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Harmonic Meromorphic Functions Involving Generalized Incomplete Beta Functions

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Abstract

In this Article, a class $M_H([\alpha_1])$ of complex valued harmonic meromorphic functions of the form $f = h + \overline{g} \in M_H$ is introduced with the use of inverse function involving generalized incomplete beta function. A subclass $M_{\overline{H}}([\alpha_1])$ of $M_H([\alpha_1])$ is considered for various properties. Using coefficient condition for functions belonging to $M_{\overline{H}}([\alpha_1])$ class, bounds, extreme points, closure theorems and integral operator for those functions are also obtained.

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1 Introduction

Hengartner and Schober [4] studied and gave the concept of the special classes of harmonic functions, which are defined on the exterior of the unit

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disk $\widetilde{U} = \{z : |z| > 1\}$. They showed that these functions are complex valued, harmonic, sense preserving, univalent mappings f, admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|,$$

where $h(z)$ and $g(z)$ are analytic in $\widetilde{U} = \{z : |z| > 1\}$.

Let M_H denote a class of functions which are harmonic meromorphic in the unit disk $U = \{z : |z| < 1\}$ and are of the form:

$$f(z) = h(z) + \overline{g(z)}, \tag{1}$$

where

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are meromorphic in U and $h(z)$ has a simple pole at the origin with residue 1 there. The class M_H is studied in [2], [5], [6] and [8]. Whereas $M_{\overline{H}}$ denotes a subclass of M_H consisting of functions $f = h + \overline{g}$, with h and g are of the form

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n| \, z^n, \; z \in U \setminus \{0\} \, \text{and} \, g(z) = -\sum_{n=1}^{\infty} |b_n| \, z^n, z \in U \tag{2}$$

and are called respectively meromorphic part and co-meromorphic part. A function $f = h + \overline{g} \in M_H$ is said to be in the class $MS^{\star}_H$ of meromorphically harmonic starlike functions in $U \setminus \{0\}$ if it satisfies the condition

$$Re \left\{ -\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} > 0 \quad (z \in U).$$

Jahangiri and Silverman studied the class $MS^{\star}_H$ in [5]. For positive real numbers α_i and β_i, $i = 1, 2, 3, ...s$, a generalized incomplete beta function, $\phi\left((\alpha_i)_{1,s}, (\beta_i)_{1,s}, z\right) \equiv \phi\left([\alpha_1], z\right)$ is defined as

$$\phi\left([\alpha_1], z\right) \; = \; z \, {}_{s+1}F_s\left((\alpha_i)_{1,s}, 1; (\beta_i)_{1,s}; z\right)$$

and its series representation is given as:

$$\begin{aligned} \phi\left([\alpha_1], z\right) &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots\ldots (\alpha_s)_n}{(\beta_1)_n \ldots\ldots (\beta_s)_n} z^{n+1} \qquad (3) \\ &= \sum_{n=o}^{\infty} \nabla^s_n\left([\alpha_1]\right) z^{n+1}, \end{aligned}$$

where

$$\nabla_n^s([\alpha_1]) := \frac{(\alpha_1)_n......(\alpha_s)_n}{(\beta_1)_n......(\beta_s)_n},\ s \in N = \{1, 2, ...\} \tag{4}$$

and $(a)_n$ is the Pochhammer symbol defined as:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a\,(a+1) \cdot \cdots \cdot (a+n-1)$$

for $n \in \mathbb{N} = \{1, 2, ...\}$.

A differential operator θ [8] on the function $\phi\left([\alpha_1], z\right)$ given in (3) is defined as

$$\theta\left(\phi\left((\alpha_i)_{1,s}, (\beta_i)_{1,s}, z\right)\right) = z\frac{d}{dz}\phi\left((\alpha_i)_{1,s}, (\beta_i)_{1,s}, z\right).$$

The series expansion of $\theta\left(\phi\left((\alpha_i)_{1,s}, (\beta_i)_{1,s}, z\right)\right)$ is given as

$$\begin{aligned} \theta\left(\phi\left([\alpha_1], z\right)\right) &= \sum_{n=o}^{\infty}(n+1)\frac{(\alpha_1)_n......(\alpha_s)_n}{(\beta_1)_n......(\beta_s)_n}z^{n+1} \\ &= \sum_{n=o}^{\infty}(n+1)\nabla_n^s([\alpha_1])\, z^{n+1}. \end{aligned} \tag{5}$$

Throughout this paper, the following notations are being used:

$$\theta\left(\phi\left([\alpha_1], 1\right)\right) = \sum_{n=o}^{\infty}(n+1)\nabla_n^s([\alpha_1]) =: \theta\left(\phi\left([\alpha_1]\right)\right) \tag{6}$$

and

$$\phi\left([\alpha_1], 1\right) = \sum_{n=o}^{\infty}\nabla_n^s([\alpha_1]) =: \phi\left([\alpha_1]\right), \tag{7}$$

provided the corresponding series are absolutely convergent, i.e if $\sum\limits_{i=1}^{s}(\beta_i - \alpha_i) > 1$ and $\sum\limits_{i=1}^{s}(\beta_i - \alpha_i) > 2$ respectively.

