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Approximation of Stable and Geometric Stable Distribution

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Abstract

Although there has been increased interest in the application of the stable and geometric stable distributions in economics and finance, further application has been limited because their probability density function does not have an explicit solution. In this paper, we present three analytic approximation methods — homotopy perturbation method, Adomian decomposition method, and variational iteration method — to resolve this problem.

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1 Introduction

The family of stable distributions has received considerable interest by financial economists since the major empirical work in the early 1960s by Mandelbrot [20] and Fama [4,5] where asset return distributions were found to be better described as a stable non-Gaussian distribution (also referred to as the Paretian distribution). Subsequently, empirical evidence reported by other researchers also suggests that some important economic variables such as stock price changes, interest rate changes, currency changes, and price expectations can be better described by stable non-Gaussian distributions (see [29]). The geometric-stable (henceforth geo-stable) distribution is particularly appropriate in modeling heavy-tailed (See [16]), when the variable of interest may be thought of as a result of a random number of independent innovations. One of the stylized facts observed for asset returns is that they are heavy tailed.

The stable distribution is described by four parameters: (1) α, which determines the tail weight or the distribution's kurtosis with $0<\alpha\leq 2$; (2) β, which determines the distribution's skewness; (3) σ, a scale parameter, and; (4) μ, a location parameter. Except in two special cases — the exponential distribution ($\sigma = 0$) and Laplace distributions ($\alpha = 2$ and $\mu = 0$) — the densities and distribution functions of geo-stable laws are not known in closed form. Other special cases include the Linnik distribution (symmetric geo-stable distribution, $\mu = 0$ and $\beta = 0$) and Mittag-Leffler distribution ($\alpha < 1$ and $\beta = 1$). The Laplace distribution is a special case of Linnik distribution (see [2,27]).

The failure of these distributions to have a closed form has limited their application. Several studies attempt to approximate the density function. Racheva-Iotova and Stoyanov [31] discuss the advantages and disadvantages of the different approximation methods. The approach suggested by Zolotarev [33] is based on an integral representation of the density function. Nolan and Rajput [26]and Nolan [25], in addition to extending Zolotarev's approach, implemented it by means of numerical integration methods. Bergstrom [3] and Feller [6] provide series expansions for the probability density function (pdf) and cumulative density function (cdf). A cdf approximation based on the Bergstrom's series expansion and Zolotarev's representation is developed by McCulloch [22]. Holt and Crow [13] combine four alternative procedures to approximate an inversion integral for computing pdf values from the characteristic function. McCulloch [23] derives an approximation for the symmetric case by interpolating between normal and Cauchy pdfs and fitting splines to the residuals. Rachev and Mittnik [30] expanded and examined the inverse Fourier approach. Using the appealing properties of the fast Fourier transform (FFT)-based density approximations, Rachev and Mittnik [30] (also see [24]) provide an algorithm which approximates the stable densities with a verified accuracy for a subset of the parameter space. Some researchers use indirect inference for estimation of stable distributions (see, for example, [7] and Lombardi and [19]).

In this paper, we provide a new strategy for obtaining an analytic approximation of the pdf for the stable and geo-stable distributions. (For the definitions and properties of these distributions, see [32] and [15,16]) Specifically, we employ three analytic approximation methods —homotopy perturbation method, Adomian decomposition method, and variational iteration method — to compute the fundamental solutions of a partial differential equation (PDE) of fractional order. These three methods offer efficient approaches for solving linear and nonlinear PDEs, integral equations, and integro-differential equations that have been applied to a wide class of problems in physics, biology, and chemical reaction. The key in our presentation is that stable and geo-stable distributions are linked to PDEs of fractional order. Despite the long history of fractional derivatives and integral equations in the fields of science, engineering, and business (see [8,9]) there have been only a few studies that have applied fractional derivatives to the stable and geo-stable distributions (see [17]).

Our strategy for deriving an analytic approximation of the pdf of the stable and geo-stable distributions requires that we develop a clear link between fractional calculus and these two distributions. Basically, the motivation of this work is to generalize and extend the approach of [17] linking PDEs of fractional order with stable distributions. After introducing new PDEs of fractional order that are related to geo-stable distributions, we then derive analytical-numeric solutions for nearly all the pdfs of the stable and geo-stable distributions. These results are particularly interesting because they not only provide the analytic approximations for the pdf of the stable and geo-stable distributions, but they also connect two seemingly different fields.

We have organized our presentation as follows. In Section 2, we interpret a PDE of fractional order, whose solution gives nearly all the stable and geo-stable distributions. In Sections 3 and 4, the analytic approximations of stable and geo-stable distributions are investigated using the homotopy perturbation, Adomian decomposition, and variational iteration methods. Some numerical experiments and convergence analysis to clarify the methods are provided in Sections 5 and 6, respectively.

2 LinkBetween Fractional PDE and Stable/Geo-Stable Distributions

2.1 Fractional PDE and Stable Distribution

In this section, we provide a brief review of the link between fractional PDE and stable distributions as presented by Li [17]. For $0 < \alpha < 3$, $D \leq 0$, consider two PDEs that are symmetric

$$\frac{\partial u}{\partial t} = D\frac{\partial^\alpha}{\partial x^\alpha}u(x,t), \quad x \in \mathbf{R}, \quad t > 0, \quad u(x,0) = u_0(x), \tag{1}$$

and

$$\frac{\partial u}{\partial t} = D\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t), \quad x \in \mathbf{R}, \quad t > 0, \quad u(x,0) = u_0(-x), \tag{2}$$

If $u(x,t)$ is the solution of equation (1), we see that $u(-x,t)$ solves equation (2). In the integral-order derivative case, we have a simple relation

$$\frac{\partial^n}{\partial(-x)^n} = (-1)^n \frac{\partial^n}{\partial x^n} \tag{3}$$

so that we need not bother to call $\frac{\partial^n}{\partial(-x)^n}$ another derivative. However, in the case of fractional order, the relation given by equation (3) does not hold. Consequently, both derivatives are necessary.

Now consider a fractional PDE given by

$$\frac{\partial u}{\partial t} = -\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) - \frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t) + \mu\frac{\partial}{\partial x}u(x,t), \tag{4}$$

where $0 < \alpha \leq 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$ and $-\infty < \mu < \infty$, also $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$. Let $H(\omega,t)$ be the Fourier transform of $u(x,t)$ with respect to t. Then equation (4) converts to the following initial value problem

$$\frac{\partial H}{\partial t} = -\frac{1+\beta}{2c}(i\omega)^\alpha H - \frac{1-\beta}{2c}(-i\omega)^\alpha H + (i\mu\omega)H, \tag{5}$$

where the initial value is $\delta(x)$. If $u(x,0) = \delta(x)$, then $H(\omega,0) = 1$. Therefore, the solution to equation (5) can be obtained as

$$H(\omega,t) = \exp\{-\frac{1+\beta}{2c}(i\omega)^\alpha t - \frac{1-\beta}{2c}(-i\omega)^\alpha t + (i\mu\omega)\}.$$

Another fractional PDE defined by Li (2003) that will be helpful for solving equation (4) according to the Laplace definition of a fractional derivative is

$$\frac{\partial u}{\partial t} = -\frac{\beta}{c}\frac{\partial^\alpha}{\partial x^\alpha} + (1-\beta)\frac{\partial^\alpha}{\partial|x|^\alpha}u(x,t) + \mu\frac{\partial}{\partial x}u(x,t), \tag{6}$$

where $u(x,0) = u_0(x)$, $-\infty < x < \infty$ and $t > 0$. It is easily verified that this equation is equivalent to equation (4). The Fourier transform of the fundamental solution of equation (6) can be written as (see Li (2003))

$$H(\omega,t) = \exp\{-|\omega|^\alpha t - i\beta sign(\omega)\tan\frac{\alpha\pi}{2}|\omega|^\alpha t + i\mu\omega t\}. \tag{7}$$

If one compares equation (7) to the cdf of a stable distribution (see [32]), one would find that they are identical for the case of a stable distribution with $\alpha \neq 1$. Consequently, $u(x,t)$ is the pdf of stable distribution ($S_\alpha(t^{\frac{1}{\alpha}},\beta,\mu t)$) according to x. This demonstates that there is a direct connection between stable distributions and a class of fractional PDEs.

