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A note of the induced topological pressure for topological systems

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Abstract

In this paper, we give an equivalent definition of the induced topological pressure [8]. We also set up a relation for two induced topological pressures with a factor map by using a method which is different from that of [9].

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1 Introduction and statement of main result

Throughout this paper, a topological dynamical system (for short TDS) means a pair (X, f), where f is a continuous map from a compact metric space (X, d) to itself. For $n \in \mathbb{N}$, the n-th Bowen metric d_n on X is defined by

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)) : i = 0, 1, \ldots, n-1\}.$$

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Recall that $C(X,\mathbb{R})$ is the Banach algebra of real-valued continuous functions of X equipped with the supremum norm. For $\varphi \in C(X,\mathbb{R})$, let $(S_n\varphi)(x) := \sum_{i=0}^{n-1}\varphi(f^i x)$.

The notion of the topological entropy plays an important role in topological dynamics and dimension theory [1, 2, 6]. In 1971, Bowen [4] considered a factor map $\pi : (X, f) \to (Y, g)$, and showed that

$$h(f) \leq h(g) + \sup_{y\in Y} h(f, \pi^{-1}(y)), \tag{1}$$

where $h(f, K)$ denotes the entropy of a compact subset $K \subseteq X$ with respect to f.

Topological pressure is a generalization to topological entropy for dynamical systems. It was first introduced by Ruelle [5] for expansive dynamical systems, and later by Walters [3, 6] for the general case. Recently, the theory for dynamical systems with different time-scalings has been developed. Jaerisch, Kesseböhmer, and Lamei [7] studied the induced topological pressure of a countable state Markov shift. In [8], the authors defined the induced topological pressure for a topological dynamical system, and established a variational principle for it. In this paper, we give an equivalent definition of the induced topological pressure. We also set up a relation for two induced topological pressures with a factor map by using a method which is different from that of [9].

Let (X, f) be a TDS and $\psi \in C(X,\mathbb{R})$ with $\psi > 0$. For $x \in X, T > 0, \epsilon > 0$, define

$$n(x, T) = \inf\{n \in \mathbb{N} : S_n\psi(x) \geq T\}$$

and

$$B_T(x, \epsilon, f) = \{y \in X : d_{n(x,T)}(x, y) < \epsilon\}; \overline{B}_T(x, \epsilon, f) = \{y \in X : d_{n(x,T)}(x, y) \leq \epsilon\}.$$

Let K be a compact set of X. A subset $F_T \subset X$ is called a (ψ, T, ϵ)-spanning set of K with respect to f, if for any $y \in K$, there exists $x \in F_T$ with $d_{n(x,T)}(x, y) \leq \epsilon$. Let $r_T(f, K, \epsilon)$ denotes the smallest cardinality of any (ψ, T, ϵ)-spanning set of K. Obviously $r_T(f, K, \epsilon) < \infty$. Define

$$r(f, K, \epsilon) = \limsup_{T\to\infty} \frac{1}{T} \log r_T(f, K, \epsilon).$$

Clearly if $0 < \epsilon_1 < \epsilon_2$, then $r_T(f, K, \epsilon_1) \geq r_T(f, K, \epsilon_2)$.

Definition 1. *We define the ψ-induced topological entropy of K (with respect to f) by*

$$h_\psi(f,K)=\lim_{\epsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\log r_T(f,K,\epsilon) \tag{2}$$

Remarks.
$h_1(f,X)=h(f)$, where $h(f)$ denotes the topological entropy of f [3, 6].