Generalized hypergeometric functions are used to define harmonic functions in various research papers such as [1] and [3]. For harmonic functions

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty}A_n^j z^n + \sum_{n=1}^{\infty}B_n^j\overline{z^n},\ j = 1, 2$$

the convolution $f_1\tilde{\star}f_2$ is defined by

$$\left(f_1\tilde{\star}f_2\right)(z) := \frac{1}{z} + \sum_{n=1}^{\infty}A_n^1A_n^2 z^n + \sum_{n=1}^{\infty}B_n^1B_n^2\overline{z^n}.$$

Corresponding to the function $\phi([\alpha_1], z)$, defined in (3), consider

$$\widetilde{\phi}_1([\alpha_1], z) = \frac{1}{z^2}\phi([\alpha_1], z), \ z \in U \setminus \{0\}$$

and its inverse function $\left(\widetilde{\phi}_1([\alpha_1], z)\right)^{-1}$ defined by

$$\widetilde{\phi}_1([\alpha_1], z) \star \left(\widetilde{\phi}_1([\alpha_1], z)\right)^{-1} = \frac{1}{z(1-z)}, \ \ z \in U \setminus \{0\}.$$

The series expansion of this inverse function is given as:

$$\begin{aligned}\left(\widetilde{\phi}_1([\alpha_1], z)\right)^{-1} &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(\beta_1)_{n+1}.......(\beta_s)_{n+1}}{(\alpha_1)_{n+1}........(\alpha_s)_{n+1}} z^n \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{1}{\nabla_{n+1}^s([\alpha_1])} z^n\end{aligned}$$

where

$$\nabla_{n+1}^s([\alpha_1]) = \frac{(\alpha_1)_{n+1}........(\alpha_s)_{n+1}}{(\beta_1)_{n+1}.......(\beta_s)_{n+1}}. \tag{8}$$

From the contiguous relation of Pochammer symbol $(a)_{n+1} = a(a+1)_{n+1}$, it is noted that

$$\nabla_{n+1}^s([\alpha_1]) = \left(\prod_{i=1}^{s} \frac{\alpha_i}{\beta_i}\right) \nabla_n^s([\alpha_1 + 1]). \tag{9}$$

Let

$$H(z) := \left\{\left(\widetilde{\phi}_1([\alpha_1], z)\right)^{-1} - \frac{1}{\nabla_1^s([\alpha_1])}\right\} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{\nabla_{n+1}^s([\alpha_1])} z^n \tag{10}$$

which is meromorphic function in U. Again, corresponding to the function $\phi([\alpha_1], z)$, defined in (3), consider

$$\widetilde{\phi}_2([\alpha_1], z) = \frac{1}{z}\phi([\alpha_1], z), \quad z \in U \setminus \{0\}$$

and its inverse function $\left(\widetilde{\phi}_2([\alpha_1], z)\right)^{-1}$ defined by

$$\widetilde{\phi}_2([\alpha_1], z) \star \left(\widetilde{\phi}_2([\alpha_1], z)\right)^{-1} = \frac{1}{z(1-z)}, \ \ z \in U \setminus \{0\}.$$

The series expansion of this inverse function is given as:

$$\begin{aligned}\left(\widetilde{\phi}_2([\alpha_1], z)\right)^{-1} &= \sum_{n=0}^{\infty} \frac{(\beta_1)_n, ..., (\beta_s)_n}{(\alpha_1)_n, ..., (\alpha_s)_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{1}{\nabla_n^s([\alpha_1])} z^n.\end{aligned}$$

Let

$$G(z) := \left\{ \left(\widetilde{\phi_2}\left([\alpha_1], z\right) \right)^{-1} - \frac{1}{\nabla_0^s\left([\alpha_1]\right)} \right\} = \sum_{n=1}^{\infty} \frac{1}{\nabla_n^s\left([\alpha_1]\right)} z^n \tag{11}$$

which is an analytic function, hence meromorphic in U. Now, harmonic meromorphic function defined as

$$F(z) = H(z) + \overline{G(z)} \in M_H, \tag{12}$$

where H and G and are of the form (10) and (11) are called respectively meromorphic part and co-meromorphic part of $F(z)$. Using convolution "$\widetilde{\star}$" of harmonic meromorphic functions $F(z) = H(z) + \overline{G(z)}$ given by (12) and $f(z) = h(z) + \overline{g(z)}$ given by (1), a linear operator $\mathcal{F}_s\left([\alpha_1]\right) f(z) : M_H \to M_H$ is defined as:

$$\begin{aligned} \mathcal{F}_s\left([\alpha_1]\right) f(z) &= F(z)\widetilde{\star} f(z) = H(z) \star h(z) + \overline{G(z) \star g(z)} \qquad (13) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{\nabla_{n+1}^s\left([\alpha_1]\right)} a_n z^n + \sum_{n=1}^{\infty} \frac{1}{\nabla_n^s\left([\alpha_1]\right)} b_n \overline{z}^n. \end{aligned}$$