2.2 Fractional PDE and Geo-Stable Distribution

We now define a new PDE of fractional order and we will prove that the fundamental solution of this PDE gives all pdfs for the geo-stable distributions. For $0 < \alpha < 3$, $C \neq 0$ consider a fractional PDE

$$\frac{\partial u}{\partial t} = C\frac{\partial^\alpha}{\partial x^\alpha}u(x,t), \quad x \in \mathbf{R}, \quad t > 0, \quad where \;\; u(x,0) = u_0(x). \tag{8}$$

Let $\hat{u}(\omega,t)$ be the Fourier transform of $u(x,t)$ with respect to x. By the definition of a fractional derivative (see [28]), we will have

$$\frac{\partial \hat{u}}{\partial t} = C(i\omega)^\alpha \hat{u}. \tag{9}$$

This can be viewed as an ordinary differential equation with independent variable t.

$$\hat{u}(\omega,t) = \exp(C(i\omega)^\alpha t)\hat{u}(\omega,0), \tag{10}$$

where $\hat{u}(\omega,0)$ is the Fourier transform of the initial value $u_0(x) = u(x,0)$. We call the inverse Fourier transform of equation (10) the fundamental solution

$$u(x,t) = F^{-1}\{\exp(C(i\omega)^\alpha t)\hat{u}(\omega,0)\}$$
$$= F^{-1}\{\exp(C(i\omega)^\alpha t)\} * F^{-1}\{\hat{u}(\omega,0)\} \; = \; K(x,t) * u_0(x)$$

where $K(x,t)$ is the solution of equation (8) if $u_0(x) = \delta(x)$.

Li [17] has proven that the fundamental solution $K(x,t)$ of equation (8) is the density of the stable distribution where $1 < \alpha \leq 2$, as $S_\alpha((-Ct\cos(\frac{\alpha\pi}{2}))^{\frac{1}{\alpha}}, 1,0)$. Now we provide two theorems for explaining the connection between geo-stable distributions and fractional PDEs. Our proof for both theorems is provided in the paper's appendix.

Theorem 2.1. The fundamental solution $K_1(x,t)$ of the equation

$$\frac{\partial u_1}{\partial t} = u_1(x,t) * C\frac{\partial^\alpha u_1}{\partial x^\alpha}, \qquad x \in \mathbb{R}, t > 0, \tag{11}$$

with the initial condition $u_1(x,0) = \delta(x)$, is the density of the geo-stable distribution $S_\alpha((-Ct\cos(\frac{\alpha\pi}{2}))^{\frac{1}{\alpha}}, 1,0)$. Note that we assume $1 < \alpha \leq 2$ and the notation " $*$ " in equation (11) shows the convolution operator. Now we define a fractional PDE by

$$\frac{\partial u_1}{\partial t} = u_1(x,t) * \left(-\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u_1(x,t) - \frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u_1(x,t) + \mu\frac{\partial}{\partial x}u_1(x,t)\right), \tag{12}$$

where $0 < \alpha \leq 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$ and $-\infty < \mu < \infty$, and also $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$. If $H_1(\omega,t)$ is the Fourier transform of $u(x,t)$ with respect to t, then equation (12) converts to the following initial value problem

$\frac{\partial H_1}{\partial t} =$
$H_1(\omega,t) \times \left(-\frac{1+\beta}{2c}(i\omega)^\alpha H_1(\omega,t) - \frac{1-\beta}{2c}(-i\omega)^\alpha H_1(\omega,t) + (i\mu\omega)H_1(\omega,t)\right)$,
where the initial value is $\delta(x)$. If $u(x,0) = \delta(x)$, then $H_1(\omega,0) = 1$.

Theorem 2.2 The fundamental solution $K_1(x,t)$ of equation (12) is the density of all geo-stable distribution $S_\alpha(t^{\frac{1}{\alpha}},\beta,\mu t)$, for $\alpha \neq 1$.

3 PDFApproximation of Stable Distributions

In this section, we derive the pdf approximation of stable distributions by using the homotopy perturbation method (HPM) (see [10,12,18]), Adomian decomposition method (ADM) (see [1]), and variational iteration method (VIM) (see [11]).

3.1 PDF Approximation of Stable Distribution Using the HPM

We illustrate the applicability of the HPM for approximating the pdf of stable distributions. Below we derive the pdf approximation of stable distributions for the following three cases using the HPM:

Case 1: $1 < \alpha \leq 2$ for equation (1).
Case 2: The proportion of the pdf approximation of stable distributions according to x
Case 3: The pdf approximation of stable distributions for $0 < \alpha \leq 2$.

For case 2 we then obtain the proportion of the pdf approximation of stable distributions according to x. Finally, the pdf approximation of stable distributions for $0 < \alpha \leq 2$ is obtained by applying the HPM to equation (4).

3.1.1 Case 1: $1 < \alpha \leq 2$ for equation (1)

Consider the following space-fractional PDE

$$\frac{\partial u}{\partial t} = D\frac{\partial^\alpha}{\partial x^\alpha}u(x,t), \quad x \in \mathbf{R}, \quad t > 0, \quad 0 < \alpha < 2, \tag{13}$$

subject to initial condition $u(x,0) = \delta(x)$, and D is a positive coefficient. To solve equation (13) with initial condition $u(x,0) = \delta(x)$ by applying the HPM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - D\frac{\partial^\alpha v}{\partial x^\alpha}\right) = 0. \tag{14}$$

Suppose the solution of equation (16) has the form

$$v = v_0 + v_1 p^1 + v_2 p^2 + v_3 p^3 + \cdots. \tag{15}$$

Substituting equation (15) into equation (14), and comparing coefficients of terms with identical powers of p, leads to:

$$\begin{aligned} p^0: &\quad \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \\ &\vdots \\ p^n: &\quad \frac{\partial v_n}{\partial t} = D\frac{\partial^\alpha v_{n-1}}{\partial x^\alpha}, \qquad v_n(x,0) = 0. \end{aligned} \tag{16}$$

For simplicity, we take $v_0(x,t) = u_0(x,t) = \delta(x)$. According to equation (16) and the fractional derivative's definition of the Dirac delta, we derive the following recurrent relation

$$v_1 = \int_0^t \left(D\frac{\partial^\alpha v_0}{\partial x^\alpha} - \frac{\partial u_0}{\partial t}\right) dt = \int_0^t D\frac{\partial^\alpha v_0}{\partial x^\alpha} dt = \frac{D}{\Gamma(-\alpha)} x^{-\alpha-1} \times t,$$

$$\vdots$$

$$v_n = \left(\frac{D^n}{\Gamma(-n\alpha)} x^{-n\alpha-1}\right)\left(\frac{1}{2 \times 3 \times \cdots \times n}\right) t^n = \frac{D^n}{\Gamma(-n\alpha)} x^{-n\alpha-1} \frac{1}{\Gamma(n+1)} t^n.$$

The solution is

$$u_0(x,t) = v_0(x) = \delta(x),$$

$$u_1(x,t) = v_0 + v_1 = \delta(x) + \tfrac{D}{\Gamma(-\alpha)} x^{-\alpha-1} \times t,$$

$$\vdots$$

so

$$u_n(x,t) = \delta(x) + \sum_{k=1}^{n} \left(\frac{D^k}{\Gamma(-k\alpha)} x^{-k\alpha-1}\right) \times \left(\frac{1}{\Gamma(k+1)} t^k\right).$$

Therefore,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \delta(x) + \textstyle\sum_{k=1}^{\infty} \left(\frac{D^k}{\Gamma(-k\alpha)} x^{-k\alpha-1}\right) \times \left(\frac{1}{\Gamma(k+1)} t^k\right). \tag{17}$$

Equation (17) appears quite similar to the series representations for the stable density (see Feller (1966)).