Definition 2. *Let (X,f) be a TDS, and let K be a compact set of X, $\varphi,\psi\in C(X,\mathbb{R})$ with $\psi>0$. For $T>0,\epsilon>0$, put*

$$\begin{aligned}&Q_{\psi,T}(f,K,\varphi,\epsilon)\\ =&\inf\{\sum_{x\in F_T}\exp(S_{n(x,T)}\varphi)(x): F_T \text{ is a } (\psi,T,\epsilon)\text{-spanning set of } K\}.\end{aligned}$$

We define the ψ-induced topological pressure of φ (with respect to f and K) by

$$P_\psi(f,K,\varphi)=\lim_{\epsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\log Q_{\psi,T}(f,K,\varphi,\epsilon) \tag{3}$$

Remarks.
(i) If $0<\epsilon_1<\epsilon_2$, then $Q_{\psi,T}(f,K,\varphi,\epsilon_1)\geq Q_{\psi,T}(f,K,\varphi,\epsilon_2)$, which implies the existence of the $P_\psi(f,K,\varphi)$ in (3).
(ii) $P_1(f,X,\varphi)=P(\varphi)$, where $P(\varphi)$ denotes the topological pressure of φ [3, 6].
(iii) It is easy to see $P_\psi(f,X,\varphi)=P_\psi(\varphi)$, where $P_\psi(\varphi)$ denotes the ψ-induced topological pressure of φ [8].

By using a method which is different from that of [9], we obtain the result of this paper, as follows.

Theorem 1.1. *Let $(X,d),(Y,\rho)$ be compact metric spaces, and let $f:X\to X, g:Y\to Y$ be continuous maps, $\pi:X\to Y$ a factor map, i.e., a continuous surjective map with $\pi\circ f=g\circ\pi$, $\varphi,\psi\in C(Y,\mathbb{R})$ with $\psi>0$. Then*

$$P_{\psi\circ\pi}(\varphi\circ\pi)\leq P_\psi(\varphi)+\sup_{y\in Y}h_{\psi\circ\pi}(f,\pi^{-1}(y)). \tag{4}$$

2 Some lemmas

In this section, we give some lemmas, which will be needed for the proof of Theorem 1.1.

Lemma 2.1. *Let* (Y, g) *be a TDS, and let* ρ *be a compatible metric on* Y, $\psi \in C(Y, \mathbb{R})$ *with* $\psi > 0, m = \min\{\psi(x) : x \in Y\}$. *For each* $y \in \overline{B}_T(x, \delta, g)$, *we have*

$$|n(x,T) - n(y,T)| \le \frac{T+m}{m^2} var(\psi, \delta) + \frac{\|\psi\|}{m}.$$

where $var(\psi, \delta) := \sup\{|\psi(x) - \psi(y)| : \rho(x, y) \le \delta\}$.

Proof Clearly $n(x,T) \le \frac{T}{m} + 1$ for any $x \in Y$. Notice for each $y \in \overline{B}_T(x, \delta, g)$,

$$m|n(x,T) - n(y,T)| - n(x,T) var(\psi, \delta) \le |S_{n(x,T)}\psi(x) - S_{n(y,T)}\psi(y)| \le \|\psi\|.$$

Then

$$|n(x,T) - n(y,T)| \le \frac{n(x,T)}{m} var(\psi, \delta) + \frac{\|\psi\|}{m} \le \frac{T+m}{m^2} var(\psi, \delta) + \frac{\|\psi\|}{m}.$$

□

Lemma 2.2. *Let* (Y, g) *be a TDS, and let* ρ *be a compatible metric on* Y, $\varphi, \psi \in C(Y, \mathbb{R})$ *with* $\psi > 0, m = \min\{\psi(x) : x \in Y\}$. *For each* $y \in \overline{B}_T(x, \delta, g)$, *we have*

$$\exp S_{n(y,T)}\varphi(y) \le e^{(\frac{T}{m}+1) var(\varphi,\delta) + \frac{T+m}{m^2}\|\varphi\| var(\psi,\delta) + \frac{\|\psi\|\|\varphi\|}{m}} \exp S_{n(x,T)}\varphi(x),$$

where $var(\psi, \delta) := \sup\{|\psi(x) - \psi(y)| : \rho(x, y) \le \delta\}$.

