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A Weighted-fractional model to European option pricing

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Abstract

This paper deals the option pricing problem in the weighted fractional Brownian motion model. Both the long-range dependence of the weighted fractional Brownian motion and the European option pricing formula are obtained. Figures are given to illustrate the effectiveness of the result and show that the weighted-fractional model to option pricing is a reasonable one.

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1 Introduction

Since it appeared in the 1970s, the Black-Scholes model [2] has become the

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most popular method for option pricing and its generalized version has provided mathematically beautiful and powerful results on option pricing. However, they are still theoretical adoptions and not necessarily consistent with empirical features of financial return series, such as nonindependence, nonlinearity, self-similarity etc, which contradict to the traditional Black-Scholes assumption. For example, Hsieth[8], Mariani et al.[11], Ramirez et al.[15] and Willinger et al.[18] showed that returns are of long-range (or short-range) dependence, which suggests strong time-correlations between different events at different time scales (e.g., see Mandelbrot [10], and Cajueiro and Tabak [7, 17]). In the search for better models for describing long-range dependence in financial return series, a fractional Brownian model (fBm) has been proposed as an improvement of the classical Black-Scholes model, see Peters [14], Hu and Øksendal [9], Ciprian Necula [13] and references therein.

As an extension of the Brownian motion, Bojdecki et al. [5] introduced and studied a rather special class of self-similar Gaussian processes which preserve many properties of the fractional Brownian motion. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called weighted fractional Brownian motion (weighted-fBm). More works for weighted-fBm can be found in Bojdecki et al. [6], Yan-An [19] and references therein. It is well known that the fractional Brownian motion is the only continuous Gaussian process which is self-similar and has stationary increments. However, contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reasons for this are the complexity of dependence structures and the non-availability of convenient stochastic integral representations for self-similar Gaussian processes without stationary increments. On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models, and such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. There it seems interesting to study the weighted-fractional Black-Scholes model.

We mainly use a probabilistic and actuarial approach for pricing option developed by M. Blat et al [4]. This approach is valid even when an equilibrium price measure does not exist (arbitrage, non-equilibrium) or is not unique (incompleteness). By selecting different asset as numeraire and the corresponding measure transformations, we generalize the classic measure transform methods to weighted fractional Brownian motion market which not only enriches the option pricing method of quasi-martingale, but also gives a new look to the derivation of weighted fractional option pricing formula.

The remainder of this paper is organized as follows. Section 2 presents the weighted-fBm version of the Black-Scholes model and investigates the longe-range dependence of weighted-fBm. In Section 3, the weighted fractional Black-Scholes formula is obtained, and figures are given to illustrate the effectiveness of the result and show that the weighted fractional model to option pricing is a reasonable one.

2 Merton weighted-fractional model

Since a financial system is a complex system with great flexibility, investors do not make their decisions immediately after receiving the financial information, but rather wait until information reaches to its threshold limit value. This behavior can lead to the features of "asymmetric leptokurtic" and "long/short memory". The weighted fractional Brownian motion may be a useful tool for capturing this phenomenon.

Whereas the original model assumes a Geometric Brownian motion for the firm value, in this paper we consider the following dynamics for V:

$$dV_t = \mu V_t dt + \sigma V_t dB_t^{a,b}, \tag{1}$$

where $B_t^{a,b}$ denotes a weighted fractional Brownian motion and the stochastic integration is divergence-type. Let $\Omega = C_0(0,T;\mathbb{R})$ be the Banach space of a real-valued continuous function on $[0,T]$ with the initial value zero and the super norm. There is a probability measure P on $(\Omega,\mathscr{F})$, where $\mathscr{F}$ is the Borel σ-algebra on Ω such that on the probability space $(\Omega,\mathscr{F},P)$, the process $B_t^{a,b}$ defined as

$$B_t^{a,b} = \omega(t), \quad \forall \omega \in \Omega,$$

is a (one dimensional) Gaussian process with mean

$$\mathbb{E}B_t^{a,b} = \mathbb{E}B_0^{a,b} = 0, \quad \forall t \in [0,T],$$

and covariance

$$\mathbb{E}\left[B_t^{a,b}B_s^{a,b}\right] = \int_0^{s\wedge t} u^a[(t-u)^b + (s-u)^b]du, \quad \forall t,s \in [0,T],$$

particularly,

$$\mathbb{E}\left[(B_t^{a,b})^2\right] = 2\int_0^t u^a(t-u)^b du, \quad \forall t \in [0,T].$$

The canonical process $\{B_t^{a,b}, t \in [0,T]\}$ is called a standard weighted-fBm if a and b satisfy the conditions

$$a > -1,\ |b| < 1,\ |b| < a+1. \tag{2}$$

For $a = 0$, the weighted-fBm reduces to the usual fractional Brownian motion with Hurst parameter $\frac{1}{2}(b+1)$, and the Brownian motion for $a = b = 0$ (up to a multiplicative constant). The weighted fractional Brownian motion has properties analogous to those of the fractional Brownian motion (self-similarity, path continuity and it is neither a Markov process nor a semimartingale).