Involving operator $\mathcal{F}_s\left([\alpha_1]\right)$, a class $M_H\left([\alpha_1]\right)$ is defined as follows:

Definition 1.1. *Let $M_H\left([\alpha_1]\right)$ denote the family of harmonic meromorphic functions $f(z) = h(z) + \overline{g(z)} \in M_H$ satisfying*

$$Re\left\{ -\frac{z\left(\mathcal{F}_s\left([\alpha_1]\right) f(z)\right)'}{\mathcal{F}_s\left([\alpha_1]\right) f(z)} \right\} > 0 \qquad (z \in U), \tag{14}$$

or

$$Re\left\{ -\frac{z\left(H(z) \star h(z)\right)' - z\left(\overline{G(z) \star g(z)}\right)'}{H(z) \star h(z) + \overline{G(z) \star g(z)}} \right\} > 0 \qquad (z \in U).$$

Denote $M_{\overline{H}}\left([\alpha_1]\right) = M_H\left([\alpha_1]\right) \cap M_{\overline{H}}$.

2 Coefficient Conditions

In this section, sufficient coefficient condition for the class $M_H\left([\alpha_1]\right)$ is established and then it is proved that this coefficient condition is necessary for its subclass $M_{\overline{H}}\left([\alpha_1]\right)$.

Theorem 2.1. *Let* $f(z) = h(z) + \overline{g(z)}$ *be of the form* (1) *and if*

$$\sum_{n=1}^{\infty} n \left\{ \frac{1}{\nabla_{n+1}^{s}([\alpha_1])} |a_n| + \frac{1}{\nabla_{n}^{s}([\alpha_1])} |b_n| \right\} \le 1 \tag{15}$$

where $\nabla_n^s([\alpha_1])$ *and* $\nabla_{n+1}^s([\alpha_1])$ *are given in* (4) *and* (8) *respectively, for positive real numbers* α_i *and* β_i *with* $\sum_{i=1}^{s}(\beta_i - \alpha_i) > 1$, *then* $f(z)$*is harmonic, orientation preserving and univalent in* $U \setminus \{0\}$ *and* $f \in M_H([\alpha_1])$.

Proof. Let the function $f(z) = h(z) + \overline{g(z)}$ given by (1) , satisfying (15). Under the condition $\sum_{i=1}^{s}(\beta_i - \alpha_i) > 1$, it follows that $0 < \nabla_{n+1}^s([\alpha_1]) < \nabla_n^s([\alpha_1]) < 1$ for all $n \ge 1$. Then for $0 < |z_1| \le |z_2| < 1$,

$$\begin{aligned}
|f(z_1) - f(z_2)| &\ge |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
&\ge \frac{|z_1 - z_2|}{|z_1||z_2|} - |z_1 - z_2| \sum_{n=1}^{\infty} (|a_n| + |b_n|) \left| z_1^{n-1} + \cdots + z_2^{n-1} \right| \\
&> \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - |z_2|^2 \sum_{n=1}^{\infty} n (|a_n| + |b_n|) \right] \\
&> \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - \sum_{n=1}^{\infty} n \left\{ \frac{1}{\nabla_{n+1}^{s}([\alpha_1])} |a_n| + \frac{1}{\nabla_{n}^{s}([\alpha_1])} |b_n| \right\} \right] \\
&\ge 0,
\end{aligned}$$

if (15) holds, then f is univalent in $U \setminus \{0\}$.

In order to show that f is sense preserving in $U \setminus \{0\}$, it only needs to show that $|h'(z)| > |g'(z)|$. For $0 < |z| = r < 1$, on using (15) , it follows that

$$\begin{aligned}
|h'(z)| &\ge \frac{1}{|z|^2} - \sum_{n=1}^{\infty} n |a_n| |z|^{n-1} \\
&> \frac{1}{r^2} - \sum_{n=1}^{\infty} n |a_n| r^{n-1} \\
&> 1 - \sum_{n=1}^{\infty} n |a_n|
\end{aligned}$$

$$\begin{aligned}
&\geq 1-\sum_{n=1}^{\infty} n\left\{\frac{1}{\nabla_{n+1}^{s}\left(\left[\alpha_{1}\right]\right)}\left|a_{n}\right|\right\} \\
&\geq \sum_{n=1}^{\infty} n\left\{\frac{1}{\nabla_{n}^{s}\left(\left[\alpha_{1}\right]\right)}\left|b_{n}\right|\right\} \\
&> \sum_{n=1}^{\infty} n\left|b_{n}\right|>\sum_{n=1}^{\infty} n\left|b_{n}\right| r^{n-1} \\
&= \sum_{n=1}^{\infty} n\left|b_{n}\right||z|^{n-1}>\left|g^{\prime}(z)\right|,
\end{aligned}$$

which proves that the map f is sense preserving in $U \setminus \{0\}$.

