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Quasi 3-Crossed Modules

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Abstract

Using simplicial groups, quasi $3-$crossed modules of groups are introduced and some of the examples and results of quasi $3-$crossed modules are given.

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1 Introduction

Crossed modules have used widely so far, and in various context since their definition by J. H. C. Whitehead in his investigation of the algebraic structure of second relative homotopy groups. Areas in which crossed modules have been applied include the theory of group presentation (see the survey [2]), algebraic $K-$theory and homological algebra. Crossed modules can be viewed

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as 3−dimensional groups and it is therefore of interest to consider counter for crossed modules of concepts from group theory.

Given the importance of chain complex in (Abelian) homological algebra and need in many parts of mathematics to extend this to the non-Abelian case it is not surprising that various non-Abelian extension of the Dold-Kan equivalence have been studied in [4, 5]. For instance Ashley [1] examined simplicial $T-$complexes and group $T-$complexes and showed that these correspond to Moore complexes which are crossed complexes. Briefly a (reduced) crossed complex is crossed module

$$\cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0.$$

Conduché [6] considered a notion of 2−crossed module where the Peiffer elements ${}^{\partial(x)}x' \cdot (xx'x^{-1})^{-1}$ are not necessarily trivial but it is covered by elements in the next level up. These objects form a category equivalent to that of simplicial groups whose Moore complex has length 2.

$$\cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow \cdots \longrightarrow 1 \longrightarrow NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0.$$

If G is not necessarily Abelian, a semi-direct decomposition can be found that is made up of images of terms in NG. This semi-direct decomposition was well known in low dimensions but it first seems to have been exploited in higher dimensions by Conduché [6] who also gives a derivation of it.

In [4] Carrasco examined a notion of a hypercrossed complex of groups and proved that the category of such hypercrossed complex is equivalent to $\mathfrak{SimpGrp}$, the category of simplicial groups. For example if one truncated hypercrossed complex at level n, throwing away terms of $n-$complex from a category equivalent to equivalent to the $n-$hyper groupoids of groups of Duskin [7] and give algebraic models for $n-$types. For simplicial group which is group. $T-$complex in the sense of Ashley [1], the equivalence gives a hypercrossed complex which is actually a crossed complex whilst a subcategory of the category of 2−crossed complex is equivalent to Conduché's category of 2−crossed module.(see [14, 15, 16].)

In this paper, we give a definition of quasi 3−crossed module of groups and some application of Peiffer commutators on Moore complexes of a simplicial group. In particular for $i \geq k \in \{0, 1, \ldots, n+2\}$ we investigate to condition of Moore complex of G. Let $NG_i = 1$, where $NG_i = \bigcap\limits_{i=0}^{n-1} \mathrm{Ker} d_i$ is a Moore complex of **G**, be a simplicial group. We examine the simplicial long exact sequence and long exact Moore sequence respectively as follows respectively

$$\cdots \underset{s_n,\ldots s_0}{\overset{d_{n+1},\ldots,d_0}{\rightleftarrows}} G_n \underset{s_{n-1},\ldots s_0}{\overset{d_n,\ldots,d_0}{\rightleftarrows}} G_{n-1} \cdots G_2 \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightleftarrows}} G_1 \underset{s_0}{\overset{d_0,d_1}{\rightleftarrows}} G_0$$

and

$$\cdots 1 \longrightarrow NG_n \longrightarrow \cdots \longrightarrow NG_2 \longrightarrow NG_1 \longrightarrow NG_0 \ .$$

Also we iterate the long exact Moore sequence which is correspond with crossed complex, 2−crossed module, square complex, 2−crossed complex in [15], categorical group. We use crossed complex which its tail consists of quasi 3−crossed module, is called 3−crossed complex. This paper aim to give between relation algebraic topology constructions and $F_{(\alpha)(\beta)}$ Peiffer commutators are defined in [11, 13]. Moreover the Peiffer commutators is important role of these algebraic topology structures with Moore complex. Observe that the Moore complex is relation between structure of algebraic topology and a simplicial group.

2 Definitions and Notations

We remind that the following definition from Loday [10].

Definition 2.1. *A categorical group or cat*^{1}*−group is a group G together with a subgroup N two homomorphism (called structural homemorphism) $s, b : G \to N$ satisfying the following conditions*
(i) $s|_N = b|_N = id_N$
(ii) $[Kers, Kerb] = 1$

A cat^n−group $(G, N_1, \ldots N_n)$ is a group G together with n subgroups $N_1 \ldots, N_n$ and $2n$ homomorphism $s_i, b_i : G \to N_i$ satisfying the following conditions.
(1) s_i, b_i restrict to identity on N_i
(2) $[\mathrm{Ker} s_i, \mathrm{Ker} b_i] = 1$
(3) $s_i s_j = s_j s_i, \quad b_i b_j = b_j b_i, \quad s_j b_i = b_j s_i \quad$ for $\ i \neq j$.

Further, recall the following original definition, which is given by Glenn in [8].