3 The proof of Theorem 1.1

Now we give the proof of Theorem 1.1. Let $m = \min\{\psi(x) : x \in Y\}$. To show the inequality, for any $\epsilon > 0$, we choose $\delta_1 > 0$ small enough so that

$$d(u, v) < 4\delta_1 \Rightarrow d_{2+[\frac{\|\psi\|}{m}]}(u, v) \le \epsilon, \tag{5}$$

where $[\frac{\|\psi\|}{m}]$ denotes the integer part of $\frac{\|\psi\|}{m}$. Clearly, we may assume

$$a := \sup_{y\in Y} h_{\psi\circ\pi}(f,\pi^{-1}(y)) < \infty.$$

Fix $\delta_1 > 0$ and $\tau > 0$. For any $y \in Y$, we choose $T_y > 0$ such that there exist a $(\psi\circ\pi, T_y, \delta_1)$-spanning set E_y of $\pi^{-1}(y)$ with minimal cardinality such that $|E_y| = r_{T_y}(f,\pi^{-1}(y),\delta_1)$ and

$$\log r_{T_y}(f,\pi^{-1}(y),\delta_1) \leq (h_{\psi\circ\pi}(f,\pi^{-1}(y)) + \tau)T_y \leq (a+\tau)T_y.$$

Denote $U_y = \{u \in X : \exists z \in E_y \text{ s.t } d_{n(z,T_y)}(u,z) < 2\delta_1\}$, then U_y is an open neighborhood of $\pi^{-1}(y)$ and

$$(X \setminus U_y) \cap \bigcap_{\gamma>0} \pi^{-1}(\overline{B_\gamma(y)}) = \emptyset,$$

where $B_\gamma(y) = \{z \in Y : \rho(y,z) < \gamma\}$. By the finite intersection property of compact sets, there is a $W_y = B_{\gamma_y}(y), (\gamma_y > 0)$ for which $\pi^{-1}(W_y) \subset U_y$. Since Y is compact, there exists $W_{y_1}, W_{y_2} \ldots W_{y_r}$ cover Y. Let $\delta_2 > 0$ be a Lebesgue number for Y for this open cover. For $T > 0$, we choose $0 < \delta < \frac{1}{2}\delta_2$ so that $\frac{T+m}{m^2}var(\psi,\delta) + \frac{\|\psi\|}{m} \leq 2 + [\frac{\|\psi\|}{m}]$. Let F_T be a (ψ, T, δ)-spanning set of Y. For each $y \in F_T, 0 \leq j < n(y,T)$, pick $\triangle_y(j) \in \{y_1, y_2 \ldots y_r\}$ such that $\overline{B_\delta(g^j(y))} \subset W_{\triangle_y(j)}$. Define recursively

$$\begin{aligned} t_0(y) &= 0; \\ t_1(y;z_0) &= n(z_0, T_{\triangle_y(0)}), z_0 \in E_{\triangle_y(0)}; \\ t_2(y;z_0,z_1) &= t_1(y;z_0) + n(z_1, T_{\triangle_y(t_1(y;z_0))}), z_1 \in E_{\triangle_y(t_1(y;z_0))}; \\ &\cdots \\ t_{s+1}(y;z_0,z_1,\ldots,z_s) &= t_s(y;z_0,z_1,\ldots,z_{s-1}) + n(z_s, T_{\triangle_y}(t_s(y;z_0,z_1,\ldots,z_{s-1}))), \end{aligned}$$

$$z_s \in E_{\triangle_y(t_s(y;z_0,z_1,\ldots,z_{s-1}))} \tag{6}$$

until one gets a $t_{q+1}(y;z_0,z_1\ldots z_q) \geq n(y,T)$. Clearly the number of q depends on the choice of $z_0, z_1 \ldots z_{q-1}$. Set $q(y;z_0,z_1\ldots z_{q-1}) = q$, we yet denote $q(y;z_0,z_1\ldots z_{q-1})$ by q for convenience. For $y \in F_T$ and