3.1.2 Case 2: The proportion of the pdf approximation of stable distributions according tox

Consider the fractional PDE

$$\frac{\partial V}{\partial t} = D\frac{\partial^\alpha}{\partial(-x)^\alpha}V(x,t), \quad x \in \mathbf{R}, \quad t > 0, \quad 0 < \alpha < 2, \tag{18}$$

subject to initial condition $V(x,0) = \delta(-x)$, and D is a positive coefficient. It is obvoius that if $u(x,t)$ is the solution to equation (14), then $u(-x,t)$ solves the fractional PDE given by (14). Then the solution of equation (18) can be obtained as

$$V(x,t) = u(-x,t) = \delta(-x) + \sum_{k=1}^{\infty}\left(\frac{D^k}{\Gamma(-k\alpha)}(-x)^{-k\alpha-1}\right)\times\left(\frac{1}{\Gamma(k+1)}t^k\right).$$

3.1.3 Case 3 The pdf approximation of stable distributions for $0 < \alpha \le 2$

Consider the fractional PDE

$$\frac{\partial u}{\partial t} = -\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) - \frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t) + \mu\frac{\partial}{\partial x}u(x,t), \tag{19}$$

where $0 < \alpha \le 2$, $\alpha \ne 1$, $-1 \le \beta \le 1$ and $-\infty < \mu < \infty$, and $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$, subject to initial condition $u(x,0) = \delta(x)$. For simplicity, we take $v_0(x,t) = u_0(x,t) = \delta(x)$. Consequently, solving the HPM related to equation (19), then the first few components of the homotopy perturbation solution for equation (19) are derived as follows

$$v_1(x,t) = \left(\frac{d_1+(-1)^\alpha d_2}{2\Gamma(-\alpha)}\right)x^{-\alpha-1}\times t,$$

$$v_2(x,t) = \left(\left(\frac{d_1^2+(-1)^\alpha d_1 d_2 + d_2^2}{2\Gamma(-2\alpha)}\right)x^{-2\alpha-1} + \left(\frac{d_1+(-1)^\alpha d_2}{2\Gamma(-\alpha-1)}\right)x^{-\alpha-2}\right)\frac{1}{2}t^2.$$

So we derive the following recurrent relation

$$v_j = \int_0^t\left(d_1\frac{\partial^\alpha v_{n-1}}{\partial x^\alpha} + d_2\frac{\partial^\alpha v_{n-1}}{\partial(-x)^\alpha} + \mu\frac{\partial v_{n-1}}{\partial x}\right)dt,$$

for $j = 3, 4, 5, \ldots$

$$u_0(x,t) = v_0(x) = \delta(x),$$

$$u_1(x,t) = v_0 + v_1 = \delta(x) + \left(\frac{d_1+(-1)^\alpha d_2}{2\Gamma(-\alpha)}\right)x^{-\alpha-1}\times t,$$

$$u_2(x,t) = v_0 + v_1 + v_2 = \delta(x) + \left(\frac{d_1+(-1)^\alpha d_2}{2\Gamma(-\alpha)}\right)x^{-\alpha-1}\times t$$
$$+\left(\left(\frac{d_1^2+(-1)^\alpha d_1 d_2 + d_2^2}{2\Gamma(-2\alpha)}\right)x^{-2\alpha-1} + \left(\frac{d_1+(-1)^\alpha d_2}{2\Gamma(-\alpha-1)}\right)x^{-\alpha-2}\right)\frac{1}{2}t^2,$$

$$\vdots$$

and so on. In this manner, the rest of the components of the homotopy perturbation solution can be obtained. If $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ and we compute more terms, then we can show that $u(x,t)$ is the pdf of the stable distribution with respect to x, or $p(x) = \lim_{n\to\infty} u_n(x,t) = S_\alpha(t^{\frac{1}{\alpha}}, \beta, \mu t)$ where $p(x)$ is the pdf of the stable distribution.

3.2 Stable Distribution pdf Appoximation Using ADM and VIM

Here we obtain the analytic approximation of the pdf of stable distributions by using the ADM and VIM for the three cases solved in Section 3.1.

3.2.1 Case 1: $1 < \alpha \leq 2$ for equation (1):

Consider the space-fractional PDE

$$\frac{\partial u}{\partial t} = D \frac{\partial^\alpha}{\partial x^\alpha} u(x,t), \quad x \in \mathbf{R}, \quad t > 0, \quad 0 < \alpha < 2, \tag{20}$$

subject to initial condition $u(x,0) = \delta(x)$, and D is a positive coefficient. First we will solve equation (38) with initial condition $u(x,0) = \delta(x)$ using the ADM. To do so, we construct the following recurrence relation:

$$u_0 = u(x,0) = \delta(x), u_{k+1} = \int_0^t \left(D \frac{\partial^\alpha u_k}{\partial x^\alpha} \right) dt, \quad k \geq 0.$$

So, the solution is obtained as:

$$u_1 = \int_0^t \left(D \frac{\partial^\alpha \delta(x)}{\partial x^\alpha} \right) dt = \frac{D}{\Gamma(-\alpha)} x^{-\alpha-1} \times t,$$

$$\vdots$$

$$u_n = \left(\frac{D^n}{\Gamma(-n\alpha)} x^{-n\alpha-1} \right) \times \left(\frac{1}{\Gamma(n+1)} t^n \right),$$

Therefore,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \delta(x) + \sum_{k=1}^{\infty} \left(\frac{D^k}{\Gamma(-k\alpha)} x^{-k\alpha-1} \right) \times \left(\frac{1}{\Gamma(k+1)} t^k \right).$$

To solve equation (20) using VIM instead, we set

$$u_{n+1} = u_n + \int_0^t \lambda \left(\frac{\partial u_n}{\partial s} - D_1 \frac{\partial^\alpha v_n}{\partial x^\alpha} \right) ds, \tag{21}$$

So

$$\delta u_{n+1} = \delta u_n + \delta \int_0^t \lambda \left(\frac{\partial u_n}{\partial s} - D_1 \frac{\partial^\alpha v_n}{\partial x^\alpha} \right) ds$$

$$= \delta u_n + \lambda \delta u_n + \int_0^t \left\{ -\frac{d\lambda}{ds} \right\} \delta u_n \, ds = 0, \tag{22}$$

the stationary conditions of equation (22) are:

$$1 + \lambda = 0, \ \lambda' = 0. \tag{23}$$

The Lagrange multiplier turns out to be $\lambda = -1$.