Definition 2.2. *An $n-$dimensional hypergroup (groupoid) $(n \geq 1)$ is a simplicial object G satisfying axioms.*
$n-HYPGP : G_m \to \Lambda_i^m(G)$ is an isomorphism for $i = 0, \ldots, m$ and all $m \geq n$. So

$$\begin{array}{ccccc} NG_i & \xhookrightarrow{inc} & G_m & \xrightarrow{iso} & \Lambda_i^m(G) \\ \| & & \| & & \| \\ 1 & \xhookrightarrow{inc} & G_m & \xrightarrow{iso} & \Lambda_i^m(G) \end{array}$$

where given a simplicial group G $n > 1$ and $0 \leq i \leq n$, denote by $\Lambda_i^m(G)$ the object universal with respect to having projections $p_j : \Lambda_i^m(G) \to G_{n-1}$ for $0 \leq i \leq n$, and $j \neq 1$ satisfying $d_j p_k = d_{k-1} p_j$ for $j < k, \quad k \neq i$.

An element of $\Lambda_i^m(G)$ is in effect, a "hollow" $n-$simplex whose face opposite ν_i is missing hence the term "open $i-$horn" for element of $\Lambda_i^m(G)$.

If the map $G_n \to \Lambda_i^m(G)$ sending to g to $(d_0 g, \ldots, d_{i-1}, -, d_{i+1}, \ldots, d_n g)$ is epic for each $i = 0, \ldots, n$ then G satisfying Kan extensions condition at dimension n. If this map is epic for all n, G is called a Kan complex.

Now we recall hypercrossed complex pairings form [11, 13].

2.1 Hypercrossed Complex Pairings

In the following a normal subgroup N_n of G_n is defined. We get the construction of a useful family of pairings. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_l, \ldots, i_1), \beta = (j_m, \ldots, j_1) \in S(n)$. The pairings that we will need,

$$\begin{array}{ccc} NG_{n-\#\alpha} \times NG_{n-\#\beta} & \xrightarrow{F_{(\alpha)(\beta)}} & NG_n \\ {\scriptstyle s_\alpha \times s_\beta} \downarrow & & \uparrow {\scriptstyle p} \\ G_n \times G_n & \xrightarrow{\mu} & G_n \end{array}$$

$$\{F_{(\alpha)(\beta)} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \longrightarrow NG_n : (\alpha\ \beta) \in P(n), \quad n \geq 0\}$$

are given as composites by the above diagram where

$$s_\alpha = s_{i_l} \dots s_{i_1} : NG_{n-\#\alpha} \longrightarrow G_n, \quad s_\beta = s_{j_m} \dots s_{j_1} : NG_{n-\#\beta} \longrightarrow G_n,$$

$p : G_n \to NG_n$ is defined by the composite projections $p(x) = p_{n-1} \dots p_0(x)$, where

$$p_j(z) = z s_j d_j(z)^{-1} \qquad \text{with} \quad j = 0, 1, \dots, n-1$$

and $\mu : G_n \times G_n \to G_n$ is given by the commutator map and $\#\alpha$ is the number of the elements in the set of α and similarly for $\#\beta$. Thus

$$\begin{aligned} F_{(\alpha)(\beta)}(x_\alpha,\ y_\beta) &= p\mu(s_\alpha \times s_\beta)(x_\alpha,\ y_\beta), \\ &= p[s_\alpha(x_\alpha),\ s_\beta(y_\beta)]. \end{aligned}$$

We now define the normal subgroup N_n of G_n to be that generated by elements of the form

$$F_{(\alpha)(\beta)}(x_\alpha,\ y_\beta),$$

where $x_\alpha \in NG_{n-\#\alpha}$ and $y_\beta \in NG_{n-\#\beta}$. We illustrate this subgroup for $n = 2$ and $n = 3$ to demonstrate what it looks like.

Example 2.3. For $n = 2$, suppose $\alpha = (1),\ \beta = (0)$ and $x_1,\ y_1 \in NG_1 = \mathrm{Ker} d_0$. It follows that

$$\begin{aligned} F^{(2)}_{(0)(1)}(x_1,\ y_1) &= p_1 p_0 [s_0(x_1),\ s_1(y_1)], \\ &= p_1[s_0(x_1),\ s_1(y_1)], \\ &= [s_0(x_1),\ s_1(y_1)]\ [s_1(y_1),\ s_1(x_1)], \end{aligned}$$

which is a generating element of the normal subgroup N_2.

For $n = 3$, the possible pairings are the following

$$\begin{array}{lll} F^{(3)}_{(1,0)(2)}, & F^{(3)}_{(2,0)(1)}, & F^{(3)}_{(0)(2,1)}, \\ F^{(3)}_{(0)(2)}, & F^{(3)}_{(1)(2)}, & F^{(3)}_{(0)(1)}. \end{array}$$

For all $x_1 \in NG_1,\ y_2 \in NG_2$, the corresponding generators of N_3 are:

$$\begin{aligned} F^{(3)}_{(1,0)(2)}(x_1,\ y_2) &= [s_1 s_0(x_1),\ s_2(y_2)]\ [s_2(y_2),\ s_2 s_0(x_1)], \\ F^{(3)}_{(2,0)(1)}(x_1,\ y_2) &= [s_2 s_0(x_1),\ s_1(y_2)]\ [s_1(y_2),\ s_2 s_1(x_1)] \\ &\quad\ [s_2 s_1(x_1),\ s_2(y_2)]\ [s_2(y_2),\ s_2 s_0(x_1)], \end{aligned}$$