$$z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))}, \ldots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))},$$

define

$$V(y;z_0,z_1,\ldots,z_q) = \{u \in X : d(f^{t+t_s(y;z_0,z_1,\ldots,z_{s-1})}(u), f^t(z_s)) < 2\delta_1$$

$$\text{for all } 0 \leq t < n(z_s, T_{\triangle_y}(t_s(y; z_0, z_1, \ldots, z_{s-1}))), 0 \leq s \leq q\}.$$

It is not hard to see that

$$\bigcup_{z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))}, \ldots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}} V(y; z_0, z_1, \ldots, z_q) \supset \pi^{-1}(\overline{B}_T(y, \delta, g)). \tag{7}$$

In fact, for any $u \in \pi^{-1}(\overline{B}_T(y, \delta, g))$, we have

$$\rho(g^j(y), g^j(\pi u)) \leq \delta, \quad \forall 0 \leq j < n(y, T).$$

Then

$$\pi(f^j(u)) = g^j(\pi u) \in \overline{B_\delta(g^j(y))} \subset W_{\triangle_y(j)}, \quad \forall 0 \leq j < n(y, T).$$

This implies that

$$f^j(u) \in \pi^{-1}(W_{\triangle_y(j)}) \subset U_{\triangle_y(j)}, \quad \forall 0 \leq j < n(y, T), \tag{8}$$

and hence, there exists $\widetilde{z_0} \in E_{\triangle_y(0)}$ with $d_{n(\widetilde{z_0}, T_{\triangle_y(0)})}(\widetilde{z_0}, u) < 2\delta_1$. If $n(\widetilde{z_0}, T_{\triangle_y(0)}) \geq n(y, T)$, let $t_1(y; \widetilde{z_0}) = n(\widetilde{z_0}, T_{\triangle_y(0)})$, we have $u \in V(y; \widetilde{z_0})$ and finish the proof. Otherwise, it follows from (8) that there exists $\widetilde{z_1} \in E_{\triangle_y(t_1(y;\widetilde{z_0}))}$ such that

$$d_{n(\widetilde{z_1}, T_{\triangle_y(t_1(y;\widetilde{z_0}))})}(\widetilde{z_1}, f^{n(\widetilde{z_0}, T_{\triangle_y(0)})}(u)) < 2\delta_1.$$

By this means, we get the minimal $q(y; \widetilde{z_0}, \widetilde{z_1} \ldots \widetilde{z_{q-1}})$ with $t_{q+1}(y; \widetilde{z_0}, \widetilde{z_1} \ldots \widetilde{z_q}) \geq n(y, T)$. This implies that $u \in V(y; \widetilde{z_0}, \widetilde{z_1}, \ldots, \widetilde{z_q})$. Since u is arbitrary, this shows (7).

Notice for each $x \in X$,

$$\begin{aligned} n(x, T) &= \inf\{n : (S_n \psi \circ \pi)(x) \geq T\} \\ &= \inf\{n : \sum_{i=0}^{n-1} \psi \circ \pi(f^i(x) \geq T\} \\ &= \inf\{n : \sum_{i=0}^{n-1} \psi(g^i \pi(x)) \geq T\} \\ &= n(\pi(x), T). \end{aligned} \tag{9}$$

If $V(y; z_0, z_1, \ldots, z_q) \cap \pi^{-1}(\overline{B}_T(y, \delta, g)) \neq \emptyset$, pick any

$$v(y; z_0, z_1, \ldots, z_q) \in V(y; z_0, z_1, \ldots, z_q) \cap \pi^{-1}(\overline{B}_T(y, \delta, g)) \neq \emptyset,$$

we have

$$\overline{B}_T(v(y;z_0,z_1,\ldots,z_q),\epsilon,f)\supset V(y;z_0,z_1,\ldots,z_q). \tag{10}$$

