}$$

We have
$$\int_a^c [(x_m - x_n)(t)]^2 \, dt \to 0,$$
And then $x_m - x_n \to 0$, So $x_n$is a Cauchy sequence in $L^2[a,b]$. By the completeness of $L^2[a,b]$ we have $xn \to x* \in L^2[a,b]$. Hence $x_* \in L^2[a,b], F, \Delta)$. So $(L^2[a,b], F, \Delta)$ is a complete Δ - *PIP* – space.

Theorem 4.1. *Let $(L^2[a,\ b], F, \Delta)$ be a complete Δ - PIP – space. Then the following conditions are satisfied:*

i) $\int_a^t (x(s) - z(s), t) ds \leq x(t) - z(t), \forall x(\cdot) \in L^2[a,b]$

ii) Let T be a linear mapping and defined as follows

$$(Tx)(t) = f(t) + \int_a^t k(t,s)\, x(s)\, ds$$

Where $f \in L^2[a,b]$ is a given function $k(t,s)$ is a continuous function defined on $a \leq t \leq b \leq c$, $a \leq s \leq t$, λ is a constant, we denote

$$\max_{a \leq t \leq b, a \leq s \leq t} k(t,s) = M.$$

Then when $\lambda M \in$ (0, 1), T has a unique fixed point in $L^2\ [a,b]$. Furthermore, for any $x_0 \in L^2[a,b]$, the iterative sequence $\{T^n x_0\}$ T - converges to the fixed point.

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