Now, in order to show that $f \in M_H([\alpha_1])$, it suffices to show that

$$Re\left\{-\frac{z\,(H(z)\star h(z))^{\prime}-z\left(\overline{G(z)\star g(z)}\right)^{\prime}}{H(z)\star h(z)+\overline{G(z)\star g(z)}}\right\}>0. \tag{16}$$

It is known that $Re\,(p(z)) > 0$, if and only if $\left|\frac{p(z)-1}{p(z)+1}\right| < 1$ for an analytic function $p(z) = 1 + p_1 z + p_2 z + ...$

Let

$$A(z):=-z\,(H(z)\star h(z))^{\prime}+z\left(\overline{G(z)\star g(z)}\right)^{\prime} \tag{17}$$

and

$$B(z):=H(z)\star h(z)+\overline{G(z)\star g(z)}. \tag{18}$$

It is observe that (16) holds if

$$|A(z)+B(z)|-|A(z)-B(z)|>0. \tag{19}$$

Now from (17) and (18), it follows that

$$\begin{aligned}
&|A(z)+B(z)| \\
&= \left|-z\,(H(z)\star h(z))^{\prime}+z\left(\overline{G(z)\star g(z)}\right)^{\prime}+H(z)\star h(z)+\overline{G(z)\star g(z)}\right| \\
&= \left|\frac{2}{z}-\sum_{n=1}^{\infty}(n-1)\frac{1}{\nabla_{n+1}^{s}\left(\left[\alpha_{1}\right]\right)}a_{n}z^{n}+\sum_{n=1}^{\infty}(n+1)\frac{1}{\nabla_{n}^{s}\left(\left[\alpha_{1}\right]\right)}b_{n}\overline{z}^{n}\right| \\
&\geq \frac{2}{|z|}-\sum_{n=1}^{\infty}(n-1)\frac{1}{\nabla_{n+1}^{s}\left(\left[\alpha_{1}\right]\right)}\left|a_{n}\right||z|^{n}-\sum_{n=1}^{\infty}(n+1)\frac{1}{\nabla_{n}^{s}\left(\left[\alpha_{1}\right]\right)}\left|b_{n}\right||z|^{n}
\end{aligned}$$

and

$$
\begin{aligned}
&|A(z)-B(z)| \\
&= \left|-z\left(H(z)\star h(z)\right)'+z\left(\overline{G(z)\star g(z)}\right)'-H(z)\star h(z)-\overline{G(z)\star g(z)}\right| \\
&= \left|\sum_{n=1}^{\infty}(n+1)\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}a_n z^n-\sum_{n=1}^{\infty}(n-1)\frac{1}{\nabla_{n}^{s}([\alpha_1])}b_n\overline{z}^n\right| \\
&\leq \sum_{n=1}^{\infty}(n+1)\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n||z|^n+\sum_{n=1}^{\infty}(n-1)\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|b_n||z|^n .
\end{aligned}
$$

Thus, from (15)

$$
\begin{aligned}
&|A(z)+B(z)|-|A(z)-B(z)| \\
&\geq \frac{2}{|z|}-2\sum_{n=1}^{\infty}n\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n||z|^n-2\sum_{n=1}^{\infty}n\frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n||z|^n \\
&\geq \frac{2}{|z|}\left\{1-\sum_{n=1}^{\infty}n\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n||z|^{n+1}-\sum_{n=1}^{\infty}n\frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n||z|^{n+1}\right\} \\
&\geq 2\left\{1-\sum_{n=1}^{\infty}\left[n\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n|+n\frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n|\right]\right\} \\
&\geq 2\left\{1-\sum_{n=1}^{\infty}n\left\{\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n|+\frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n|\right\}\right\} \\
&\geq 0.
\end{aligned}
$$

This proves the result. □

Now, we prove that the condition (15) is necessary for functions $f\in M_{\overline{H}}([\alpha_1])$.

Theorem 2.2. *Let $f(z)=h(z)+\overline{g(z)}\in M_{\overline{H}}$ with h and g are of the form (2) , then $f\in M_{\overline{H}}([\alpha_1])$ if and only if*

$$\sum_{n=1}^{\infty}n\left\{\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n|+\frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n|\right\}\leq 1, \tag{20}$$

where $\nabla_{n}^{s}([\alpha_1])$ and $\nabla_{n+1}^{s}([\alpha_1])$ are given in (4) and (8) respectively.

Proof. In view of Theorem 2.1, it only needs to prove the "only if"' part of the Theorem. Since $M_{\overline{H}}([\alpha_1]) \subset M_H([\alpha_1])$, it suffices to show that $f \notin M_{\overline{H}}([\alpha_1])$ if the condition (20) does not hold. If $f \in M_{\overline{H}}([\alpha_1])$ then

$$Re\left\{-\frac{z\left(H(z)\star h(z)\right)' - z\left(\overline{G(z)\star g(z)}\right)'}{H(z)\star h(z)+\overline{G(z)\star g(z)}}\right\} > 0$$

is equivalent to

$$Re\left\{\frac{\frac{2}{z} - 2\sum_{n=1}^{\infty} n\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n| z^n - 2\sum_{n=1}^{\infty} n\frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n| \overline{z}^n}{\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n| z^n - \sum_{n=1}^{\infty} \frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n| \overline{z}^n}\right\} > 0. \qquad (21)$$