In fact, for any $v\in V(y;z_0,z_1,\ldots,z_q)$, we have for all

$$0\le t<n(z_s,T_{\triangle_y}(t_s(y;z_0,z_1,\ldots,z_{s-1})))$$

and $0\le s\le q$,

$$d(f^{t+t_s(y;z_0,z_1,\ldots,z_{s-1})}(v),f^t(z_s))<2\delta_1.$$

Since $v(y;z_0,z_1,\ldots,z_q)\in V(y;z_0,z_1,\ldots,z_q)$, we get

$$d(f^{t+t_s(y;z_0,z_1,\ldots,z_{s-1})}(v(y;z_0,z_1,\ldots,z_q)),f^t(z_s))<2\delta_1.$$

Hence

$$d(f^j(v(y;z_0,z_1,\ldots,z_q)),f^j(v))<4\delta_1,\quad 0\le j\le t_{q+1}(y;z_0,z_1,\ldots,z_q).$$

By Lemma 2.1, we have

$$\begin{aligned}n(v(y;z_0,z_1,\ldots,z_q),T)&=n(\pi(v(y;z_0,z_1,\ldots,z_q),T)\\&\le n(y,T)+\frac{T+m}{m^2}var(\psi,\delta)+\frac{\|\psi\|}{m}\\&\le n(y,T)+2+[\frac{\|\psi\|}{m}].\end{aligned}$$

Now that $n(y,T)\le t_{q+1}(y;z_0,z_1,\ldots,z_q)$, it follows from (5) that

$$d_{n(y,T)+2+[\frac{\|\psi\|}{m}]}(v(y;z_0,z_1,\ldots,z_q),v)\le\epsilon.$$

Therefore

$$d_{n(v(y;z_0,z_1,\ldots,z_q),T)}(v(y;z_0,z_1,\ldots,z_q),v)\le\epsilon.$$

That is, we show (10). Combing (7) and (10), we obtain

$$\begin{aligned}\bigcup_{z_0\in E_{\triangle_y(0)},z_1\in E_{\triangle_y(t_1(y;z_0))},\ldots,z_q\in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}}&\overline{B}_T(v(y;z_0,z_1,\ldots,z_q),\epsilon,f)\\&\supset\pi^{-1}(\overline{B}_T(y,\delta,g)).\end{aligned}$$

Let

$$E_T=\{v(y;z_0,z_1,\ldots,z_q):y\in F_T,z_0\in E_{\triangle_y(0)},z_1\in E_{\triangle_y(t_1(y;z_0))},$$

$$\ldots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}\}.$$

Clearly E_T is a $(\psi \circ \pi, T, \epsilon)$-spanning set of X. For $y \in F_T$, there exists a permissible $(z_0^{'}, z_1^{'}, \ldots, z_q^{'})$ such that the number of permissible $(z_0, z_1, \ldots, z_q)$ is at most

$$N_y = \prod_{s=0}^{q} r_{T_{\triangle_y(t_s(y;z_0^{'},z_1^{'},\ldots,z_{s-1}^{'}))}}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{'}, z_1^{'}, \ldots, z_{s-1}^{'}))), \delta_1), \qquad (11)$$

where $t_0(y; z_{-1}^{'}) = 0$.

To show (11), we give some notions which will be needed in next proof. Following (6), we suppose $q(y; z_0, z_1 \ldots z_{q-1}) \geq 1$. For each $1 \leq s \leq q(y; z_0, z_1 \ldots z_{q-1})$, if $z_{s-1} \in E_{\triangle_y(t_{s-1}(y;z_0,z_1,\ldots,z_{s-2}))}$, we call z_{s-1} *directs* $E_{\triangle_y(t_s(y;z_0,z_1,\ldots,z_{s-1}))}$ and $E_{\triangle_y(t_{s-1}(y;z_0,z_1,\ldots,z_{s-2}))}$ is a *corresponding set* of $E_{\triangle_y(t_s(y;z_0,z_1,\ldots,z_{s-1}))}$. We say a permissible $(z_0, z_1, \ldots, z_q)$ is a $q+1$-*string*, and z_q is a *terminal point* of the permissible $(z_0, z_1, \ldots, z_q)$. For each $z \in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}$, if z is a terminal point of a $q+1$-string, we also say $E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}$ is a *terminal set of* $q+1$-*step*.