and all $x_2 \in NG_2,\ y_1 \in NG_1$,

$$F^{(3)}_{(0)(2,1)}(x_2,\ y_1) = [s_0(x_2),\ s_2 s_1(y_1)]\ [s_2 s_1(y_1),\ s_1(x_2)]\ [s_2(x_2),\ s_2 s_1(y_1)],$$

whilst for all $x_2,\ y_2 \in NG_2$,

$$\begin{array}{rcl} F^{(3)}_{(0)(1)}(x_2,\ y_2) & = & [s_0(x_2),\ s_1(y_2)]\ [s_1(y_2),\ s_1(x_2)]\ [s_2(x_2),\ s_2(y_2)], \\ F^{(3)}_{(0)(2)}(x_2,\ y_2) & = & [s_0(x_2),\ s_2(y_2)], \\ F^{(3)}_{(1)(2)}(x_2,\ y_2) & = & [s_1(x_2),\ s_2(y_2)]\ [s_2(y_2),\ s_2(x_2)]. \end{array}$$

□

We have examined the long exact Moore sequence

$$\cdots \longrightarrow NG_n \longrightarrow \cdots \longrightarrow NG_2 \longrightarrow NG_1 \longrightarrow NG_0 \qquad *$$

for case $i \geq 1$ and for $0 \leq i \leq n+1$. That is $\cdots 1 \to NG_0 = G_0$.

3 Illustrative Examples: Pre$-2-$Crossed Modules and Quasi $3-$Crossed Modules of a Simplicial Group with Moore Complex

Before giving definition of quasi 3−crossed module it will be helpful to have notion of a pre−crossed module and introduce description of pre−2−crossed modules.

A *pre−crossed module* of groups consists of a group, M, a N-group M, and a group homomorphism $\partial : M \longrightarrow N$, such that for all $m \in M,\quad n \in N$ $CM1)\qquad \partial({}^n m) \ = \ n\partial(m)n^{-1}$. Now we may describe that definition of a pre−2−crossed module of group.

A *pre−2−crossed modules* consists of complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial} N$$

together with action of N on L and M so that $\partial_2,\ \partial_1$ are morphism of $N-$group where the group acts an itself by ${}^x y$, action of M on L written $m \cdot l$ such that with this action

$$L \xrightarrow{\partial_2} M$$

is a pre−crossed module and there is a second action of M on L via N denoted ${}^m l$, so that for all $l \in L,\quad m \in M$, and $n \in N$ that ${}^{n m} l = {}^{nm} l$. Further there is a $N-$equivalent function

$$\{,\} : M \times M \to L$$

called Peiffer commutator, which satisfying the following conditions:
2CM1$_p$ $\partial_2\{x,y\} = {}^{\partial_1(x)}y\ xy^{-1}x^{-1}$
2CM2$_p$ (i) $\{xx',y\} = {}^{\partial_1(x)}\{x',y\}\ \{x,x'y(x')^{-1}\}$
(ii) $\{x,yy'\} = \{x,y\}\ {}^{xyx^{-1}}\{x,y'\}$
2CM3$_p$ ${}^{n}\{x,y\} = \{{}^{n}x,{}^{n}y\}$
for all $x,y \in M$, $n \in N$, $l \in L$. Let G be a simplicial group with the Moore complex NG. Then the complex of groups

$$NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0$$

is a pre−2−crossed module, where the Peiffer commutator map is defined as follows:

$$\begin{array}{rcll}\{\ ,\ \}: & NG_1 \times NG_1 & \longrightarrow & NG_2 \\ & (x_0,\ x_1) & \longmapsto & s_0(x_0)s_1(x_1)s_0(x_0)^{-1}s_1(x_0)s_1(x_1)^{-1}s_1(x_0)^{-1}.\end{array}$$

It is obvious to pre−crossed module condition is satisfied. Indeed it is sufficient to show that ∂_2, ∂_1 are pre−crossed modules and pre−2−crossed module axioms are verified. That is NG_0 acts on NG_1 via s_0 and NG_1 acts on NG_2 via s_1 and NG_0 acts on NG_2 via s_1s_0. Thus

$$\partial_1({}^{x_0}x_1) = \partial_1(s_0(x_0)x_1s_0(x_0)^{-1}) = x_0\partial_1(x_1)x_0^{-1}$$

$$\partial_2({}^{x_1}x_2) = \partial_2(s_1(x_1)x_2s_1(x_1)^{-1}) = x_1\partial_2(x_2)x_1^{-1}$$

2CM1$_p$:

$$\begin{array}{rcl}\partial_2\{x_0,\ x_1\} & = & \partial_2(s_0(x_0)s_1(x_1)s_0{x_0}^{-1}s_1(x_0)s_1(x_1)^{-1}s_1(x_0)^{-1}), \\ & = & s_0d_1(x_0)x_1s_0d_1(x_0)^{-1}x_0(x_1)^{-1}(x_0)^{-1}, \\ & = & {}^{\partial_1(x_0)}x_1 \quad x_0(x_1)^{-1}(x_0)^{-1}.\end{array}$$

Other two conditions are clear and where ∂_1, ∂_2 are restrictions of d_1 d_2 respectively.

Now we can give definition of a quasi 3−crossed modules of groups.