Since

$$\left|\frac{\xi(z)}{\eta(z)}\right| \geq Re\left\{\frac{\xi(z)}{\eta(z)}\right\} \geq 0,$$

hence the condition (21) holds if

$$\frac{1 - \sum_{n=1}^{\infty} n\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n| |z|^{n+1} - \sum_{n=1}^{\infty} n\frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n| |z|^{n+1}}{1 + \sum_{n=1}^{\infty} \frac{1}{\nabla_{n+1}^{s}([\alpha_1])}|a_n| |z|^{n+1} + \sum_{n=1}^{\infty} \frac{1}{\nabla_{n}^{s}([\alpha_1])}|b_n| |z|^{n+1}} \geq 0. \qquad (22)$$

Now, if the condition (20) does not holds, then the numerator of above equation is negative for z sufficiently close to 1. Which contradicts the required condition for $f \in M_{\overline{H}}([\alpha_1])$ and this proves the required result. □

Corollary 2.3. *If $f \in M_{\overline{H}}([\alpha_1])$ then*

$$\sum_{n=1}^{\infty} |a_n| \leq \left(\prod_{i=1}^{s} \frac{\alpha_i}{\beta_i}\right)(\zeta(2))\left[\{\theta(\phi([\alpha_1+1]))\} - \{\phi([\alpha_1+1])\}\right]$$

and

$$\sum_{n=1}^{\infty} |b_n| \leq (\zeta(2))\left[\{\theta(\phi([\alpha_1]))\} - \{\phi([\alpha_1])\}\right],$$

where $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is called a Zeta function and $\theta(\phi([\alpha_1]))$, $\phi([\alpha_1])$ are given in (6) *and* (7) *provided $\sum_{i=1}^{s} (\beta_i - \alpha_i) > 2$.*

Proof. On using relation given in (9), (6) and (7), from Theorem 2.2 it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty}|a_n| &\leq \sum_{n=1}^{\infty}\frac{\nabla_{n+1}^{s}\left([\alpha_1]\right)}{n}\\
&= \sum_{n=1}^{\infty}\left[\frac{1}{n^2}(n+1)-\frac{1}{n^2}\right]\left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)\nabla_{n}^{s}\left([\alpha_1+1]\right)\\
&= \left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)\left[\sum_{n=1}^{\infty}\frac{1}{n^2}(n+1)\nabla_{n}^{s}\left([\alpha_1+1]\right)-\sum_{n=1}^{\infty}\frac{1}{n^2}\nabla_{n+1}^{s}\left([\alpha_1+1]\right)\right]\\
&\leq \left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)(\zeta(2))\left[\left\{\theta\left(\phi\left([\alpha_1+1]\right)\right)\right\}-\left\{\phi\left([\alpha_1+1]\right)\right\}\right]
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty}|b_n| &\leq \sum_{n=1}^{\infty}\frac{\nabla_{n}^{s}\left([\alpha_1]\right)}{n}\\
&= \sum_{n=1}^{\infty}\left[\frac{1}{n^2}(n+1)-\frac{1}{n^2}\right]\nabla_{n}^{s}\left([\alpha_1]\right)\\
&\leq \sum_{n=1}^{\infty}\frac{1}{n^2}(n+1)\nabla_{n}^{s}\left([\alpha_1]\right)-\sum_{n=1}^{\infty}\frac{1}{n^2}\nabla_{n}^{s}\left([\alpha_1]\right)\\
&\leq (\zeta(2))\left[\left\{\theta\left(\phi\left([\alpha_1]\right)\right)\right\}-\left\{\phi\left([\alpha_1]\right)\right\}\right].
\end{aligned}$$

□

3 Bounds

In this section, bounds for functions belonging to the class $M_{\overline{H}}\left([\alpha_1]\right)$ are determined with the use of Theorem 2.2 and Corollary 2.3.

Theorem 3.1. *Let* $f \in M_{\overline{H}}\left([\alpha_1]\right)$, *then for* $\sum_{i=1}^{s}(\beta_i-\alpha_i)>1$ *and* $0<|z|=r<1$

$$\frac{1}{r}-r\left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)\leq|f(z)|\leq\frac{1}{r}+r\left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right).$$

Proof. Let $f \in M_{\overline{H}}([\alpha_1])$, taking the absolute value of f defined in (2) and using Theorem 2.2, it follows that