Now we show (11). Let

$$p = \max\{q(y; z_0, z_1, \ldots, z_{q-1}) : z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))},$$
$$\ldots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}\},$$

and

$$\mathcal{E} := \{E_{y_1}, E_{y_2}, \ldots, E_{y_r}\}.$$

If $p = 0$, it is clear that (11) holds.
If $p = 1$, there exists terminal sets of 2-step. We assume $E_{01}, \ldots, E_{0p_1} \in \mathcal{E}, (1 \leq p_1 \leq |E_{\triangle_y(0)}|)$ are all terminal sets of 2-step and $|E_{01}| = \max\{|E_{0i}| : 1 \leq i \leq p_1\}$. Then the sum of the number of all 1-strings and the number of all 2-strings is at most $|E_{\triangle_y(0)}||E_{01}|$. Let $z_0^{'} \in E_{\triangle_y(0)}$ directs E_{01}. Then the permissible $(z_0^{'})$ such that (11) holds.
If $p = 2$, there exists terminal sets of 3-step. We assume

$$E_{0l_11}, \ldots, E_{0l_1s_1}; E_{0l_21}, \ldots, E_{0l_2s_2}; \ldots; E_{0l_t1} \ldots E_{0l_ts_t}, (1 \leq t \leq p_1)$$

are all terminal sets of 3-step and satisfy the following:
(i) For each $1 \leq i \leq t, 1 \leq j \leq s_i$, E_{0l_i} is a corresponding set of E_{0l_ij}, where $1 \leq s_i \leq |E_{0l_i}|$.

(ii) For each $1 \leq i \leq t$, $|E_{0l_i1}| = \max\{|E_{0l_ij}| : 1 \leq j \leq s_i\}$.
There exists $1 \leq k \leq t$ with

$$|E_{0l_k}||E_{0l_k1}| = \max\{|E_{0l_i}||E_{0l_i1}| : 1 \leq i \leq t\}.$$

Considering that the possibility of the existent terminal set of 2-step, if

$$|E_{0l_k}||E_{0l_k1}| \geq |E_{01}|,$$

we obtain that the number of permissible $(z_0, z_1, \dots, z_q)$ is at most

$$|E_{\triangle_y(0)}||E_{0l_k}||E_{0l_k1}|.$$

Choose $z_0^{'} \in E_{\triangle_y(0)}$ with $z_0^{'}$ directs E_{0l_k}, $z_1^{'} \in E_{0l_k}$ with $z_1^{'}$ directs E_{0l_k1}. Then the permissible $(z_0^{'}, z_1^{'})$ such that (11) holds. If $|E_{01}| \geq |E_{0l_k}||E_{k1}|$, we have the number of permissible $(z_0, z_1, \dots, z_q)$ is at most $|E_{\triangle_y(0)}||E_{01}|$ and permissible $(z_0^{'})$ with $z_0^{'}$ directs E_{01} such that (11) holds.
Proceeding in this way, if $p > 2$, for each $1 \leq i \leq t$, we assume there exists a permissible $(z_1^{(i)}, \dots, z_q^{(i)})$ such that the number of permissible $(z_1, \dots, z_q)$ with $z_1 \in E_{0l_i}$ is at most

$$\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(i)},z_1^{(i)},\dots,z_{s-1}^{(i)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(i)}, z_1^{(i)}, \dots, z_{s-1}^{(i)}))), \delta_1),$$