Definition 3.1. *A quasi* 3−*crossed module of group consists of a complex* N−*groups*

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

and $\partial_3, \partial_2, \partial_1$ morphism of $N-$groups, where the group N acts on itself by conjugation, such that

$$K \xrightarrow{\partial_3} L$$

is a crossed module and

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

is a pre$-2-$crossed module. Thus L acts on K and we require that for all $k \in K$, $l \in L$, $m \in M$ and $n \in N$ that

$$^{(^n m)}(^l k) = {}^n\left(^m(^l k)\right).$$

Furthermore there is a $N-$equivalent function

$$\{\ ,\ \} : L \times L \to K$$

Mutlu mapping is defined as follows

$$\{l_1, l_2\} = H(l_1, l_2) = [s_0(l_1), s_1(l_2)][s_1(l_2), s_1(l_1)][s_2(l_1), s_2(l_2)],$$

if the following conditions are verified.

3CM1$_q$ ∂_2, ∂_1 *are pre$-$crossed module,* ∂_3 *is a crossed module*
3CM2$_q$ $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ *is a pre$-2-$crossed module*
3CM3$_q$ $\partial_3 H(l_1, l_2) = {}^{s_0 d_2(l_1)} s_1 d_2(l_2)\ \ {}^{s_1 d_2(l_1)} s_1 d_2(l)^{-1}\ \ {}^{l_1} l_2 l_2^{-1}$
3CM4$_q$ (a) $H(l, \partial_3(k)) = [l, s_2(k)]$
(b) $H(\partial_3(k) l,) = [s_2(k), l]$
3CM5$_q$ $H(l, \partial_3(k)) H(\partial_3(k) l,) = 1$
3CM6$_q$ $H(\partial_3(k_1),\ \partial_3(k_2)) = [k_1, k_2]$

where $l_1, l_2 \in L$ and $k_1, k_2 \in K$

Theorem 3.2. ***(a)*** *Let $NG_i = 1$ for $\forall i \geq 1$ in the long exact Moore sequence if and only if the long exact Moore sequence become only group i.e., G_0 be a group.*
(b) *Let $NG_i = 1$ for $\forall i \geq 2$ in the long exact Moore sequence if and only if the long exact Moore sequence be crossed module that is $\cdots 1 \to NG_1 \to NG_0$ is a crossed module.*

(c) *Let* $NG_i = 1$ *for* $\forall i \geq 3$ *in the long exact Moore sequence if and only if the long exact Moore sequence become a* $2-$*crossed module i.e,* $\cdots 1 \to NG_2 \to NG_1 \to NG_0$ *is a* $2-$*crossed module.*

(d) *Let* $NG_i = 1$ *for* $\forall i \geq 4$ *in the long exact Moore sequence if and only if the long exact Moore sequence be quasi* $3-$*crossed module* $\cdots 1 \to NG_3 \to NG_2 \to NG_1 \to NG_0$ *is a* $3-$*quasi crossed module (*$3-$*crossed complex).*

(e) *Let* $NG_i = 1$ *for* $\forall i \geq n+1$ *in the long exact Moore sequence if and only if the long exact Moore sequence become an* $n-$*crossed complex that is,* $\cdots 1 \to NG_n \to NG_{n-1} \to \cdots \to NG_3 \to NG_2 \to NG_1 \to NG_0$ *is an* $n-$*crossed complex.*

(f) *Let* $NG_i = 1$ *for* $\forall i \geq n+2$ *in the long exact Moore sequence if and only if the long exact Moore sequence be a* $T-$*complex.*

(g) *Let* $F_{(\alpha)(\beta)}(x_\alpha, y_\beta) = 1$ *hypercrossed complex pairings are described in [11, 13] if and only if the long exact Moore sequence be a crossed complex.*

Proof: **(a)** Suppose that $NG_i = 1$ for $\forall i \geq 1$ and so the long exact Moore sequence obtains as follows: $\cdots 1 \to 1 \to 1 \to \cdots \to 1 \to 1 \to 1 \to NG_0 = G_0$. This is because of the long exact simplicial sequence

$$\cdots \underset{s_0,\ldots,s_2}{\overset{d_0,\ldots,d_3}{\rightleftarrows}} 1 \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightleftarrows}} 1 \underset{s_0}{\overset{d_0,d_1}{\rightleftarrows}} G_0$$

. Therefore $NG_0 = \mathrm{Ker}d_0^1$ is a subgroup of G_0.

On the other hand, if $a \in \mathrm{Ker}d_0^1$, then $NG_1 = 1$ since $d_0(a) = 1$. Moreover it is a cat^0-group. (see Brown-Loday [3].)

(b) If $NG_i = 1$ for $\forall i \geq 2$ $(1 \leq i \leq n+2)$ in the long exact Moore sequence then the long exact Moore sequence obtains as follows i.e., $\cdots 1 \to 1 \to 1 \to \cdots \to 1 \to 1 \to NG_1 \to NG_0 = G_0$ be a crossed module. (see [12] and [10].) On other word, recall that $F_{\alpha,\beta}(x_\alpha, y_\beta) = 1$ in [11, 13], then for $\alpha = (0)$, $\beta = (1)$

$$\begin{aligned} F^{(2)}_{(0)(1)}(x_1, y_1) &= NG_1 \times NG_1 \to NG_2 \\ F^{(2)}_{(0)(1)}(x_1, y_1) &= s_0(x_1)s_1(y_1)s_0(x_1)s_1(x_1 y_1^{-1} x_1^{-1}) = 1 \end{aligned}$$

since $NG_1 \to NG_0$ be crossed module i.e, NG_0 acts on NG_1 together with ${}^{x_1}y_1 = s_0(x_1)y_1 s_0(x)^{-1}$ verifies crossed axioms such as,

$$\partial_1({}^{x_1}y_1) = x_1 \partial_1(y) {x_1}^{-1}$$

and

$$^{\partial_1(x_1)}y_1 = s_0d_1(x_1)y_1s_0d_1(x_1)^{-1} = x_1y_1x_1^{-1} \ \ (\partial_1 \ \text{ by restriction } \ d_1).$$