For simplicity throughout this paper we use the notation $x \vee y := \max\{x, y\}$ and $F \asymp G$ with the meaning that there are positive constants c_1 and c_2 so that

$$c_1 G(x) \leq F(x) \leq c_2 G(x)$$

in the common domain of F and G.

Theorem 2.1. *Under the condition* (2) *we have*

$$E\left[\left(B_t^{a,b} - B_s^{a,b}\right)^2\right] \asymp (t \vee s)^a |t-s|^{b+1} \tag{3}$$

for $s, t \geq 0$. *In particular, we have*

$$E\left[\left(B_t^{a,b} - B_s^{a,b}\right)^2\right] \leq C_{a,b}|t-s|^{a+b+1} \tag{4}$$

for $a \leq 0$.

Proof. For all $t > s > 0$ we have

$$\begin{aligned} Q(t,s) :&= \left[\left(B_t^{a,b} - B_s^{a,b}\right)^2\right] = 2\int_s^t u^a(t-u)^b du \\ &= 2t^{a+b+1}\int_{\frac{s}{t}}^1 r^a(1-r)^b dr. \end{aligned}$$

Consider the function

$$x \mapsto f(x) = \int_x^1 r^a(1-r)^b dr, \quad x \in [0,1]$$

for all $a, b > -1$. We have

$$\lim_{x\to 1} \frac{f(x)}{(1-x)^{1+b}} = \frac{1}{1+b}$$

for all $a, b > -1$, which gives

$$\int_x^1 r^a(1-r)^b dr \asymp (1-x)^{1+b}, \quad x \in [0,1].$$

In particular, for $a \le 0$ we have $(1-x)^{1+b} \le (1-x)^{1+a+b}$. This completes the proof. □

Thus, Kolmogorov's continuity criterion implies that weighted - fBm is Hölder continuous of order δ for any $\delta < 1+b$.

Recall that a process X is the long-range dependence if

$$\sum_{n\ge\alpha} \rho_n(\alpha) = \infty, \tag{5}$$

for any $\alpha > 0$, and it is short-range dependence if

$$\sum_{n\ge\alpha} |\rho_n(\alpha)| < \infty. \tag{6}$$

where

$$\rho_n(\alpha) = E\left[(X_{\alpha+1} - X_\alpha)(X_{n+1} - X_n)\right], \qquad \alpha > 0.$$

Theorem 2.2. *Let $B^{a,b}$ be a weighted-fBm with $a > -1, -1 < b < 1$ and $|b| < 1+a$.*

(i) If $b > 0$, then $B^{a,b}$ is long-range dependence;

(ii) If $b < 0$, then $B^{a,b}$ is short-range dependence.

Proof. For any $\alpha > 0$ and $n \geq \alpha + 1$ we have

$$\rho_n(\alpha) = E\left[(B^{a,b}_{\alpha+1} - B^{a,b}_{\alpha})(B^{a,b}_{n+1} - B^{a,b}_{n})\right] = \int_{\alpha}^{\alpha+1} u^a\left[(n+1-u)^b - (n-u)^b\right]du.$$

If $b > 0$, we have

$$0 < (n+1-u)^b - (n-u)^b = (n+1-u)^b\left[1 - \left(1 - \frac{1}{n+1-u}\right)^b\right] \sim (n+1-u)^{b-1}$$

for all $\alpha \leq u \leq \alpha + 1$, and

$$\rho_n(\alpha) \sim \int_{\alpha}^{\alpha+1} u^a(n+1-u)^{b-1}du \geq \frac{1}{1+a}\left((\alpha+1)^{1+a} - \alpha^{1+a}\right)(n+1-\alpha)^{b-1},$$