where $z_1^{(i)} \in E_{0l_i}$ and $z_0^{(i)}$ directs $E_{0l_i}, q := q(z_0^{(i)}, z_1^{(i)}, \dots, z_{q-1}^{(i)})$.
There exists $1 \leq k \leq t$ with

$$\begin{aligned}
&\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(k)},z_1^{(k)},\dots,z_{s-1}^{(k)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(k)}, z_1^{(k)}, \dots, z_{s-1}^{(k)}))), \delta_1) \\
= \ &\max\{\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(i)},z_1^{(i)},\dots,z_{s-1}^{(i)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(i)}, z_1^{(i)}, \dots, z_{s-1}^{(i)}))), \delta_1) : 1 \leq i \leq t\}.
\end{aligned}$$

If

$$\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(k)},z_1^{(k)},\dots,z_{s-1}^{(k)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(k)}, z_1^{(k)}, \dots, z_{s-1}^{(k)}))), \delta_1) \geq |E_{01}|,$$

then the number of permissible $(z_0, z_1, \dots, z_q)$ is at most

$$\prod_{s=0}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(i)},z_1^{(i)},\dots,z_{s-1}^{(i)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(i)}, z_1^{(i)}, \dots, z_{s-1}^{(i)}))), \delta_1).$$

This implies $(z_0^{(k)}, z_1^{(k)}, \ldots, z_q^{(k)})$ such that (11) holds.
If

$$\prod_{s=1}^{q} r_{T_{\triangle_y(t_s(y;z_0^{(k)},z_1^{(k)},\ldots,z_{s-1}^{(k)}))}}(f, \pi^{-1}(\triangle_y(t_s(y;z_0^{(k)}, z_1^{(k)}, \ldots, z_{s-1}^{(k)}))), \delta_1) \leq |E_{01}|,$$

we have the number of permissible $(z_0, z_1, \ldots, z_q)$ is at most $|E_{\triangle_y(0)}||E_{01}|$ and permissible $(z_0^{'})$ with $z_0^{'}$ directs E_{01} such that (11)holds and finish the proof of (11).

Let

$$v \in V(y; z_0^{'}, z_1^{'}, \ldots, z_q^{'}), N = \max\{n(z, T_{y_i}) : z \in E_{y_i}, i = 1, 2 \ldots r\}.$$

Then

$$\begin{aligned}
\log N_y &= \sum_{s=0}^{q} \log r_{T_{\triangle_y(t_s(y;z_0^{'},z_1^{'},\ldots,z_{s-1}^{'}))}}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{'}, z_1^{'}, \ldots, z_{s-1}^{'}))), \delta_1) \\
&\leq (a+\tau)(T_{\triangle_y(0)} + T_{\triangle_y(t_1(y;z_0^{'}))} + \ldots + T_{\triangle_y(t_q(y;z_0^{'},z_1^{'},\ldots,z_q^{'}))}) \\
&\leq (a+\tau)(S_{n(z_0^{'},T_{\triangle_y(0)})}\psi \circ \pi(z_0^{'}) + \ldots + S_{n(z_q^{'},T_{\triangle_y(t_q(y;z_0^{'},z_1^{'},\ldots,z_q^{'})})}\psi \circ \pi(z_q^{'})) \\
&\leq (a+\tau)[(n(y,T)+N)Var(\psi \circ \pi, 2\delta_1) + S_{n(y,T)+N}\psi \circ \pi(v)]. \qquad (12)
\end{aligned}$$