(see [12, 13]) Also the long exact simplicial sequence $\cdots 1 \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightrightarrows}} 1 \underset{s_0}{\overset{d_0,d_1}{\rightrightarrows}} G_0$ be correspond to cat$^1-$group which is proved in [10] by Loday. Recall that the structural morphism s and b are given by $d_1 = s, \quad b = d_0$. Axiom (i) of cat$^1-$group follows that relations between face and degeneracy maps. To prove axiom (ii) it is sufficient to see for $x \in \mathrm{Ker}d_1$ and $y \in \mathrm{Ker}d_0$ the element $[s_1(x_1)s_0(x_0)^{-1},\ s_1(y_1)]$ of NG_2 where s_0, s_1 are degeneracy maps and in fact its image by d_2 is $[1, y]$. So $[\mathrm{Ker}d_1,\ \mathrm{Ker}d_0] = 1$, since $\partial_2 NG_2 = 1$ and also 1$-$truncated hypercrossed complex, 1$-$hypercrossed complex and 1$-$crossed complex see Carrosco and Cegarra [5] and [12] respectively.

(c) Let $NG_i = 1$ for $\forall i \geq 3$ then the long simplicial sequence

$$\cdots \underset{s_0,\dots,s_3}{\overset{d_0,\dots,d_4}{\rightleftarrows}} 1 \underset{s_0,s_1,s_2}{\overset{d_0,d_1,d_2,d_3}{\rightrightarrows}} G_2 \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightrightarrows}} G_1 \underset{s_0}{\overset{d_0,d_1}{\rightrightarrows}} G_0$$

be and the long exact sequence of Moore complex $\cdots 1 \to NG_2 \to NG_1 \to NG_0$ is a 2$-$crossed module with $F^{(3)}_{(\alpha)(\beta)}(x_\alpha, y_\beta) = 1$ for $\alpha, \beta \in P(3)$. So the Peiffer lifting is defined as follows:

$$\{,\} : NG_1 \times NG_1 \to NG_2$$

$$\{x_1, y_1\} \mapsto s_0(x_1)s_1(y_1)s_0(x_1)s_1(x_1y_1^{-1}x_1^{-1}) = 1$$

and thus 2$-$crossed module conditions are also satisfied in [12, 13]. For sufficient condition, it is obvious from 2CM2, 2CM4(a) and (b) of 2$-$crossed modules axioms give us $F^{(3)}_{(\alpha)(\beta)}(x_\alpha, y_\beta) = 1$ implies $NG_3 = 1$. Moreover Ellis and Stenier showed crossed square equivalent to cat$^2-$group. Here we say cat$^2-$group axioms verified i.e., axioms (a) and (b) of cat$^2-$group follows from relations between face and degeneracy maps. To prove axiom (b) it is sufficient to see for $x \in \mathrm{Ker}d_i$ and $y \in \mathrm{Ker}d_j$ the element $\prod\limits_{I,J}[K_I, K_J]$ of NG_3 and its image by d_3 is $[x, y] = 1$. As $NG_3 = 1$ it follows that $[\mathrm{Ker}d_i,\ \mathrm{Ker}d_j] = 1$ for see details in [13, 17].

(d) Let **G** be a simplicial group with the Moore complex $N\mathbf{G}$. Then the Moore complex of groups

$$NG_3/\partial_4(NG_4 \cap D_4) \xrightarrow{\partial_3} NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0$$

is a quasi 3−crossed module of groups, where also D_4 is the normal subgroup generated by the degenerate elements.

Now we can define Mutlu map is define as follows:

$$\begin{array}{rccl} \{\ ,\ \}: & NG_2 \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\ & (x_2,\ y_2) & \longmapsto & [s_0(x_2), s_1(y_2)][s_1(x_2), s_1(y_2)][s_2(x_2), s_2(y_2)] \end{array}$$

here the right hand side denotes a coset in $NG_3/\partial_4(NG_4 \cap D_4)$ represented by an element in NG_3.