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z} + \sum_{n=1}^{\infty} |a_n| z^n - \overline{\sum_{n=1}^{\infty} |b_n| z^n}\right| \\
&\leq \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \\
&\leq \frac{1}{r} + r\sum_{n=1}^{\infty}(|a_n| + |b_n|) \\
&\leq \frac{1}{r} + r\nabla_1^s([\alpha_1]) \sum_{n=1}^{\infty} n \frac{1}{\nabla_n^s([\alpha_1])}(|a_n| + |b_n|) \\
&\leq \frac{1}{r} + r\nabla_1^s([\alpha_1]) \sum_{n=1}^{\infty} n \left[\frac{1}{\nabla_{n+1}^s([\alpha_1])} |a_n| + \frac{1}{\nabla_n^s([\alpha_1])} |b_n|\right] \\
&\leq \frac{1}{r} + r\left(\prod_{i=1}^{s} \frac{\alpha_i}{\beta_i}\right)
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z} + \sum_{n=1}^{\infty} |a_n| z^n - \overline{\sum_{n=1}^{\infty} |b_n| z^n}\right| \\
&\geq \frac{1}{r} - \sum_{n=1}^{\infty} |a_n| r^n - \sum_{n=1}^{\infty} |b_n| r^n \\
&\geq \frac{1}{r} - r\sum_{n=1}^{\infty}(|a_n| + |b_n|) \\
&\geq \frac{1}{r} - r\nabla_1^s([\alpha_1]) \sum_{n=1}^{\infty} n \frac{1}{\nabla_n^s([\alpha_1])}(|a_n| + |b_n|) \\
&\geq \frac{1}{r} - r\nabla_1^s([\alpha_1]) \sum_{n=1}^{\infty} n \left[\frac{1}{\nabla_{n+1}^s([\alpha_1])} |a_n| + \frac{1}{\nabla_n^s([\alpha_1])} |b_n|\right] \\
&\geq \frac{1}{r} - r\left(\prod_{i=1}^{s} \frac{\alpha_i}{\beta_i}\right).
\end{aligned}$$

This proves the required result. □

Using inequalities obtained in Corollary 2.3, following functional bounds are estimated.

Theorem 3.2. *If* $f \in M_{\overline{H}}([\alpha_1])$ *then for* $0 < |z| = r < 1$

$$|f(z)| \leq \frac{1}{r} + r\zeta(2)\left[\begin{array}{r}\left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)[\{\theta(\phi([\alpha_1+1]))\} - \{\phi([\alpha_1+1])\}] \\ +[\{\theta(\phi([\alpha_1]))\} - \{\phi([\alpha_1])\}]\end{array}\right]$$

and

$$|f(z)| \geq \frac{1}{r} - r\zeta(2)\left[\begin{array}{r}\left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)[\{\theta(\phi([\alpha_1+1]))\} - \{\phi([\alpha_1+1])\}] \\ +[\{\theta(\phi([\alpha_1]))\} - \{\phi([\alpha_1])\}]\end{array}\right]$$

where $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2}$ *is called a Zeta function and* $\theta(\phi([\alpha_1]))$, $\phi([\alpha_1])$ *are given in* (6) *and* (7) *provided* $\sum_{i=1}^{s}(\beta_i - \alpha_i) > 2$.

Proof. Let $f \in M_{\overline{H}}([\alpha_1])$, using Corollary 2.3 and taking the absolute value of f of the form (2), it follows that

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z} + \sum_{n=1}^{\infty}|a_n|z^n - \overline{\sum_{n=1}^{\infty}|b_n|z^n}\right| \\
&\leq \frac{1}{r} + \sum_{n=1}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n \\
&\leq \frac{1}{r} + r\sum_{n=1}^{\infty}(|a_n| + |b_n|) \\
&\leq \frac{1}{r} + r\zeta(2)\left[\begin{array}{r}\left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)[\{\theta(\phi([\alpha_1+1]))\} - \{\phi([\alpha_1+1])\}] \\ +[\{\theta(\phi([\alpha_1]))\} - \{\phi([\alpha_1])\}]\end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z} + \sum_{n=1}^{\infty}|a_n|z^n - \overline{\sum_{n=1}^{\infty}|b_n|z^n}\right| \\
&\geq \frac{1}{r} - \sum_{n=1}^{\infty}|a_n|r^n - \sum_{n=1}^{\infty}|b_n|r^n \\
&\geq \frac{1}{r} - r\sum_{n=1}^{\infty}(|a_n| + |b_n|) \\
&\geq \frac{1}{r} - r\zeta(2)\left[\begin{array}{r}\left(\prod_{i=1}^{s}\frac{\alpha_i}{\beta_i}\right)[\{\theta(\phi([\alpha_1+1]))\} - \{\phi([\alpha_1+1])\}] \\ +[\{\theta(\phi([\alpha_1]))\} - \{\phi([\alpha_1])\}]\end{array}\right]
\end{aligned}$$

This proves the required result. □

Remark: The functional bounds obtained in Theorem 3.2 are best possible for $f \in M_{\overline{H}}([\alpha_1])$.

In view of above remark, the following result is obtained which is sharp:

Corollary 3.3. *Let $f \in M_{\overline{H}}([\alpha_1])$ then*

$$\left\{ w : |w| < 1 - \zeta(2) \left[\begin{array}{r} \left(\prod_{i=1}^{s} \frac{\alpha_i}{\beta_i} \right) [\{\theta(\phi([\alpha_1+1]))\} - \{\phi([\alpha_1+1])\}] \\ + [\{\theta(\phi([\alpha_1]))\} - \{\phi([\alpha_1])\}] \end{array} \right] \right\} \subseteq C \setminus f(U \setminus \{0\}).$$

4 Extreme Points

In this section, extreme points for the class $M_{\overline{H}}([\alpha_1])$ are provided.