which deduces the seises

$$\sum_{n\geq\alpha}\rho_n(\alpha) = \infty.$$

If $b < 0$, we have

$$0 < (n-u)^b - (n+1-u)^b = (n-u)^b\left[1 - \left(1 + \frac{1}{n-u}\right)^b\right] \sim (n-u)^{b-1}$$

for all $\alpha \leq u \leq \alpha + 1$, and

$$|\rho_n(\alpha)| \sim \int_{\alpha}^{\alpha+1} u^a(n-u)^{b-1}du \leq \frac{1}{1+a}\left((\alpha+1)^{1+a} - \alpha^{1+a}\right)(n-\alpha-1)^{b-1},$$

which deduces the seises

$$\sum_{n\geq\alpha}|\rho_n(\alpha)| < \infty.$$

This completes the proof. □

In what follows we model long-range dependence of financial assets under the assumption $b > 0$, and denote by $\Phi(\cdot)$ the cumulative probability distribution function of a standard normal random variable:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}u^2\right)du$$

and by $\varphi(\cdot) = \Phi'(\cdot)$ the density function.

3 Pricing using Fair Premium

Consider a financial market in which we have two securities: a bond (Security 1) with (instantaneous) interest rate which is also interpreted as the risk-free rate of interest, and a stock (Security 2) which is described by the stochastic price process (pay-out) V_t at time t. A time interval $[0,T]$is considered with 0 being the initial or present time and T being the terminal time. The price of Security 2 is denoted by V_0. We are interested in calculating the pricing of a European call option $C(K,T)$, say, written on Security 2 with strike price K and time to maturity T.

Definition 3.1. *The value $\{V_t\}$ results in an expected (instantaneous) rate of return μ and T is defined as*

$$e^{\mu T} = \frac{E[V_T]}{V_0} \tag{7}$$

Since nothing has been assumed about the process $\{V_t\}$, μ will in general depend on T.

Lemma 3.1. *(M. Bladt et al [4]) The fair premium, and hence the call option price, $C(K,T)$, of a European call option with time to maturity T and strike price K is given by*

$$C(K,T) = E[(e^{-\mu T}V_T - e^{-rT}K)1_{\{e^{-\mu T}V_T > e^{-rT}K\}}] \tag{8}$$

and the put option price, $P(K,T)$, of a European put option with time to maturity T and strike price K is given by

$$C(K,T) = E[(e^{-rT}K - e^{-\mu T}V_T)1_{\{e^{-\mu T}V_T < e^{-rT}K\}}]. \tag{9}$$

According to Alós *et al* [1] (see also Yan-An [19]), we have the following.

Lemma 3.2. *The solution to Equation* (1) *is given by*

$$V_t = V_0 \exp\left(\mu t - \sigma^2 \int_0^t u^a (t-u)^b du + \sigma B_t^{a,b}\right). \tag{10}$$

Theorem 3.1. *The fair premium, and hence the call option price, $C(K,T)$, of a European call option with time to maturity T and strike price K, is given by*

$$C(K,T) = V_0\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{11}$$

where

$$d_1 = \frac{\ln\frac{V_0}{K} + rT + \sigma^2\int_0^T u^a(T-u)^b du}{\sigma\sqrt{2\int_0^T u^a(T-u)^b du}}, \tag{12}$$

and

$$d_2 = \frac{\ln\frac{V_0}{K} + rT - \sigma^2\int_0^T u^a(T-u)^b du}{\sigma\sqrt{2\int_0^T u^a(T-u)^b du}}. \tag{13}$$

Proof. Fix $T > 0$, for $t \in [0,T]$, the weighted fractional Brownian motion $B_t^{a,b}$ is a centered Gaussian process with variance $2\int_0^T u^a(T-u)^b du$. According to (10), we have

$$\frac{V_t}{V_0} = \exp\left(\mu t - \sigma^2\int_0^t u^a(t-u)^b du + \sigma B_t^{a,b}\right).$$

Then

$$\log\frac{V_t}{V_0} = \mu s - \sigma^2\int_0^t u^a(t-u)^b du + \sigma B_t^{a,b}), \tag{14}$$

which means $\log\frac{V_T}{V_0}$ is a Gaussian process with mean $\mu s - \sigma^2\int_0^t u^a(t-u)^b du$ and variance $2\sigma^2\int_0^t u^a(t-u)^b du$. The distribution of V_T at T is in fact the only thing we need since only the price at the terminal date matters. Then noticing that $e^{-\mu T}V_T > e^{-rT}K$ is equivalent to

$$S_T > \frac{\log\frac{K}{V_0} + \sigma^2\int_0^T u^a(T-u)^b du - rT}{\sigma}.$$

