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Analysis of Phase-lag for Diagonally Implicit Runge-Kutta-Nyström Methods

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Abstract

We analyse the phase errors introduced by diagonally implicit Runge-Kutta-Nystrom (DIRKN) methods when linear homogeneous test equation is integrated. It is shown that the homogeneous phase errors dominate if long interval integration are performed. Dispersion relations for the special class of DIRKN methods are derived and also both dissipative and zero-dissipative DIRKN methods are constructed. These methods are applied to linear differential equations with oscillating solutions and compared with the current of the same type DIRKN methods.

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1 Introduction

In this paper the phase-lag (or dispersion) analysis of Diagonally Implicit Runge-Kutta-Nystrom (DIRKN) method is discussed. The methods proposed here are devised for the accurate integration of the second-order ODEs, in which the derivative does not appear explicitly,

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0, \quad (1)$$

for which it is known in advance that their solution is oscillating. Such problems often arise in different areas of engineering and applied sciences such as celestial mechanics, quantum mechanics, elastodynamics, theoretical physics and chemistry, and electronics.

The approach for constructing such methods are based on the approach introduced in Houwen and Sommeijer [1]. The dispersion and dissipation relations are developed and imposed in deriving of our new methods. The special class of DIRKN methods are studied and derived which give very efficient implementation due to the diagonally implicit structure.

We note that recently a large number of papers have been published proposing methods with high-order phase-lag. A few of them deal with first-order differential equations, see for example Bursa and Nigro [2], Houwen and Sommeijer [3]. The others are for the second-order equations, see for example Senu et al. [4] which are focusing on the explicit RKN method. For the implicit RKN methods, see for example Houwen and Sommeijer [1], Sommeijer [5], Senu et al. [6] and Al-Khasawneh et al. [7]. In Houwen and Sommeijer [1], they developed low order methods which have high order phase-lag up to order 10.

Next, Sharp et al. [8], derived two- and three-stage with order three and four with relate to dispersion order. Recently Senu et al. [9] proposed a new DIRKN method with reduced phase-lag.

2 Dispersion and Dissipation of RKN Methods

An s -stage Runge-Kutta-Nyström (RKN) method for the numerical integration of the IVP is given by

$$q_{n+1} = q_n + hq'_n + h^2 \sum_{i=1}^{m} b_i f(t_n + c_i h, Q_i) \tag{2}$$

$$q'_{n+1} = q'_n + h \sum_{i=1}^{m} b'_i f(t_n + c_i h, Q_i) \tag{3}$$

where

$$Q_i = q_n + c_i h q'_n + h^2 \sum_{j=1}^{m} a_{ij} f(t_n + c_j h, Q_j).$$

The RKN parameters a_{ij}, b_j, b'_j and c_j are assumed to be real and s is the number of stages of the method. Introduce the s-dimensional vectors c, b and b' and $s \times s$ matrix A, where $c = [c_1, c_2, \cdots, c_s]^T$, $b = [b_1, b_2, \cdots, b_s]^T$, $b' = [b'_1, b'_2, \cdots, b'_s]^T$, $A = [a_{ij}]$ respectively. The RKN method above can be expressed in Butcher notation by the table of coefficients

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & b'^T \end{array}$$

Now, the phase-lag analysis of the method (2) is investigated using the homogeneous test equation (see [1])

$$q'' = (i\lambda)^2 q(t), i = \sqrt{-1}. \quad (4)$$

By applying the general method (2) to the test equation (4) we obtain the following recursive relation

$$\begin{bmatrix} q_{n+1} \\ hq'_{n+1} \end{bmatrix} = D \begin{bmatrix} q_n \\ hq'_n \end{bmatrix}, \quad D = \begin{bmatrix} A(z^2) & B(z^2) \\ A'(z^2) & B'(z^2) \end{bmatrix}, \quad z = \lambda h \quad (5)$$

where A, A', B and B' are polynomials in z^2, completely determined by the parameters of the method (2). The characteristic equation for equation (5) is given by

$$\xi^2 - \mathrm{tr}(D(z^2))\xi + \det(D(z^2)) = 0. \quad (6)$$

The exact solution of (4) is given by

$$q(t_n) = 2|\sigma|\cos(\chi + nz). \quad (7)$$

The numerical solution of (5) is

$$q_n = 2|c||\rho|^n \cos(\omega + n\varphi). \quad (8)$$

Equations (7) and (8) led us to the following definition.