Theorem 4.1. *Let $f = h + \overline{g}$, where h and g are of the form (2) then $f \in M_{\overline{H}}([\alpha_1])$, if and only if f can be expressed as*

$$f(z) = \sum_{n=0}^{\infty} (x_n h_n(z) + y_n g_n(z)), \tag{23}$$

where $z \in U \setminus \{0\}$ and

$$h_0(z) = \frac{1}{z}, \quad h_n(z) = \frac{1}{z} + \frac{\nabla_{n+1}^{s}([\alpha_1])}{n} z^n, n = 1, 2... \tag{24}$$

for

$$g_0(z) = \frac{1}{z}, \quad g_n(z) = \frac{1}{z} - \frac{\nabla_{n}^{s}([\alpha_1])}{n} \overline{z}^n, n = 1, 2... \tag{25}$$

and

$$\sum_{n=0}^{\infty} (x_n + y_n) = 1, \qquad x_n \geq 0 \quad and \quad y_n \geq 0. \tag{26}$$

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (x_n h_n(z) + y_n g_n(z)) \\ &= x_0 h_0 + y_0 g_0 + \sum_{n=1}^{\infty} x_n \left(\frac{1}{z} + \frac{\nabla_{n+1}^{s}([\alpha_1])}{n} z^n \right) + y_n \left(\frac{1}{z} - \frac{\nabla_{n}^{s}([\alpha_1])}{n} \overline{z}^n \right) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \left(\frac{\nabla_{n+1}^{s}([\alpha_1])}{n} x_n \right) - \left(\frac{\nabla_{n}^{s}([\alpha_1])}{n} y_n \right) \right\} z^n. \end{aligned}$$

Thus by Theorem 2.2, it follows that $f \in M_{\overline{H}}([\alpha_1])$, since

$$\sum_{n=1}^{\infty}\left\{n\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}\left(\frac{\nabla_{n+1}^{s}([\alpha_1])}{n}x_n\right)-n\frac{1}{\nabla_{n}^{s}([\alpha_1])}\left(\frac{\nabla_{n}^{s}([\alpha_1])}{n}y_n\right)\right\}$$

$$=\sum_{n=1}^{\infty}(x_n+y_n)=(1-x_0-y_0)\leq 1.$$

Conversely, suppose that $f \in M_{\overline{H}}([\alpha_1])$. Set

$$\begin{aligned} x_n &= n\frac{1}{\nabla_{n+1}^{s}([\alpha_1])}\left|a_n\right|, \\ y_n &= n\frac{1}{\nabla_{n}^{s}([\alpha_1])}\left|b_n\right| \end{aligned}$$

which satisfy (26), thus

$$\begin{aligned} f(z) &= \frac{1}{z}+\sum_{n=1}^{\infty}\left|a_n\right|z^n-\sum_{n=1}^{\infty}\left|b_n\right|\overline{z}^n \\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{\nabla_{n+1}^{s}([\alpha_1])}{n}x_n z^n-\sum_{n=1}^{\infty}\frac{\nabla_{n}^{s}([\alpha_1])}{n}y_n\overline{z}^n \\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\left[h_n-\frac{1}{z}\right]x_n+\sum_{n=1}^{\infty}\left[g_n-\frac{1}{z}\right]y_n \\ &= \frac{1}{z}\left[1-\sum_{n=1}^{\infty}x_n-\sum_{n=1}^{\infty}y_n\right]+\sum_{n=1}^{\infty}h_n x_n+\sum_{n=1}^{\infty}g_n y_n \\ &= x_0 h_0+y_0 g_0+\sum_{n=1}^{\infty}h_n x_n+\sum_{n=1}^{\infty}g_n y_n \\ &= \sum_{n=0}^{\infty}(x_n h_n+y_n g_n). \end{aligned}$$

This proves the Theorem. □

Remark: The extreme points for the class $M_{\overline{H}}([\alpha_1])$ are given by (24) and (25).

5 Closure Theorems

In this section, convolution of the class $M_{\overline{H}}([\alpha_1])$ and convex linear combination of its members are defined and studied.

Theorem 5.1. *Let $f \in M_{\overline{H}}([\alpha_1])$ and $F \in M_{\overline{H}}([\alpha_1])$, then the convolution function*

$$f \tilde{\star} F = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n A_n| z^n - \sum_{n=1}^{\infty} |b_n B_n| \overline{z}^n \in M_{\overline{H}}([\alpha_1]).$$

Proof. Since $F \in M_{\overline{H}}([\alpha_1])$, then by Theorem 2.2, $|A_n| \leq 1$ and $|B_n| \leq 1$, hence

$$\sum_{n=1}^{\infty} n \left\{ \frac{1}{\nabla_{n+1}^s([\alpha_1])} |a_n A_n| + \frac{1}{\nabla_n^s([\alpha_1])} |b_n B_n| \right\}$$
$$\leq \sum_{n=1}^{\infty} n \left\{ \frac{1}{\nabla_{n+1}^s([\alpha_1])} |a_n| + \frac{1}{\nabla_n^s([\alpha_1])} |b_n| \right\} \leq 1$$

by Theorem 2.2 as $f \in M_{\overline{H}}([\alpha_1])$. Thus, again by Theorem 2.2, $f \tilde{\star} F \in M_{\overline{H}}([\alpha_1])$. □