($\mathbf{3CM1}_q$) Let ∂_2, ∂_1 are pre−crossed modules and so NG_1 acts on NG_2 via s_1 and NG_0 acts on NG_1 via s_0. Thus $\partial_1({}^{x_0}y_1) = \partial_1(s_0(x_0)y_1s_0(x_0)^{-1}) = x_0\partial_1(y_1)x_0^{-1} = {}^{x_0}\partial_1(y_1)$ and $\partial_2({}^{y_1}y_2) = \partial_2(s_1(y_1)y_2s_1(y_1)^{-1}) = y_1\partial_2(y_2)y_1^{-1} = {}^{y_1}\partial_2(y_2)$.
It is readily checked that the morphism $\partial_3 : NG_3/\partial_4(NG_4 \cap D_4) \to NG_2$ is a crossed module i.e., NG_2 acts on $NG_3/\partial_4(NG_4 \cap D_4)$ via s_2 and we have $\partial_4F_{(2)(3)}(x_3, y_3) = s_2\partial_3(x_3)y_3s_2\partial_3(y_3)x_3y_3^{-1}x_3^{-1} = 1$ via mod $\partial_4(NG_4 \cap D_4)$. Thus $\partial_4F_{(2)(3)}(x_3, y_3) = s_2\partial_3(x_3)y_3s_2\partial_3(y_3)$
$x_3y_3^{-1}x_3^{-1}$mod $\partial_4(NG_4 \cap D_4)$ so $\partial_3({}^{x_3}y_3) = \partial_3(s_3(x_3)y_3s_3(x_3)^{-1}) = x_3\partial_3(y_3)x_3^{-1}$ and ${}^{\partial_3x_3}y_3 = s_2\partial_3(x_3)y_3s_2\partial_3(x_3)^{-1} = x_3y_3x_3^{-1}$ is obtained.
($\mathbf{3CM2}_q$) $NG_2 \to NG_1 \to NG_0$ is a pre−2−crossed module, where Peiffer map is defined as above.
$((x_0,\ x_1) \longmapsto s_0(x_0)s_1(x_1)s_0(x_0)^{-1}s_1(x_0)s_1(x_1)^{-1}s_1(x_0)^{-1})$
($\mathbf{3CM3}_q$)

$$\partial_3H(x_2, y_2) \;=\; {}^{s_0d_2(x_2)}s_1d_2(y_2)\ \ {}^{s_1d_2(x_2)}s_1d_2(y_2)^{-1}\ \ {}^{x_2}y_2y_2^{-1}.$$

($\mathbf{3CM4}_q$) (a) Using the hypercrossed complex parings are defined in [11, 13] and then

$$\begin{aligned}1 \equiv \partial_4 F^{(4)}_{(0)(3,1)}(x_3, y_2) &= [s_0d_3(x_3), s_1(y_2)][s_1(y_2), s_1d_3(x_3)] \\ &\quad [s_2d_3(x_3), s_2(y_2)][s_2(y_2), x_3] \\ &\quad \mod \partial_4(NG_4 \cap D_4)\end{aligned}$$

is calculated. Thus we have

$$\begin{aligned}H(\partial_3(x_3), y_2) &= [s_0d_3(x_3), s_1(y_2)][s_1(y_2), s_1d_3(x_3)] \\ &\quad [s_2d_3(x_3), s_2(y_2)] \mod \partial_4(NG_4 \cap D_4)\end{aligned}$$

and therefore we obtain

$$\begin{aligned}H(\partial_3(x_3), y_2) &= [x_3, s_2(y_2)] \mod \partial_4(NG_4 \cap D_4) \\ &= {}^{x_3}y_2\, y_2^{-1}. \text{ (definition of the action)}\end{aligned}$$

(b) Again using the hypercrossed complex parings in [11, 13] then

$$\begin{aligned}1 \equiv \partial_4 F^{(4)}_{(0,3)(1)}(y_2, x_3) &= [s_0(y_2), s_1d_3(x_3)][s_1d_3(x_3), s_1(y_2)] \\ &\quad [s_2(y_2), s_2d_3(x_3)][x_3, s_2(y_2)] \\ &\quad \mod \partial_4(NG_4 \cap D_4)\end{aligned}$$

is found. This equality also holds

$$\begin{aligned}H(y_2, \partial_3(x_3)) &= [s_0(y_2), s_1d_3(x_3)][s_1d_3(x_3), s_1(y_2)] \\ &\quad [s_2(y_2), s_2d_3(x_3)] \mod \partial_4(NG_4 \cap D_4)\end{aligned}$$

and so this implies that

$$H(y_2, \partial_3(x_3)) = [s_2(y_2), x_3] \mod \partial_4(NG_4 \cap D_4)$$

which is commutated. Thus the results of (a) and (b) of **3CM**4_q is given as above.

3CM5_q

$$H(\partial_3(x_3), y_2)H(y_2, \partial_3(x_3)) = [x_3, s_2(y_2)][s_2(y_2), x_3] = 1$$

3CM6_q Using by [11, 13] we may also be written this equation as

$$\begin{aligned}1 \equiv \partial_4 F^{(4)}_{(0)(1)}(x_3, y_3) &= [s_0d_3(x_3), s_1d_3(y_3)][s_1d_3(x_3), s_1d_3(y_3)] \\ &\quad [s_2d_3(x_3), s_2d_3(y_3)][y_3, x_3] \\ &\quad \mod \partial_4(NG_4 \cap D_4).\end{aligned}$$

Using the equation is obtained as

$$\begin{aligned}H(\partial_3(x_3), \partial_3(y_3)) &= [s_0d_3(x_3), s_1d_3(y_3)][s_1d_3(y_3), s_1d_3(x_3)] \\ &\quad [s_2d_3(x_3), s_2d_3(y_3)]\end{aligned}$$

$\mod \partial_4(NG_4 \cap D_4)$.