Definition 2.1. *For the RKN method corresponding to the characteristic equation (6) the quantities*

$$\phi(z) = z - \varphi, \quad \alpha(z) = 1 - |\rho|$$

are the phase-lag (or dispersion) and dissipation (or amplification error) respectively. If $\phi(z) = O(z^{q+1})$ *, then the RKN method is said to have phase-lag order* q *and if* $\alpha(z) = O(z^{r+1})$ *,then the RKN method is said to have dissipation order* r *. If at a point* z *,* $\alpha(z) = 0$ *, then the RKN method has zero dissipation.*

From Definition 2.1 it follows that

$$\phi(z) = z - \cos^{-1}\left(\frac{R(z^2)}{2\sqrt{S(z^2)}}\right), \quad \alpha(z) = 1 - \sqrt{S(z^2)}$$

where $R(z^2)$ and $S(z^2)$ defined by

$$T(z^2)=\frac{2+\alpha_1 z^2+\cdots+\alpha_s z^{2s}}{(1+\hat{\lambda}z^2)^s}, \tag{9}$$

$$U(z^2)=\frac{1+\beta_1 z^2+\cdots+\beta_s z^{2s}}{(1+\hat{\lambda}z^2)^s}, \tag{10}$$

where $\hat{\lambda}=2\lambda^2$ is diagonal element in the Butcher tableau.

In this section we will consider the development of consistent and dispersion relations for three-stage with dispersion order up to ten. Based on the functions of $T(v^2)$ and $U(v^2)$ defined as in (9) and (10), a few properties of the functions T and U are summarized in the following theorem which is introduced by Houwen and Sommeijer [1]. The development of consistency, dispersion and dissipation relations is according to the following theorem.

Theorem 2.1

(a) The function $T(v^2)$ and $U(v^2)$ are consistent, dispersive and dissipative of orders $p,q,$ and r, respectively

$$e^{iv}[2\cos(v)-T(v^2)]+U(v^2)-1=O(v^{p+2}) \tag{11}$$

$$T(v^2)-2\sqrt{U(v^2)}\cos(v)=O(v^{q+2}) \tag{12}$$

$$U(v^2)-1=O(v^{r+1}) \tag{13}$$

(b) An RKN method of algebraic order p, dispersion of order q, and dissipation order r possess functions T and U that are consistent, dispersive, and dissipative of orders $p,q,$ and r.

(c) If $U(v^2)\equiv 1$ then the order of consistency and dispersion of T and U are equal.

Proof. (see Houwen and Sommeijer [1]).

Now, expending the functions $T(v^2)$ and $U(v^2)$ as a Taylor series and substitute in the equations (11), (12), and (13) in Theorem 1, then we have

$$e^{iv}[2\cos(v)-T(v^2)]+U(v^2)-1=$$
$$\left(6\lambda^2+\beta_1-\alpha_1-1\right)v^2+\left(-\alpha_1+12\lambda^2-1\right)iv^3+$$
$$.\qquad \left(1/2\alpha_1+6\alpha_1\lambda^2-24\lambda^4+\beta_2-6\beta_1\lambda^2-6\lambda^2+\frac{7}{12}-\alpha_2\right)v^4$$
$$\left(6\alpha_1\lambda^2-48\lambda^4-2\lambda^2-\alpha_2+\frac{1}{4}+\frac{1}{6}\alpha_1\right)iv^5+O(h^6)$$