It follows from (9) and Lemma 2.1 that

$$n(y, T) \leq n(v, T) + 2 + [\frac{\|\psi\|}{m}]$$

and

$$\begin{aligned}
&(a+\tau)[(n(y,T)+N)Var(\psi \circ \pi, 2\delta_1) + S_{n(y,T)+N}\psi \circ \pi(v)] \\
\leq&(a+\tau)[(n(y,T)+N)Var(\psi \circ \pi, 2\delta_1) + S_{n(v,T)}\psi \circ \pi(v) + (2 + [\frac{\|\psi\|}{m}] + N)\|\psi\|] \\
\leq&(a+\tau)[(\frac{T}{m} + 1 + N)Var(\psi \circ \pi, 2\delta_1) + T + \|\psi\| + (2 + [\frac{\|\psi\|}{m}] + N)\|\psi\|], \qquad (13)
\end{aligned}$$

where $Var(\psi \circ \pi, \delta) = \sup\{|\psi \circ \pi(x) - \psi \circ \pi(y)| : d(x, y) < \delta, x, y \in X\}$. Let

$v := v(y; z_0, z_1, \ldots, z_q)$. Combining (13) and Lemma 2.2, we have

$$\begin{aligned}
&\sum_{v\in E_T} \exp(S_{n(v,T)}\varphi\circ\pi(v)) \\
\le& \sum_{y\in F_T}\ \sum_{v\in V(y;z_0,z_1,\ldots,z_q)\cap\pi^{-1}(\overline{B}_T(y,\delta,g))} \exp(S_{n(v,T)}\varphi\circ\pi(v)) \\
\le& \sum_{y\in F_T}\ \sum_{v\in V(y;z_0,z_1,\ldots,z_q)\cap\pi^{-1}(\overline{B}_T(y,\delta,g))} \exp(|S_{n(v,T)}\varphi\circ\pi(v) - S_{n(y,T)}\varphi(y)| + S_{n(y,T)}\varphi(y)) \\
\le& \sum_{y\in F_T} \exp S_{n(y,T)}\varphi(y) \\
&\sum_{v\in V(y;z_0,z_1,\ldots,z_q)\cap\pi^{-1}(\overline{B}_T(y,\delta,g))} \exp[n(y,T)var(\varphi,\delta) + |n(v,T)-n(y,T)|\|\varphi\|] \\
\le& \exp\{(a+\tau)[(\frac{T}{m}+1+N)Var(\psi\circ\pi, 2\delta_1) + T + \|\psi\| + (2+[\frac{\|\psi\|}{m}]+N)\|\psi\|]\} \\
&\exp[(\frac{T}{m}+1)var(\varphi,\delta)]\exp[(2+[\frac{\|\psi\|}{m}])]\|\varphi\|]\sum_{y\in F_T}\exp S_{n(y,T)}\varphi(y) \qquad (14)
\end{aligned}$$

Now that $\delta \to 0$ as $T \to \infty$, it is easy to see

$$\limsup_{T\to\infty}\frac{1}{T}\log Q_{\psi\circ\pi,T}(f,\varphi\circ\pi,\epsilon) \le (a+\tau)(\frac{1}{m}Var(\psi\circ\pi,2\delta_1)+1) + P_\psi(\varphi),$$

where

$$\begin{aligned}
& Q_{\psi\circ\pi,T}(f,\varphi\circ\pi,\epsilon) \\
=\ & \inf\{\sum_{v\in E_T}\exp(S_{n(v,T)}\varphi)(v) : E_T \ is\ a\ (\psi\circ\pi,T,\epsilon)\text{-spanning set of } X\}.
\end{aligned}$$

Notice $Var(\psi\circ\pi, 2\delta_1) \to 0$ as $\delta_1 \to 0$. Since $\delta_1 \to 0$ as $\epsilon \to 0$, we have

$$P_{\psi\circ\pi}(\varphi\circ\pi) \le P_\psi(\varphi) + a + \tau.$$

As $\tau \to 0$, we obtain

$$P_{\psi\circ\pi}(\varphi\circ\pi) \le P_\psi(\varphi) + \sup_{y\in Y} h_{\psi\circ\pi}(f,\pi^{-1}(y)).$$

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