Using the new recently developed algorithm for the Lagrange multiplier (see [14]), for $m = 1$ we obtain:

$$1 + (-1)^{m-1} \lambda^{(m-1)} = 0 \Rightarrow 1 + \lambda = 0. \tag{24}$$

The extremum of the functional (21) is given by:

$$\frac{\partial(\lambda f)}{\partial y} - \frac{d}{ds} \left(\frac{\partial(\lambda f)}{\partial y'} \right) \Rightarrow \lambda' = 0, \tag{25}$$

Using equations (24) and (25), we get the same λ as equation (23). Substituting $\lambda = -1$ into equation (21), we get the following variational iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n}{\partial s} - D_1 \frac{\partial^\alpha u_n}{\partial x^\alpha} \right) ds, \tag{26}$$

where $u_0(x,t) = u(x,0) = \delta(x)$.

Therefore,

$$u(x,t) = \lim_{n \to \infty} u_n(x,t) = \delta(x) + \sum_{k=1}^{\infty} \left(\frac{D^k}{\Gamma(-k\alpha)} x^{-k\alpha - 1} \right) \times \left(\frac{1}{\Gamma(k+1)} t^k \right).$$

3.2.2 Case 2: The proportion of the pdf approximation of stable distributions according to x:

Consider the fractional PDE

$$\frac{\partial V}{\partial t} = D \frac{\partial^\alpha}{\partial(-x)^\alpha} V(x,t), \quad x \in \mathbf{R}, \ \ t > 0, \quad 0 < \alpha < 2, \tag{27}$$

subject to initial condition $V(x,0) = \delta(-x)$, and D is a positive coefficient. It is obvious that if $u(x,t)$ is the solution of equation (20), then $u(-x,t)$ solves the fractional PDE given by equation (27). So

$$V(x,t) = u(-x,t) = \delta(-x) + \sum_{k=1}^{\infty} \left(\frac{D^k}{\Gamma(-k\alpha)} (-x)^{-k\alpha - 1} \right) \times \left(\frac{1}{\Gamma(k+1)} t^k \right).$$

3.2.3 Case 3: The pdf approximation of stable distributions for $0 < \alpha \leq 2$

Consider the fractional PDE

$$\frac{\partial u}{\partial t} = -\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) - \frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t) + \mu\frac{\partial}{\partial x}u(x,t), \tag{28}$$

where $0 < \alpha \leq 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$ and $-\infty < \mu < \infty$, and $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$, subject to the initial condition $u(x,0) = \delta(x)$. The recurrence relation of ADM for equation (28) can be constructed as

$$u_0 = u(x,0) = \delta(x), u_{k+1} = \int_0^t \left(D_1\frac{\partial^\alpha u_k}{\partial x^\alpha} + D_2\frac{\partial^\alpha u_k}{\partial(-x)^\alpha} + \mu\frac{\partial u_k}{\partial x}u_k\right)dt, \qquad k \geq 0.$$

where $D_1 = -\frac{1+\beta}{2c}$ and $D_2 = -\frac{1-\beta}{2c}$.

So we derive the following recurrent relation

$$u_j = \int_0^t \left(D_1\frac{\partial^\alpha u_{j-1}}{\partial x^\alpha} + D_2\frac{\partial^\alpha u_{j-1}}{\partial(-x)^\alpha} + \mu\frac{\partial u_{j-1}}{\partial x}\right)dt,$$

for $j = 3, 4, 5, \ldots$. To solve equation (28) by means of the VIM, we set

$$u_{n+1} = u_n + \int_0^t \lambda\left(\frac{\partial u_n}{\partial s} - D_1\frac{\partial^\alpha v_n}{\partial x^\alpha} - D_2\frac{\partial^\alpha v_n}{\partial(-x)^\alpha} - \mu\frac{\partial v_n}{\partial x}\right)ds, \tag{29}$$

where $u_0(x,t) = u(x,0) = \delta(x)$. So we derive the following recurrent relation

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n}{\partial s} - D_1\frac{\partial^\alpha u_n}{\partial x^\alpha} - D_2\frac{\partial^\alpha u_n}{\partial(-x)^\alpha} - \mu\frac{\partial u_n}{\partial x}\right)ds.$$

In this manner the rest of the components of the VIM can be obtained. If $u(x,t) = \lim_{n\to\infty}u_n(x,t)$ and we compute more terms, then we can show that $u(x,t)$ is the pdf of the stable distribution with respect to x, as $S_\alpha\left(t^{\frac{1}{\alpha}}, \beta, \mu t\right)$ (the solution converges to the stable distribution's pdf).

4 PDF Approximation of Geo-Stable Distributions

We repeat in this section the derivation of the approximation for the pdfs as in Section 3 but do so for the geo-stable pdfs rather than the stable distributions. We use the same three analytic approximation methods (HPM, ADM, and VIM).

4.1 PDF Approximation of Geo-Stable Distributions Via HPM

We shall illustrate the applicability of HPM to geo-stable pdfs for the three cases in Section 3.

4.1.1 Case 1: $1 < \alpha \leq 2$ for equation (11):

Consider the space-fractional PDE

$$\frac{\partial u}{\partial t} = u(x,t) * C\frac{\partial^\alpha u}{\partial x^\alpha}, \qquad x \in \mathbf{R}, \quad t > 0, \tag{30}$$

subject to initial condition $u(x,0) = \delta(x)$ and C is a positive coefficient. To solve equation (30) with initial condition $u(x,0) = \delta(x)$ using HPM, we construct the following homotopy:

$$H(v,p) = (1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - v(x,t) * C\frac{\partial^\alpha v}{\partial x^\alpha}\right) = 0. \tag{31}$$

According to equation (31), we will have

$$v_1(x,t) = C\frac{x^{-\alpha-1}}{2^2\Gamma(-\alpha)} \times t,$$

$$v_2(x,t) = \left(\frac{C^2}{2^3}\frac{x^{-2\alpha-1}}{\Gamma(-2\alpha)} \times \frac{1}{2}t^2\right) + \int_0^t \left(\int_0^s \left(C\frac{(s-x)^{-\alpha-1}}{2^2\Gamma(-\alpha)} \times t \times C\frac{\partial^\alpha}{\partial x^\alpha}\delta(x)\right)dx\right)dt,$$

$$v_{n+1}(x,\mathrm{t}) = \int_0^t \left(\sum_{i=0}^n \sum_{j=0}^n p^{i+j} v_i(x,t) * C\frac{\partial^\alpha v_j}{\partial x^\alpha}\right), \qquad i+j = n-1, \qquad p = 1.$$

Therefore, the solution is

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

$$= \delta(x) + \sum_{k=1}^\infty \left(\int_0^t \left(\sum_{i=0}^k \sum_{j=0}^k p^{i+j} v_i(x,t) * C\frac{\partial^\alpha v_j}{\partial x^\alpha}\right)\right),$$

for $i+j = k-1, p = 1$.

4.1.2 Case 2: The pdf approximation of geo-stable distributions for $0 < \alpha \leq 2$

Consider the fractional PDE

$$\frac{\partial u}{\partial t} = u(x,t) * \left(-\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) - \frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t) + \mu\frac{\partial}{\partial x}u(x,t)\right) \tag{32}$$

where $0 < \alpha \leq 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$ and $-\infty < \mu < \infty$ and $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$. Given the defintion for the HPM, the homotopy for equation (32) can be constructed as

$$H(v,p) = (1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right)$$
$$+p\left(\frac{\partial v}{\partial t} + v(x,t) * d_1 \frac{\partial^\alpha v}{\partial x^\alpha} + v(x,t) * d_2 \frac{\partial^\alpha v}{\partial(-x)^\alpha} + \mu v(x,t) * \frac{\partial v}{\partial x}\right) = 0,$$

where $d_1 = -\frac{1+\beta}{2c}$ and $d_2 = -\frac{1-\beta}{2c}$.