and $T(v^2)-2\sqrt{U(v^2)}\cos(v)=$

$$\left(-6\lambda^2+\alpha_1+1-\beta_1\right)v^2+\Big(33\lambda^4+\alpha_2-6\alpha_1\lambda^2-\beta_2+$$
$$\frac{1}{2}\beta_1-\frac{1}{12}+\frac{1}{4}\beta_1^{\,2}+3\beta_1\lambda^2-3\lambda^2\Big)v^4+\Big(24\alpha_1\lambda^4-125\lambda^6-$$
$$6\alpha_2\lambda^2+\alpha_3-\frac{1}{8}\beta_1^{\,2}-\beta_3-\frac{3}{4}\beta_1^{\,2}\lambda^2-\frac{3}{2}\beta_1\lambda^2-\frac{15}{2}\beta_1\lambda^4+$$
$$3\beta_2\lambda^2+\frac{1}{360}+\frac{1}{2}\beta_1\beta_2-\frac{1}{8}\beta_1^{\,3}+\frac{15}{2}\lambda^4+\frac{1}{4}\lambda^2-\frac{1}{24}\beta_1+\frac{1}{2}\beta_2\Big)v^6+$$
$$\Big(\frac{1}{720}\beta_1-\frac{1}{24}\beta_2+\frac{1}{2}\beta_3-\frac{3}{8}\beta_1^{\,2}\beta_2+\frac{1}{8}\beta_1\lambda^2+\frac{15}{4}\beta_1\lambda^4-$$
$$\frac{3}{2}\beta_2\lambda^2-\frac{15}{2}\beta_2\lambda^4+\frac{35}{2}\beta_1\lambda^6+3\beta_3\lambda^2+\frac{3}{8}\beta_1^{\,2}\lambda^2+\frac{15}{8}\beta_1^{\,2}\lambda^4+$$
$$\frac{3}{8}\beta_1^{\,3}\lambda^2-\frac{1}{4}\beta_1\beta_2+\frac{1}{2}\beta_1\beta_3+24\alpha_2\lambda^4-80\alpha_1\lambda^6-6\alpha_3\lambda^2-$$
$$\frac{1}{120}\lambda^2+\frac{1}{4}\beta_2^{\,2}+\frac{5}{64}\beta_1^{\,4}+\frac{1}{16}\beta_1^{\,3}+\frac{1}{96}\beta_1^{\,2}-\frac{5}{8}\lambda^4-\frac{35}{2}\lambda^6+$$
$$\frac{1605}{4}\lambda^8-\frac{3}{2}\beta_1\beta_2\lambda^2-\frac{1}{20160}\Big)v^8+O(v^{10})$$

and

$$U(v^2)-1=\left(-6\lambda^2+\beta_1\right)v^2+\left(24\lambda^4+\beta_2-6\beta_1\lambda^2\right)v^4+\Big(24\beta_1\lambda^4-80a^6-$$
$$6\beta_2\lambda^2+\beta_3\Big)v^6+\left(-6\beta_3\lambda^2-80\beta_1\lambda^6+240\lambda^8+24\beta_2\lambda^4\right)v^8+$$
$$\left(-80\beta_2\lambda^6+24\beta_3\lambda^4+240\beta_1\lambda^8-672\lambda^{10}\right)v^{10}+O(v^{12}).$$

Based on the above Taylor expansion of consistency and dispersion for DIRKN, then it can be summarized in the following set of equations. The dispersion relations in terms of α_i and β_i for $m=3, p=4$ DIRKN method are given as follow

order 4 $$\alpha_1 = 12\lambda^2 - 1, \beta_1 = 6\lambda^2, \beta_2 = 12\lambda^4, \alpha_2 = \frac{1}{12} + 24\lambda^4 - 6\lambda^2 \tag{14}$$

order 6 $$-8\lambda^6 + 12\lambda^4 + \frac{1}{360} + \alpha_3 - \beta_3 - \frac{\lambda^2}{2} = 0 \tag{15}$$

order 8 $$4\lambda^6 - \lambda^4 + \frac{1}{60}\lambda^2 + \frac{1}{2}\beta_3 - \frac{1}{20160} = 0 \tag{16}$$

order 10 $$(-\frac{1}{24} - 3\lambda^2)\beta_3 + \frac{17}{3}\lambda^6 - 24\lambda^8 + \frac{1}{1814400} - \frac{\lambda^4}{15} = 0 \tag{17}$$

From the above dispersion relations, dissipation conditions and together with algebraic conditions [10], we developed two DIRKN methods of three-stage and algebraic order four. One has phase-lag order six called DIRKN3(4,6)NEW method (see Table 1) and one has a zero-dissipation ($\beta_i = 0, i = 1, 2, 3$) property called DIRKN3(4,4)NEW method (see Table 2).