From the Lemma 3.1, the call option price, $C(K,T)$, of a European call option with time to maturity T and strike price K is given by

$$C(K,T) = E[(e^{-\mu T}V_T - e^{-rT}K)1_{\{e^{-\mu T}V_T > e^{-rT}K\}}]. \tag{15}$$

First get that with $y = \frac{\log \frac{K}{V_0} + \sigma^2 \int_0^T u^a (T-u)^b du - rT}{\sigma}$,

$$
\begin{aligned}
&E[e^{\mu T} V_T 1_{\{e^{-\mu T} V_T > e^{-rT} K\}}] \\
&= e^{-\mu T} \int_y^\infty V_0 e^{\mu T - \sigma^2 \int_0^T u^a (T-u)^b du + \sigma x} \frac{1}{\sqrt{4\pi \int_0^T u^a (T-u)^b du}} e^{-\frac{x^2}{4\int_0^T u^a (T-u)^b du}} dx \\
&= V_0 \int_y^\infty \frac{1}{\sqrt{4\pi \int_0^T u^a (T-u)^b du}} e^{-\frac{(x - 2\sigma \int_0^T u^a (T-u)^b du)^2}{4\int_0^T u^a (T-u)^b du}} dx \\
&= V_0 P(Z > y),
\end{aligned}
$$

where $Z \sim N(2\sigma \int_0^T u^a (T-u)^b du, 2\int_0^T u^a (T-u)^b du)$. Furthermore

$$
P(Z > y) = \Phi\Big(\frac{\ln \frac{V_0}{K} + rT + \sigma^2 \int_0^T u^a (T-u)^b du}{\sigma \sqrt{2\int_0^T u^a (T-u)^b du}}\Big).
$$

On the other hand

$$
\begin{aligned}
&E[e^{-rT} K 1_{\{e^{-\mu T} V_T > e^{-rT} K\}}] \\
&= e^{-rT} K \Phi\Big(\frac{\ln \frac{V_0}{K} + rT - \sigma^2 \int_0^T u^a (T-u)^b du}{\sigma \sqrt{2\int_0^T u^a (T-u)^b du}}\Big).
\end{aligned}
$$

Then the proof of this theorem is complete. □

Corollary 3.1. *The put option price, $P(K,T)$, of a European put option with time to maturity T and strike price K is given*

$$
P(K,T) = K e^{-rT} \Phi(-d_2) - V_0 \Phi(-d_1). \tag{16}
$$

In Figure 1, 2, 3, 4, we plot the prices of the call option and the put option at time zero as a function of time to maturity for three values of $\sigma \in \{0.2, 0.3, 0.5\}$ and three values of the parameter $a \in \{-0.2, 0.0, 0.2\}$ with a fixed $b = 0.4$.

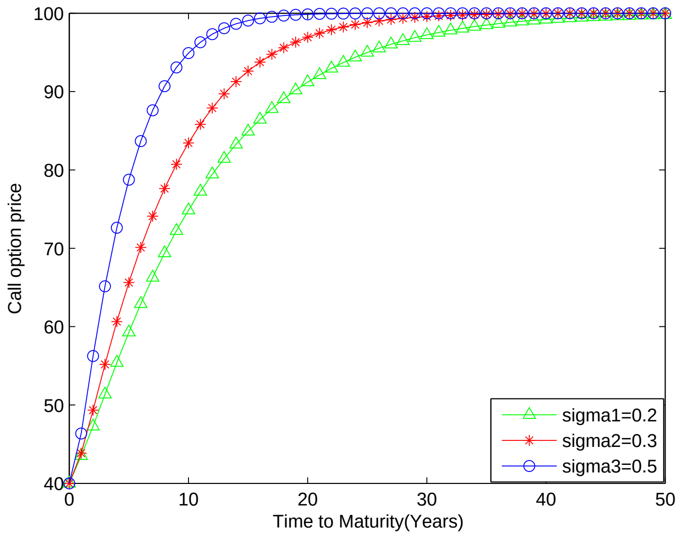

Figure 1: Price of call option at time zero resulting in the weighted-fractional Merton model against maturity time T when $r = 0.06$, $a = 0.2$, $b = 0.4$, $K = 60$, $V_0 = 100$ and $0 < T < 50$.