Theorem 5.2. *Let the functions $f_j(z)$ defined as*

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_{n,j}| z^n - \sum_{n=1}^{\infty} |b_{n,j}| \overline{z}^n \tag{27}$$

be in the class $M_{\overline{H}}([\alpha_1])$ for every $j = 1, 2, 3.......$, then the function

$$\psi(z) = \sum_{n=1}^{\infty} c_j f_j(z)$$

is also in the class $M_{\overline{H}}([\alpha_1])$, where $\sum_{n=1}^{\infty} c_j = 1$,$c_j \geq 0$ $(j = 1, 2, 3.......)$.

Proof. It is noted that

$$\psi(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} c_j |a_{n,j}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} c_j |b_{n,j}| \right) \overline{z}^n.$$

Since $f_j(z) \in M_{\overline{H}}([\alpha_1])$ for every $j = 1, 2, 3.......$, then by Theorem 2.2, it follows that

$$\sum_{n=1}^{\infty} \left[n \frac{1}{\nabla_{n+1}^s([\alpha_1])} \left(\sum_{j=1}^{\infty} c_j |a_{n,j}| \right) + n \frac{1}{\nabla_n^s([\alpha_1])} \left(\sum_{j=1}^{\infty} c_j |b_{n,j}| \right) \right]$$
$$= \sum_{j=1}^{\infty} c_j \left(\sum_{n=1}^{\infty} \left\{ n \frac{1}{\nabla_{n+1}^s([\alpha_1])} \right\} |a_{n,j}| + \left\{ n \frac{1}{\nabla_n^s([\alpha_1])} \right\} |b_{n,j}| \right)$$
$$\leq \sum_{j=1}^{\infty} c_j \leq 1$$

hence, $\psi(z) \in M_{\overline{H}}([\alpha_1])$, which is the desired result. $\square$

6 Integral Operator

In this section, it is shown that the class $M_{\overline{H}}([\alpha_1])$ is closed under Integral operator.

Definition 6.1. *An integral operator $I : M_{\overline{H}} \to M_{\overline{H}}$ is defined as:*

$$If(z) = \frac{c}{z^{c+1}} \int_0^z t^c h(t)dt + \overline{\frac{c}{z^{c+1}} \int_0^z t^c g(t)dt}, \ for \ \ c > 0, \ z \in U \setminus \{0\}. \quad (28)$$

Theorem 6.2. *Let $f \in M_{\overline{H}}([\alpha_1])$ and $If(z)$ be defined in (28), then $If(z) \in M_{\overline{H}}([\alpha_1])$.*

Proof. From the series representation of I, it follows that

$$If(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{n+c+1} |a_n| z^n - \sum_{n=1}^{\infty} \frac{c}{n+c+1} |b_n| \overline{z}^n.$$

Since, $f \in M_{\overline{H}}([\alpha_1])$, by Theorem 2.2, $If(z) \in M_{\overline{H}}([\alpha_1])$, since

$$\sum_{n=1}^{\infty} \frac{c}{n+c+1} n \left\{ \frac{1}{\nabla_{n+1}^s([\alpha_1])} |a_n| + \frac{1}{\nabla_n^s([\alpha_1])} |b_n| \right\}$$

$$\leq \sum_{n=1}^{\infty} n \left\{ \frac{1}{\nabla_{n+1}^s([\alpha_1])} |a_n| + \frac{1}{\nabla_n^s([\alpha_1])} |b_n| \right\} \leq 1.$$

$\square$

References

[1] H.A. Al-Kharsani and R.A. Al-Khal, Univalent Harmonic Functions, *Journal of Inequa. in pu. and App. Math.*, **8**, (Art59), (2007), 8 pp.

[2] H. Bostanci and M. Ozturk, New classes of salagean type meromorphic harmonic functions, *Inter. jour. of math. scie.*, **2**, (2008), 52-57.

[3] J. Dziok and H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric functions, *Appl. Math. Comp.*, **103**, (1999), 1-13.

[4] W. Hengartner and G. Schober, Univalent Harmonic Functions, *Trans. Amer. Math. Soc.*, **299**, (1987), 1-31.

[5] J.M. Jahangiri and H. Silverman, Harmonic Meromorphic univalent Functions with Negative Coefficients, *Bull. Korean Math. Soc.*, **36**, (1999), 291-301.

[6] J.M. Jahangiri, Harmonic Meromorphic Starlike Functions, *Bull. Korean Math. Soc.*, **37**, (2001), 763-770.

[7] E.D. Rainvillie, *Special Functions*, Chelsea Publishing Company, New York, 1960.

[8] K.Al. Shaqsi and M. Darus, On meromorphic harmonic functions with respact to k - symmetric points, *Jour. of Inequ. and Apps.*, **Art. 259205** (2008).