Table 1: The DIRKN3(4,6)NEW method

–0.2031515178	0.02063526960		
$\frac{1}{2} - \frac{\sqrt{3}}{6}$	0.001693829777	0.02063526960	
$\frac{1}{2} + \frac{\sqrt{3}}{6}$	–0.0040532720	0.2944222365	0.02063526960
	0	$\frac{1}{4} + \frac{\sqrt{3}}{12}$	$\frac{1}{4} - \frac{\sqrt{3}}{12}$
	0	$\frac{1}{2}$	$\frac{1}{2}$

Table 2: The DIRKN3(4,4)NEW method

$$
\begin{array}{c|ccc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{6}-\frac{\sqrt{3}}{12} & & \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & 0 & \frac{1}{6}-\frac{\sqrt{3}}{12} & \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & 0 & \frac{\sqrt{3}}{6} & \frac{1}{6}-\frac{\sqrt{3}}{12} \\
\hline
 & 0 & \frac{1}{4}+\frac{\sqrt{3}}{12} & \frac{1}{4}-\frac{\sqrt{3}}{12} \\
 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Table 3: Summary of the characteristic the fourth order DIRKN methods

Method	q	d	$\lVert \tau^{(p+1)} \rVert_2$	$\lVert \tau'^{(p+1)} \rVert_2$
DIRKN3(4,6)NEW	6	1.19×10^{-4}	1.88×10^{-3}	1.70×10^{-3}
DIRKN3(4,4)NEW	4	-	2.06×10^{-3}	1.48×10^{-3}
DIRKN3(4,4)IMONI	4	-	3.75×10^{-2}	3.22×10^{-2}
DIRKN3(4,4)HS	4	1.43×10^{-4}	6.35×10^{-4}	1.59×10^{-4}
DIRKN3(4,6)SHARP	6	1.02×10^{-2}	1.85×10^{-3}	6.26×10^{-4}

Notations : q – Dispersion order, d – Dissipation constant

$\lVert \tau^{(p+1)} \rVert_2$ – Error coefficient for y_n, $\lVert \tau'^{(p+1)} \rVert_2$ – Error coefficient for y'_n

3 Problems Tested

In this section we use our methods to solve homogeneous and inhomogeneous problems whose exact solution consists of oscillating function.

Problem 1(*Homogenous problem*)

$$\frac{d^2y(t)}{dt^2} = -100y(t), \quad y(0) = 1, \quad y'(0) = -2$$

Exact solution $y(t) = -\frac{1}{5}\sin(10t) + \cos(10t)$

Problem 2 (*Inhomogeneous problem*)

$$\frac{d^2y(t)}{dt^2} = -y(t) + t, \quad y(0) = 1, \quad y'(0) = 2$$

Exact solution $y(t) = \sin(t) + \cos(t) + t$

Source : Allen and Wing [11]

The following notations are used in Tables 4-5:

- **DIRKN3(4,6)NEW** : A three-stage fourth-order dispersive order six method with 'small' dissipation constant errors derived in this paper.
- **DIRKN3(4,4)NEW** : A three-stage fourth-order zero-dissipative and dispersive order four method derived in this paper.
- **DIRKN3(4,4)IMONI** : A three-stage fourth-order derived by Imoni, Otunta and Ramamohan [12].
- **DIRKN3(4,4)HS** : A three-stage fourth-order dispersive order four derived by van der Houwen and Sommeijer [5].
- **DIRKN3(4,6)SHARP** : A three-stage fourth-order dispersive order six as in Sharp, Fine and Burrage [8].

4 Numerical Results

The results for the four problems above are tabulated in Tables 4 and 5. One measure of the accuracy of a method is to examine the maximum error. Tables 4 and 5 show the absolute maximum error for DIRKN3(4,6)NEW, DIRKN3(4,4)NEW, DIRKN3(4,4)IMONI, DIRKN3(4,4)HS, and DIRKN3(3,6)SHARP methods when solving Problems 1-2 with three different step values. From numerical results in Tables 4 and 5, we observed that the new method DIRKN3(4,6)NEW is more accurate compared with the others methods which has dispersive order six and small dissipation constant.