For simplicity, taking $v_0(x,t) = u_0(x,t) = \delta(x)$, we derive the following recurrent relation

$$v_{n+1} = \int_0^t \left(\left(\sum_{i=0}^{n}\sum_{j=0}^{n} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial x^\alpha}\right)\right.$$
$$\left.\left(+\sum_{i=0}^{n}\sum_{j=0}^{n} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial(-x)^\alpha}\right) + \left(\sum_{i=0}^{n}\sum_{j=0}^{n} p^{i+j} v_i(x,t) * \mu \frac{\partial v_j}{\partial x}\right)\right) dt,$$

where $i + j = n - 1$ and $p = 1$.

$$u_{n+1}(x,t) = v_0 + v_1 + v_2 + \cdots + v_{n+1},$$

$$u_{n+1}(x,t) = \delta(x) + \sum_{k=1}^{n} \int_0^t \left(\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial x^\alpha}\right)\right.$$
$$\left.+\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial(-x)^\alpha}\right)\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * \mu \frac{\partial v_j}{\partial x}\right)\right) dt,$$

$$i + j = k - 1, \ \ p = 1,$$

and so on. In this manner, the rest of the components of the homotopy perturbation solution can be obtained. If $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ and we compute more terms, then we can show that $u(x,t)$ is the stable distribution's pdf with respect to x, as $S_\alpha(t^{\frac{1}{\alpha}}, \beta, \mu t)$ (the solution converges to the of geo-stable distribution's pdf). Therefore,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$
$$= \delta(x) + \sum_{k=1}^{\infty} \int_0^t \left(\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial x^\alpha}\right)\right.$$

$$+\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial(-x)^\alpha}\right)+\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * \mu \frac{\partial v_j}{\partial x}\right)\Bigg) dt,$$

where $i+j=k-1$and $p=1$.

4.2 PDF Approximation of Geo-Stable Distributions Via ADM

4.2.1 Case 1: $1<\alpha\leq 2$ for equation (11)

Consider the space-fractional PDE

$$\frac{\partial u}{\partial t}=u(x,t) * C\frac{\partial^\alpha u}{\partial x^\alpha}, \qquad x\in \mathbf{R}, \qquad t>0, \tag{33}$$

subject to initial condition $u(x,0)=\delta(x)$ and C is a positive coefficient. To solve equation (33) with initial condition $u(x,0)=\delta(x)$ using the ADM, we construct the following recurrence relation:

$$u_0=u(x,0)=\delta(x), u_{k+1}=\int_0^t\left(\sum_{i=0}^{k} A_i\right)dt: \qquad k=0,1,2,\cdots,$$

where

$$A_i=\frac{1}{i!}\left[\frac{d^i}{d\lambda^i}\left[\left(\sum_{j=0}^{i}\lambda^i u_i\right)* C\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{j=0}^{i}\lambda^i u_i\right)\right]\right]_{\lambda=0} : \qquad n=0,1,2,\cdots.$$

Therefore,

$$u(x,t)=\lim_{n\to\infty} u_n(x,t)$$

$$=\delta(x)+\sum_{k=1}^{\infty}\left(\int_0^t\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * C\frac{\partial^\alpha v_j}{\partial x^\alpha}\right)\right),$$

where $i+j=k-1$ and $p=1$.

4.2.2 Case 2: pdf approximation of geo-stable distributions for $0<\alpha\leq 2$

Consider the fractional PDE

$$\frac{\partial u}{\partial t}=u(x,t) * \left(-\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)-\frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t)+\mu\frac{\partial}{\partial x}u(x,t)\right), \tag{34}$$

where $0 < \alpha \leq 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$ and $-\infty < \mu < \infty$ and $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$. To solve equation (34) with initial condition $u(x,0) = \delta(x)$ using the ADM, we construct the following recurrence relation:

$$u_0 = u(x,0) = \delta(x),$$

$$u_{k+1} = \int_0^t \left(\sum_{i=0}^{k} A_i\right) dt: \qquad k = 0,1,2,\cdots,$$

where

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\lambda^i}\left[\begin{array}{l}\left(\sum_{j=0}^{i} \lambda^i u_i\right) * d_1 \frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{j=0}^{i} \lambda^i u_i\right)\\ +\left(\sum_{j=0}^{i} \lambda^i u_i\right) * d_2 \frac{\partial^\alpha}{\partial(-x)^\alpha}\left(\sum_{j=0}^{i} \lambda^i u_i\right)\end{array}\right.\right.$$

$$\left.\left.+\left(\sum_{j=0}^{i} \lambda^i u_i\right) * \mu \frac{\partial}{\partial x}\left(\sum_{j=0}^{i} \lambda^i u_i\right)\right]\right]_{\lambda=0} : \qquad n = 0,1,2,\cdots.$$

and $d_1 = -\frac{1+\beta}{2c}$ and $d_2 = -\frac{1-\beta}{2c}$.

Therefore,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

$$= \delta(x) + \sum_{k=1}^{\infty} \int_0^t \left(\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial x^\alpha}\right)\right.$$

$$+\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial(-x)^\alpha}\right)$$

$$\left.+\left(\sum_{i=0}^{k}\sum_{j=0}^{k} p^{i+j} v_i(x,t) * \mu \frac{\partial v_j}{\partial x}\right)\right) dt,$$

where $i + j = k - 1$ and $p = 1$.

4.3 PDF Approximation of Geo-Stable Distributions Via VIM

4.3.1 Case 1: $1 < \alpha \leq 2$ for equation (11)

Consider the space-fractional PDE

$$\frac{\partial u}{\partial t} = u(x,t) * C \frac{\partial^\alpha u}{\partial x^\alpha}, \qquad x \in \mathbf{R}, \quad t > 0, \tag{35}$$

subject to initial condition $u(x,0) = \delta(x)$, and C is a positive coefficient. To solve equation (35) by means of the VIM, we set

$$u_{n+1} = u_n + \int_0^t \lambda\left(\frac{\partial u_n}{\partial s} - v_n(x,t) * C\frac{\partial^\alpha v_n}{\partial x^\alpha}\right) ds.$$

Therefore,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \delta(x) + \sum_{k=1}^{\infty}\left(\int_0^t\left(\sum_{i=0}^{k}\sum_{j=0}^{k} v_i(x,t) * C\frac{\partial^\alpha v_j}{\partial x^\alpha}\right)\right),$$

where $i + j = k - 1$ and $p = 1$.

4.3.2 Case 2: pdf approximation of geo-stable distributions for $0 < \alpha \le 2$

Consider the fractional PDE

$$\frac{\partial u}{\partial t} = u(x,t) * \left(-\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) - \frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x,t) + \mu\frac{\partial}{\partial x}u(x,t)\right), \quad (36)$$

where $0 < \alpha \le 2$, $\alpha \ne 1$, $-1 \le \beta \le 1$ and $-\infty < \mu < \infty$, also $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$, and subject to initial condition $u(x,0) = \delta(x)$. To solve equation (36)using the VIM, we set

$$u_{n+1} = u_n + \int_0^t \lambda\left(\frac{\partial u_n}{\partial s} - u_n(x,t) * d_1\frac{\partial^\alpha u_n}{\partial x^\alpha}\right.$$
$$\left. - u_n(x,t) * d_2\frac{\partial^\alpha u_n}{\partial(-x)^\alpha} - u_n(x,t) * \mu\frac{\partial u_n}{\partial x}\right) ds.$$

So we derive the following recurrent relation

$$v_{n+1} = \int_0^t (\left(\sum_{i=0}^{n}\sum_{j=0}^{n} p^{i+j} v_i(x,t) * d_1\frac{\partial^\alpha v_j}{\partial x^\alpha}\right)$$
$$+\left(\sum_{i=0}^{n}\sum_{j=0}^{n} p^{i+j} v_i(x,t) * d_1\frac{\partial^\alpha v_j}{\partial(-x)^\alpha}\right)$$
$$+\left(\sum_{i=0}^{n}\sum_{j=0}^{n} p^{i+j} v_i(x,t) * \mu\frac{\partial v_j}{\partial x}\right))dt,$$

where $i + j = k - 1$ and $p = 1$.