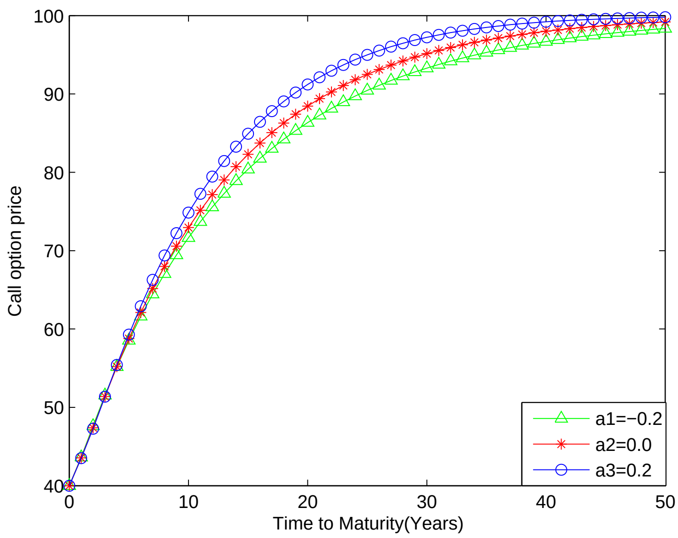

Figure 2: Price of call option at time zero resulting in the weighted-fractional Merton model against maturity time T when $b = 0.4$, $r = 0.06$, $\sigma = 0.2$, $K = 60$, $V_0 = 100$ and $0 < T < 50$.

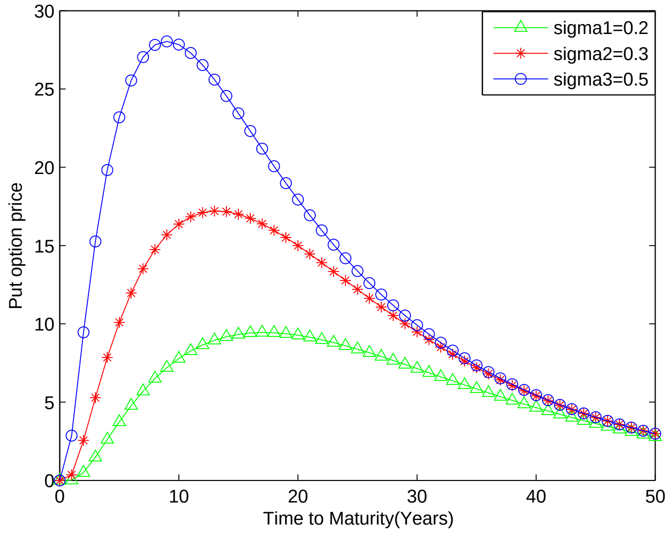

Figure 3: Price of put option at time zero resulting in the weighted-fractional Merton model against maturity time T when $r = 0.06$, $a = 0.2$, $b = 0.4$, $K = 60$, $V_0 = 100$ and $0 < T < 50$.

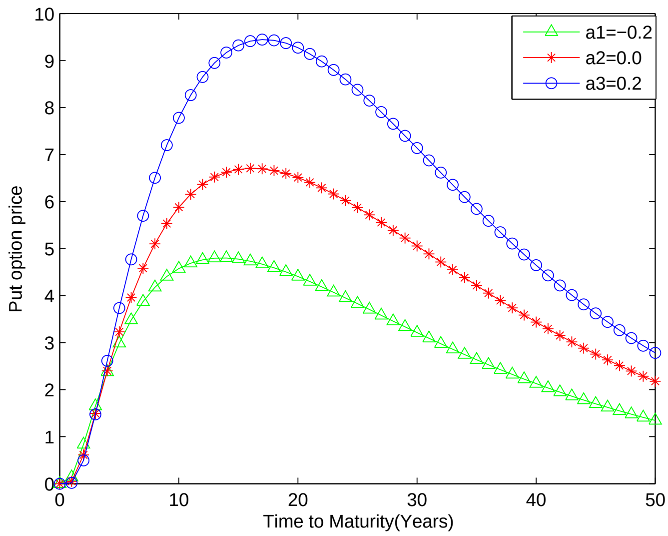

Figure 4: Price of put option at time zero resulting in the weighted-fractional Merton model against maturity time T when $b = 0.4$, $r = 0.06$, $\sigma = 0.2$, $K = 60$, $V_0 = 100$ and $0 < T < 50$.

In the above four Figures, for fixed T, we see that the price of European call option is increasing with respect to σ and a.

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