Table 4: Comparison results ours methods with the methods in the literature for Problem 1

h	**Method**	**T=100**	**T=1000**	**T=4000**
0.0025	**DIRKN3(4,6)NEW**	6.6480(-10)	1.0432(-7)	7.7282(-7)
	DIRKN3(4,4)NEW	8.9105(-9)	7.8235(-7)	2.7745(-6)
	DIRKN3(4,4)IMONI	1.5646(-2)	1.4622(-1)	4.7069(-1)
	DIRKN3(4,4)HS	1.2561(-7)	1.3689(-6)	5.8314(-6)
	DIRKN3(4,6)SHARP	3.0150(-7)	3.0229(-6)	1.2120(-5)
0.005	**DIRKN3(4,6)NEW**	1.4042(-9)	1.2816(-8)	5.2981(-7)
	DIRKN3(4,4)NEW	1.4169(-6)	1.4194(-5)	5.6083(-5)
	DIRKN3(4,4)IMONI	2.0121(-2)	1.8480(-1)	5.6322(-1)
	DIRKN3(4,4)HS	6.6977(-7)	6.6966(-6)	2.7338(-5)
	DIRKN3(4,6)SHARP	2.5569(-6)	2.5624(-5)	1.0255(-4)
0.01	**DIRKN3(4,6)NEW**	1.2746(-7)	1.2641(-6)	5.0385(-6)
	DIRKN3(4,4)NEW	2.2671(-5)	2.2696(-4)	9.0759(-4)
	DIRKN3(4,4)IMONI	5.9680(-2)	4.6223(-1)	9.2860(-1)
	DIRKN3(4,4)HS	3.2305(-5)	3.2361(-4)	1.2955(-3)
	DIRKN3(4,6) SHARP	3.1342(-4)	3.1448(-3)	1.2637(-2)

Table 5: Comparison results ours methods with the methods in the literature for Problem 2

h	Method	T=100	T=1000	T=4000
0.065	**DIRKN3(4,6)NEW**	2.4734(-8)	2.4734(-8)	5.1346(-8)
	DIRKN3(4,4)NEW	5.4183(-7)	5.5891(-6)	2.2438(-5)
	DIRKN3(4,4)IMONI	5.2936(-3)	5.3213(-2)	2.0104(-1)
	DIRKN3(4,4)HS	6.8021(-7)	6.8361(-6)	2.7394(-5)
	DIRKN3(4,6) SHARP	4.0017(-6)	4.1061(-5)	1.6419(-4)
0.125	**DIRKN3(4,6)NEW**	3.9303(-7)	3.9303(-7)	1.8558(-6)
	DIRKN3(4,4)NEW	7.4257(-6)	7.6351(-5)	3.0694(-4)
	DIRKN3(4,4)IMONI	1.0214(-2)	1.01103(-1)	3.6319(-1)
	DIRKN3(4,4)HS	1.0871(-5)	1.0930(-4)	4.3835(-4)
	DIRKN3(4,6)SHARP	1.3006(-4)	1.3398(-3)	5.3657(-3)
0.25	**DIRKN3(4,6)NEW**	6.2304(-6)	1.2593(-5)	6.4522(-5)
	DIRKN3(4,4)NEW	1.1913(-4)	1.2214(-3)	4.9117(-3)
	DIRKN3(4,4)IMONI	1.9124(-2)	1.8683(-1)	6.2958(-1)
	DIRKN3(4,4)HS	1.7332(-4)	1.7444(-3)	7.0007(-3)
	DIRKN3(4,6) SHARP	4.4802(-3)	4.6441(-2)	1.9520(-1)

Notation : 1.2345(-4) means 1.2345×10^{-4}

5 Conclusion

In this paper the analysis of phase-lag diagonally implicit RKN methods with three-stage fourth-order. New methods which are dispersive order four and six are presented. We have also performed various numerical tests. From the results tabulated in Table 4-5, we conclude that the new method DIRKN3(4,6)NEW which is high dispersion order is more accurate for integrating second-order equations possessing an oscillatory solution when compared with the new zero-dissipative DIRKN3(4,4)NEW and also with the current DIRKN methods DIRKN3(4,4)IMONI, DIRKN3(4,4)HS and DIRKN3(4,6)SHARP. We also found that the phase-lag is more important rather than the dissipation property when dealing with oscillatory problems.

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