In this manner, the rest of the components of the VIM can be obtained. If $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ and we compute more terms, then we can show that $u(x,t)$ is the stable distribution's pdf with respect to x, as $GS_\alpha\left(t^{\frac{1}{\alpha}}, \beta, \mu t\right)$ (the solution converges to the pdf of geo-stable distribution

$$
\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\
&= \delta(x) + \sum_{k=1}^{\infty} \int_0^t \left(\left(\sum_{i=0}^{k} \sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial x^\alpha} \right) \right. \\
&\quad + \left(\sum_{i=0}^{k} \sum_{j=0}^{k} p^{i+j} v_i(x,t) * d_1 \frac{\partial^\alpha v_j}{\partial(-x)^\alpha} \right) \\
&\quad \left. + \left(\sum_{i=0}^{k} \sum_{j=0}^{k} p^{i+j} v_i(x,t) * \mu \frac{\partial v_j}{\partial x} \right) \right) \mathrm{dt},
\end{aligned}
$$

where $i + j = k - 1$ and $p = 1$.

5 Numerial Experiments

The numerical solutions we derived (the truncated series of equations in the prior two sections) can contain large errors which are not always acceptable in real-world applications. Such errors raise some basic issues with regard to the properties of the analytic approximation of the pdf for stable and geo-stable distributions. Large errors are attributable to the effect of the precision used in the calculations, the convergence of the method, and the effect of the initial conditions. By increasing the precision (8 digits, 16 digits, etc), the absolute error will decrease because the truncation error is decreased.

Convergence is a condition that may be imposed on the numerical solution which ensures that the output of the simulation is a correct representation of the model we solve; that is, the numerical solution must tend towards the exact solution of the mathematical model when *n* (the number of terms in the obtained series) tends to infinity. Figure 1 demonstrates that by increasing the value of *n* (i.e., increasing the terms of analytic approximation (truncated series)), the analytic approximation tends to the stable distribution's pdf. Also, the HPM results are improved by using of the Padé approximation (PA). More details about Figure 1 are provided in Figure 2 where we show the pdf of stable distributions for different values of α and t.

To demonstrate why the initial conditions are important in the numerical scheme, we changed the initial conditions to assess the impact on the approximation solutions. In Figure 3 the pdf of geo-stable distributions is obtained via the HPM for different values of α, $t = 1$, and $t = 5$. It is clear from an examination of Figure 2 that the results of the analytic approximation with PA are better than the results of the analytic approximation without PA. Moreover, if the values of α exceed 1, then HPM with PA is very suitable. All of the computations for the analytic approximation of the pdf of the stable and geo-stable distributions will be done by three terms of the truncated series; if we calculate the additional terms of analytic solution, the results will be better.

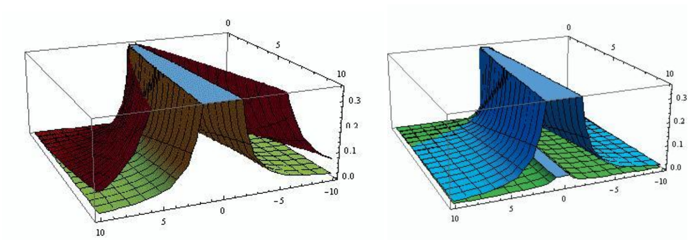

Figure 1: The analytic approximation of pdf of stable distributions with HPM for different values of v_is (v_1 and v_2) (left). The analytic approximation of pdf of stable distributions with HPM and PA for different values of v_is ($v_1$andv_2) (right) α = 0.8

6 Convergance Analysis

In this section, we study the convergence of two perturbation methods when applied to a space-fractional PDE. The two perturbation methods we study are the HPM and the homotopy analysis method (HAM). Because the results obtained by applying the HPM, ADM, and VIM applied to the space-fractional PDE produced the same result, for convergence of the methods, it is sufficient to demonstrate the convergence of just one of them.

To investigate the convergence of solutions obtained using the HAM and HPM, we must consider convergence in two ways: (1) the convergence of the series solutions we obtain to some fixed and finite value for each x in the domain of the nonlinear problem and (2) whether or not such a convergent series converges to the solution of the nonlinear problem. Let's first look at the convergence of the series solutions we obtain to some fixed and finite value for each x in the domain of the nonlinear problem. The answer to this convergence is provided by Liao [18] who demonstrated that a convergent series solution obtained via HAM exists.

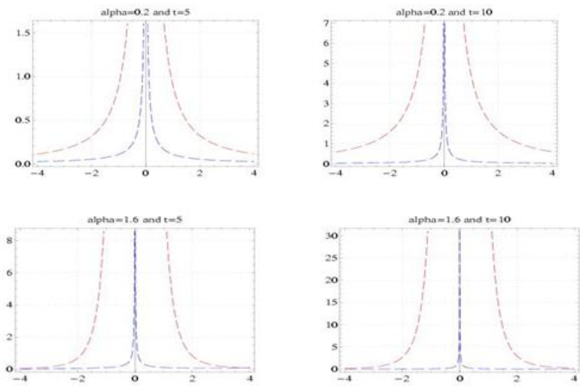

Figure 2: The plot of analytic approximation of pdf of stable distributions by HPM and HPM with PA for different values of α and t (red is HPM and blue is HPM with PA)

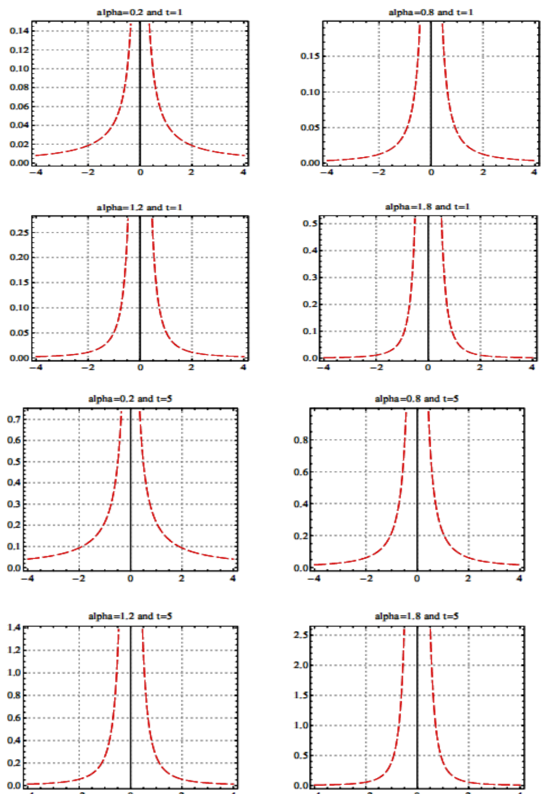

Figure 3: (Left to Right)The plot of analytic approximation of pdf of geo-stable distribution by HPM different values of α and $t = 1$ and $t = 5$

In practice, a series may not converge over the whole domain of the problem. In such cases, the following result may be useful. If the partial sum $S_k(x)$ is defined as follow

$$S_0(x) = u_0(x),\ \dots\ S_k(x) = u_0(x) + \sum_{m=1}^{k} u_m(x), \tag{37}$$

then the following necessary and sufficient conditions for the convergence of the series solution in the HAM are proven by Liao [18]:

Necessary conditions for convergence: For a specific nonlinear differential equation$N[u] = 0$, let $u(x)$ and $u_m(x)$ be the terms of HAM and $u(x) = \sum_{m=1}^{\infty} u_m(x)$, respectively, and let X be the domain of interest. Then, in order for $u(x)$ to converge, $\lim_{m\to\infty}|u_m(x)| = 0$ for all $x \in \Omega$, and there must exist a positive integer r such that $|u_m(x)| \le |u_{m-1}|$ for all $m > r$, and all $x \in \Omega$.