Here we yield

$$H(\partial_3(x_3), \partial_3(y_3)) \equiv [x_3, y_3] \mod \partial_4(NG_4 \cap D_4).$$

(e) If $NG_i = 1$ for $\forall i \geq n+1$ in the long exact Moore sequence, then the long exact Moore sequence be an $n-$crossed complex with $F^{(n+1)}_{(\alpha)(\beta)}(x,y) = 1$. Recalling by [11, 14] we have the trivial map as follows:

$$F^{(n+1)}_{(\alpha)(\beta)}(x,y) = NG_{(n+1))-\#\alpha} \times NG_{(n+1))-\#\beta} \to NG_{n+1}.$$

So NG_n also be an abelian group for $n \geq 2$ since

$$\begin{aligned} 1 &= \partial_{n+1} F^{(n+1)}_{(n-1)(n)}(x,y) \\ &= s_{n-1}d_n(x)ys_{n-1}d_(x)^{-1}xy^{-1}x^{-1} \\ &= {}^{\phi^{(n+1)}_{n-1}d_n(x)}y \ \ xy^{-1}x^{-1} \\ &= [y,x]. \end{aligned}$$

Here NG is a simplicial chain complex where NG_n is abelian for $n \geq 2$, $\phi^{(n+1)}_{n-1}$ is action of NG_0 on NG_n for each $n \geq 1$ and ∂_n is NG_0-group homomorphism defined as

$$\cdots \longrightarrow NG_n/\partial_{n+1}K_{n+1} \longrightarrow NG_{n-1}/\partial_n K_n \longrightarrow \cdots \longrightarrow NG_3/\partial_4 K_4 \longrightarrow$$

$$NG_2/\partial_3 K_3 \longrightarrow NG_1/\partial_2 K_2 \longrightarrow NG_0$$

is obviously a crossed complex, $K_i = NG_i \cap D_i$.

To prove the opposite of it $NG_n/\partial_{n+1}K_{n+1}$ be abelian group for $n \geq 2$, then

$$\begin{aligned} \partial_{n+1} F^{(n+1)}_{(n-1)(n)}(x,y) &= s_{n-1}d_n(x)ys_{n-1}d_n(x)^{-1}xy^{-1}x^{-1} \\ &= [y,x] = 1. \end{aligned}$$

Thus $F^{(n-1)}_{(\alpha)(\beta)}(x,y) = 1$ implies that $NG_{n+1} = 1$. This is also an $n-$truncated complex. (see Carrasco and Cegarra[5].)

(f) Let $NG_i = 1$ for $\forall i \geq n+2$ in the long exact Moore sequence if and only if the long Moore sequence be a $T-$complex. To proof see Ashley [1] and Carrasco and Cegarra [5].

(g) Let $F^{(n-1)}_{(\alpha)(\beta)}(x_\alpha, y_\beta) = 1$ in the long Moore sequence if and only if the long exact Moore sequence become a crossed complex. □

Example 3.3. *$3-$truncated complex is a quasi $3-$crossed module.*

Therefore we have following results.

Corollary 3.4. *If $NG_3/\partial_4(NG_4 \cap D_4) = 1$, then $NG_2 \to NG_1 \to NG_0$ corresponds a 2−crossed module. (see [12, 13])*

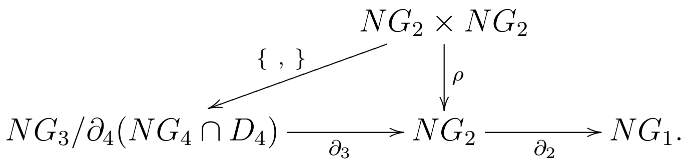
Corollary 3.5. *If $NG_0 = 1$, then*

$$NG_3/\partial_4(NG_4 \cap D_4) \xrightarrow{\partial_3} NG_2 \xrightarrow{\partial_2} NG_1$$

is a 2−crossed module with defined Peiffer map as

$$\begin{array}{rccl} \{\ \ ,\ \ \}: & NG_2 \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\ & (x_2,\ y_2) & \longmapsto & s_1(x_2)s_2(y_2)s_1(x_2)^{-1}s_2(y_2)s_2(x_2)^{-1}s_2(y_2)^{-1}. \end{array}$$