Sufficient conditions for convergence: For a specific nonlinear differential equation $N[u] = 0$, let $u(x) = \sum_{m=1}^{\infty} u_m(x)$, and $S_k(x)$ is as equation (37), and let Ω be the domain of interest. If for any real $\delta > 0$ there exists a positive integer r such that $|u(x) - S_k(x)| < \delta$ for all $k > r$ and all $x \in \Omega$, then the series solution $u(x)$ converges.

The second way to consider convergence is to determine whether or not such a convergent series converges to the solution of the nonlinear problem. The series obtained via HPM (the series solutions) are convergent for most cases. However, the convergent rate depends on the nonlinear operator $N(u)$. Moreover, He [10] made the following suggestions: (1) the second derivative of $N(u)$ with respect to u must be small because the parameter may be relatively large, i.e. $p \to 1$, and (2) the norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

For a convergent series to the solution (the analytic approximation of the pdf for the stable and geo-stable distributions), we present conditions where the series solutions will be convergent. The series solution $\sum_{k=0}^{\infty} u_k$, defined by the HPM, converges if $\exists\ 0 < \gamma < 1$ such that $\parallel v_{k+1} \parallel \le \gamma \parallel v_k \parallel$, $\forall k \ge k_0$, for some $k_0 \in N$. Also, suppose that the S_k is the partial sum of sequence $\{u_i\}_{i=0}^{\infty}$ (as equation (37)). If we can show that $\{S_k\}_0^{\infty}$ is a Cauchy sequence, then the sequence $\{S_k\}_0^{\infty}$ is convergent. This is because **R** is the complete space and any Cauchy sequence in a complete space is convergent.

For every $m, n \in N, n \ge m > k_0$, we have

$$\parallel S_n - S_m \parallel \le \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m-k_0+1} \parallel v_0 \parallel,$$

and since $0 < \gamma < 1$, we get $\lim_{n,m\to} \parallel S_n - S_m \parallel = 0$. Therefore, $\{S_k\}_0^{\infty}$ is a Cauchy sequence. It is known that the convergence region for the obtained truncated series solution in HPM may be limited and needs enhancements to enlarge the region of convergence (see Figure 4). We use the PA for increasing the convergence region of the HPM analytical solution. Figures 5 and 6 show that the HPM with the enhancement of PA is very effective, convenient, and quite accurate for such types of space-fractional PDEs. Application of PA to the truncated series solution obtained by HPM, ADM, and VIM will be an effective tool to increase the region of convergence and accuracy of the approximate solution even for large values of t. The rational approximations $[N/M]$ can be obtained by applying PA with respect to t to the obtained series solution such that $N + M \leq$ {highest power of the variable t in the truncated series solution}.

7 Conclusions

In this paper, we provide a new strategy for obtaining an analytic approximation for the pdf of the stable and geo-stable distributions by studying the space-fractional PDEs, the fundamental solutions of which are the pdf of tthese distributions. We show that three analytic approximation methods — homotopy perturbation method, Adomian decomposition method, and variational iteration method — can be used successfully for finding the solutions of a space-fractional PDE and that these solutions are the pdf of stable and geo-stable distributions. This suggests that these three analytic approximation methods are very powerful and efficient methods for finding the analytical solutions of the pdf for the stable and geo-stable distributions for a large class of them. One disadvantage of these methods is that the region of convergence is not large. However, by applying the Padé approximation method to the truncated series solution obtained by the HPM, we obtain an effective tool to increase the region of convergence and accuracy of the approximate solution even for large values of t.

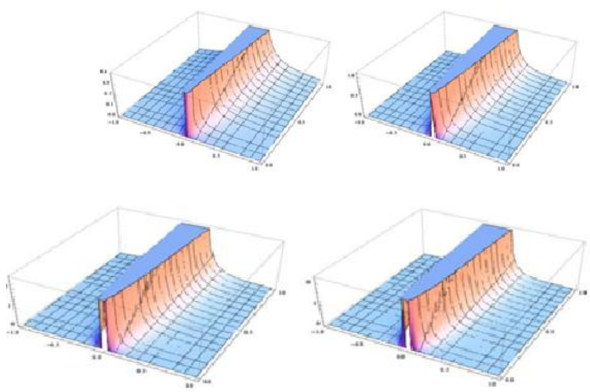

Figure 4: The plots of analytic approximation for pdf of geo-stable distribution with HPM, where $\alpha = 0.2$ and $\alpha = 0.8$, from left to right, respectively (both of left); The plots of analytic approximation for pdf of geo-stable distribution with HPM, where $\alpha = 1.2$ and $\alpha = 1.8$, from left to right, respectively (both of right)

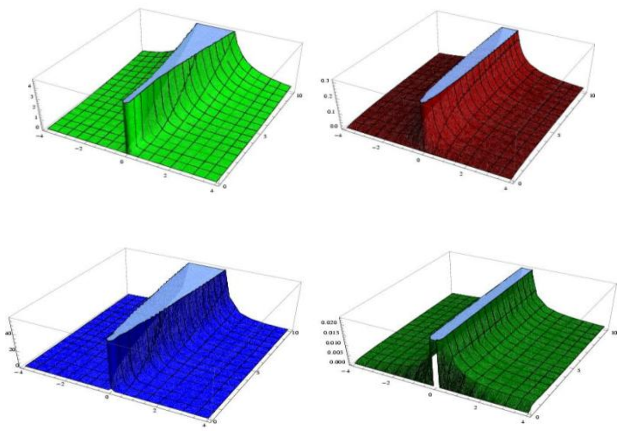

Figure 5: The plots of analytic approximation for pdf of stable distributions with HPM and HPM with PA, where $\alpha = 0.2$, from left to right, respectively (green and brown from left); The plots of analytic approximation of the pdf for stable distributions with HPM and HPM with PA, where $\alpha = 0.9$, from left to right, respectively (blue an green from right)

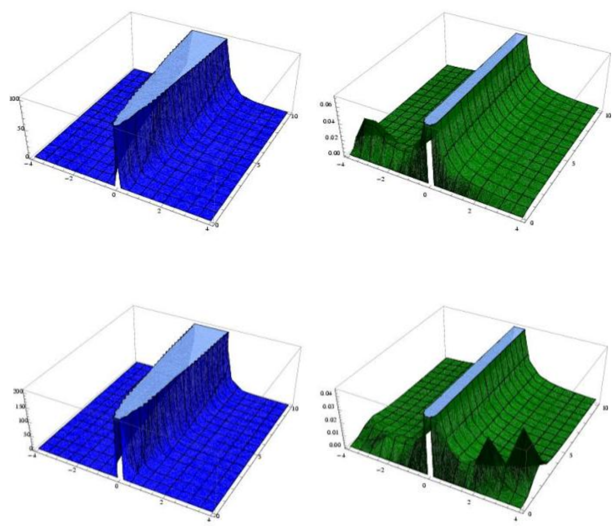

Figure 6: The plots of analytic approximation for pdf of stable distributions with HPM and HPM with PA, where $\alpha = 1.2$, from left to right, respectively (up); The plots of analytic approximation for pdf of stable distributions with HPM and HPM with PA, where $\alpha = 1.9$, from left to right, respectively (down)

Another disadvangage is that, in the small neighborhood of zero these methods do not exhibit good performance. By demonstrating that the terms of the series by HPM, ADM, and VIM hold true for a contraction, then the convergence of the analytic approximation of the pdf for the stable and geo-stable distributions will be guaranteed. In addition, an algorithm for evaluating the analytic approximation of the pdf for these distributions can be obtained.