Proof: Indeed the function is satisfied 2−crossed module axioms.
2CM1: To prove easier since $\partial_3\{x_2, y_2\} = {}^{\partial_2(x_2)}y_2 \ \ x_2 y_2^{-1} x_2^{-1}$.
2CM2: Let $\partial_4 F^{(4)}_{(1)(2)}(x_2, y_2) = d_4(F_{(1)(2)}(x_2,\ y_2)) = [s_1d_3x_2, s_2d_3y_2]$ $[s_2d_3y_2, s_2d_3x_2][x_2, y_2]$. So $\partial_4 F^{(4)}_{(1)(2)}(x_2, y_2) = 1 \ \ \mathrm{mod}\partial_4(NG_4 \cap D_4)$.
Then $\{\partial_3(x_2), \partial_3(y_2)\} = [y_2, x_2]$ is obtained. (see [11, 13])
2CM3: (i) $\{x_2x_2', y_2\} = {}^{\partial_2 x}\{x_2', y_2\} \ \ \{x_2, x_2y_2x_2^{-1}\}$
(ii) $\{x_2, y_2y_2'\} = \{x_2, y_2\} \ \ {}^{x_2y_2x_2^{-1}}\{x_2, y_2'\}$
2CM4: (a) Let $\partial_4 F^{(4)}_{(1)(3,2)}(y_3, x_2) = d_4(F_{(1)(3,2)}(y_3,\ x_2)) = [s_1d_3y_3, s_2x_2]$ $[s_2x_2,\ s_2d_3y_3][y_3,\ s_2x_2] = 1 \ \ \mathrm{mod}\ \partial_4(NG_4 \cap D_4)$. Then $\{\partial_3(y_3), x_2\} = [s_2(x_2), y_2] = {}^{x_2}y_3\ y_3^{-1}$ is obtained by the definition of action.
(b) Let $F^{(4)}_{(3,1)(2)}(x_2, y_3) = d_4(F_{(3,1)(2)}(x_2,\ y_3)) = [s_1x_2, s_2d_3y_3][s_2d_3y_3, s_2x_2]$ $[s_2x_2, y_3][y_3, s_1x_2]$. So $F^{(4)}_{(3,1)(2)}(x_2, y_3) = 1 \ \ \mathrm{mod}\partial_4(NG_4 \cap D_4)$. Then $\{x_2,\ \partial_3(y_3)\} = [s_1(x_2), y_3][y_3s_2(x_2)] = x \cdot y\ {}^{x_2}y_3\ y_3^{-1}$
is found.
2CM5 $\{x_2,\ \partial_3(y_3\}\{\partial_3(y_3), x_2\} = (x \cdot y) \ \ {}^{x}y\ y^{-1} = {}^{\partial_2(x_2)}y_2 \cdot y_2^{-1}$
is calculated by the definition of the action.
2CM6 ${}^{n}\{x, y\} = \{{}^{n}x \ \ {}^{n}y\}$.
Now we consider the following diagram of morphisms



□

The group NG_2 acts, in two way on the group $NG_3/\partial_4(NG_4 \cap D_4)$ by conjugation via s_1 and via s_2 both within G_3. The action via s_1 will be denoted by $x \cdot y = s_1(x)ys_1(x)^{-1}$ and the action via s_2 will be denoted by ${}^{x}y = s_2(x)ys_2(x)^{-1}$. The action of NG_1 on NG_3 is given as follows: from equality $[s_1(x)^{-1}s_2s_1d_2(x), y] \equiv 1 \mod NG_3/\partial_4(NG_4 \cap D_4)$, there is a commutative diagram

$$\begin{array}{ccc} NG_3/\partial_4(NG_4 \cap D_4) \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\ \downarrow & & \downarrow \\ NG_3/\partial_4(NG_4 \cap D_4) \times NG_1 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \end{array}$$

given by

$$\begin{array}{ccc} (y \times x) & \longmapsto & x \cdot y = s_1(x)ys_1(x)^{-1} \\ \downarrow & & \downarrow \\ (x \times \partial_2(y)) & \longmapsto & {}^{\partial_2 x}y = s_2s_1d_2(x)ys_2s_1d_2(x)^{-1} \end{array}$$

which gives an equality

$${}^{\partial_2 x}y = s_2s_1d_2(x)ys_2s_1d_2(x)^{-1} = s_1(x)ys_1(x)^{-1}.$$

Let us define the map ρ by $\rho(x, x') = {}^{\partial_2(x)}x' \;\; x(x')x^{-1}$ for $x, x' \in NG_2$, that is the Peiffer commutator in NG_2 corresponding $\{x, x'\}$. Thus if the map ρ is a trivial map then $\partial_2 : NG_2 \to NG_1$ is a crossed module.

Now if the long Moore sequence is iterated as follows, then two results are obtained where $K_i = NG_i \cap D_i$.

$$\cdots 1 \longrightarrow NG_n/\partial_{n+1}K_{n+1} \longrightarrow NG_{n-1}/\partial_n K_n \cdots NG_1/\partial_2 K_2 \longrightarrow NG_0$$

Corollary 3.6.

$$\cdots 1 \longrightarrow NG_k/\partial_{k+1}K_{k+1} \xrightarrow{\partial_k} NG_{k-1} \xrightarrow{\partial_{k-1}} NG_{k-2} \longrightarrow 1 \longrightarrow \cdots \longrightarrow 1$$

is a $2-$*crossed module with defined Peiffer commutator*

$$\{x_{k-1},\ y_{k-1}\} = s_{k-1}(x_{k-1})s_k(y_{k-1})s_{k-1}(x_{k-1})^{-1}s_k(x_{k-1}y_{k-1}^{-1}x_{k-1}^{-1})$$

the $2-$*crossed module conditions are clearly verified.*

Corollary 3.7.

$$\cdots 1 \longrightarrow NG_k/\partial_{k+1}K_{k+1} \xrightarrow{\partial_k} NG_{k-1} \xrightarrow{\partial_{k-1}}$$

$$NG_{k-2} \xrightarrow{\partial_{k-2}} NG_{k-3} \xrightarrow{\partial_{k-3}} 1 \longrightarrow \cdots \longrightarrow 1$$

is a quasi $3-$*crossed modules, where the Mutlu map is defined as follows:*

$$\{x_{k-1},\ y_{k-1}\} = F^{(k)}_{(0)(1)}(x_{k-1}, y_{k-1})$$

It is obvious that quasi $3-$*crossed modules conditions are satisfied.*

We can follow the same procure as we make in Corollary 3.7 in order to get to result.

Corollary 3.8. *The category of quasi* $3-$*crossed modules is equivalent to the category of simplicial groups with Moore complex of length 3.*

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