Appendix: Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1:

Based on the relation between the geo-stable and stable distributions, we have

$$\psi(t) = [1 - \log\phi(t)]^{-1},$$

or

$$\phi(t) = \exp\left(\frac{\psi(t)-1}{\psi(t)}\right). \tag{38}$$

However, $\phi(t)$ is the fundamental solution of equation (6), if $u_0(x) = \delta(x)$, then $K(x,t) = \phi(t)$ (see [17]). So $\frac{\partial K}{\partial t} = C(i\omega)^{\alpha} K(x,t)$. According to the relation (38),

$$\frac{\partial}{\partial t}\left(\exp\left(\frac{K_1(x,t)-1}{K_1(x,t)}\right)\right) = C(i\omega)^{\alpha}\exp\left(\frac{K_1(x,t)-1}{K_1(x,t)}\right). \tag{39}$$

the left side of relation (39) is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\exp\left(\frac{K_1(x,t)-1}{K_1(x,t)}\right)\right) &= \frac{\partial}{\partial t}\left(\frac{K_1(x,t)-1}{K_1(x,t)}\right) \times \exp\left(\frac{K_1(x,t)-1}{K_1(x,t)}\right),\\ &= \frac{\frac{\partial K_1}{\partial t}.K_1 - \frac{\partial K_1}{\partial t}K_1 + \frac{\partial K_1}{\partial t}}{(K_1(x,t))^2} \times \exp\left(\frac{K_1(x,t)-1}{K_1(x,t)}\right)\\ &= \left(\frac{\partial K_1}{\partial t} \times \frac{1}{(K_1(x,t))^2}\right) \times \exp\left(\frac{K_1(x,t)-1}{K_1(x,t)}\right),\end{aligned}$$

so,

$$\frac{\partial K_1}{\partial t} = (K_1(x,t)) \times (C(i\omega)^{\alpha} K_1(x,t)).$$

By the definition of the inverse Fourier transform, we have

$$\left\{\frac{\partial K_1}{\partial t}\right\} = F^{-1}\{(K_1(x,t)) \times (C(i\omega)^{\alpha} K_1(x,t))\}, \frac{\partial F^{-1}\{K_1\}}{\partial t}$$

$$= F^{-1}\{(K_1(x,t))\} * F^{-1}\{C(i\omega)^{\alpha} K_1(x,t)\},$$

$$\frac{\partial u_1(x,t)}{\partial t} = u_1(x,t) * C\frac{\partial^{\alpha}}{\partial x^{\alpha}} u_1(x,t),$$

where $F^{-1}\{K_1(x,t)\} = u_1(x,t)$.

Now, we can obtain the initial condition. Since

$$K_1(x,t) = [1 - \log K(x,t)]^{-1}, \quad \text{and} \quad K(x,0) = 1,$$

then

$$K_1(x,0) = [1 - \log 1]^{-1} = 1.$$

Also,
$F^{-1}\{K_1(x,0)\} = \delta(x)$. ■

Proof of Theorem 2.2:
If $H\{u(x,t)\}$ is the Fourier transform of $u(x,t)$, where $u(x,t)$ is probability density function of the stable distribution, then

$$\frac{\partial u}{\partial t} = -\frac{1+\beta}{2c}\frac{\partial^{\alpha}}{\partial x^{\alpha}} u(x,t) - \frac{1-\beta}{2c}\frac{\partial^{\alpha}}{\partial(-x)^{\alpha}} u(x,t) + \mu\frac{\partial}{\partial x} u(x,t), \tag{40}$$

where $0 < \alpha \leq 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$, and $-\infty < \mu < \infty$, also $c = \cos\frac{\alpha\pi}{2}$ and $s = \sin\frac{\alpha\pi}{2}$. If $H(\omega,t)$ is the Fourier transform of $u(x,t)$ with respect to t, then equation (40) converts to the following initial value problem

$$\frac{\partial H}{\partial t} = -\frac{1+\beta}{2c}(i\omega)^{\alpha} H - \frac{1-\beta}{2c}(-i\omega)^{\alpha} H + (i\mu\omega)H, \tag{41}$$

where the initial value is $\delta(x)$. If $u(x,0) = \delta(x)$, then $H(\omega,0) = 1$, the solution of equation (41) can be obtained as

$$H(\omega,t) = \exp\{-\frac{1+\beta}{2c}(i\omega)^{\alpha} t - \frac{1-\beta}{2c}(-i\omega)^{\alpha} t + (i\mu\omega)\}. \tag{42}$$

Also, the connection between $H(\omega,t)$ and $H(\omega,t)$ is

$$H_1(\omega,t) = [1 - \log H(\omega,t)]^{-1}, \tag{43}$$

or

$$H(\omega,t) = \exp(\frac{H_1(\omega,t) - 1}{H_1(\omega,t)}),$$

where $H_1(\omega,t) = F\{u_1(x,t)\}$ and $u_1(x,t)$ is the pdf of geo-stable distributions. From relation (42) and (43), we have

$$\frac{\partial H_1}{\partial t} \times \frac{1}{(H_1(\omega,t))^2} = -\frac{1+\beta}{2c}(i\omega)^{\alpha} - \frac{1-\beta}{2c}(-i\omega)^{\alpha} + (i\mu\omega)$$

or

$$\frac{\partial H_1}{\partial t} = \big(H_1(\omega,t)\big) \times \big(-\frac{1+\beta}{2c}(i\omega)^\alpha H_1(\omega,t) - \frac{1-\beta}{2c}\big(-i\omega\big)^\alpha H_1(\omega,t)\big)$$
$$+(i\mu\omega)H_1(\omega,t).$$

By the definition of the inverse Fourier transform, we will have

$$\frac{\partial u_1}{\partial t} = u_1(x,t) * \left(-\frac{1+\beta}{2c}\frac{\partial^\alpha}{\partial x^\alpha}u_1(x,t) - \frac{1-\beta}{2c}\frac{\partial^\alpha}{\partial(-x)^\alpha}u_1(x,t) + \mu\frac{\partial}{\partial x}u_1(x,t)\right),$$

where $F^{-1}\{H_1(\omega,t)\} = u_1(x,t)$.

Now, we are going to get the initial condition. Since

$$H_1(\omega,t) = [1 - \log H(\omega,t)]^{-1},$$

and $H(\omega,0) = 1$, then $H_1(\omega,0) = [1 - \log 1]^{-1} = 1$. Also,

$$F^{-1}\{H_1(x,0)\} = \delta(x). \qquad \blacksquare$$

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