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Radiatic Dimension of a Graph

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Abstract

Let $G(V, E)$ be a simple, finite, connected graph. An injective mapping $f : V(G) \to Z^+$ such that for every two distinct vertices $u, v \in V(G)$, $|f(u) - f(v)| \geq diam(G) + 1 - d(u, v)$ is called a radio labeling of G. The radio number of f, denoted by $rn(f)$ is the maximum number assigned to any vertex of G. The radio number of G, is the minimum value of $rn(f)$ taken over all radio labelings f of G. A graph G on n vertices is radio graceful if and only if $rn(G) = n$. In this paper, we define the *radiatic dimension of* G to be the smallest positive integer k, such that the sequence of injective functions $f_i : V(G) \to \{1, 2, 3, \ldots, n\}$, $1 \leq i \leq k$, satisfy the condition that for every two distinct vertices $u, v \in V(G)$, $|f_i(u) - f_i(v)| \geq diam(G) + 1 - d(u, v)$ for some i and denote it by $rd(G)$. Hence a graph is radio graceful if and only if $rd(G) = 1$. In this paper we study the radiatic dimension of some standard graphs and characterize graphs of diameter 2 that are radio graceful.

Mathematics Subject Classification: 05C78, 05C12, 05C15
Keywords: Radio labeling, Radio graceful graphs

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1 Introduction

All the graphs considered here are undirected, finite, connected and simple. We use standard terminology, the terms not defined here may be found in [13, 14, 15]. A radio labeling of a graph G is an injection $f : V(G) \to Z^+$ such that for all $u, v \in V(G), |f(u) - f(v)| \geq diam(G) + 1 - d(u, v)$. The radio number of f, denoted by $rn(f)$ is the maximum number assigned to any vertex of G. The radio number of G, is the minimum value of $rn(f)$ taken over all radio labelings f of G. A radio labeling of C_{10}, with radio number 18 is shown in the Figure 1. A graph G on n vertices is radio graceful if and only if $rn(G) = n$. An example of a radio graceful graph is shown in the Figure 2.

Radio labeling was originally introduced in 2001 by G. Chartrand, David Erwin, Ping Zhang and F. Hararay [6]. In this paper they showed that if G is a connected graph of order n and diameter 2, then $n \leq rn(G) \leq 2n - 2$, and that for every pair k, n of integers with $n \leq k \leq 2n - 2$, there exists a connected graph of order n and diameter 2 with $rn(G) = k$. Also, in the same paper a characterization of connected graphs of order n and diameter 2 with prescribed radio number is presented. The upper and lower bounds for the radio number of cycles was discussed by Ping Zhang [7] in 2002. The bounds are shown to be tight for certain cycles. In 2004, Liu and Xie [2] investigated the radio number of square of cycles. The results obtained by Zhang [7] were shown to be inaccurate and and independent proof for the results of [2] with better bounds was given by B. Sooryanarayana and P. Raghunath [11].

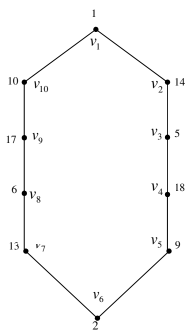

Figure 1: A radio labeling of C_{10}

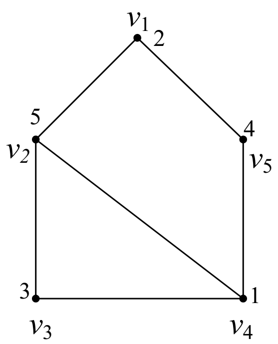

Figure 2: A radio graceful graph

In 2005, D.D.F. Liu and X. Zhu [4] completely determined radio numbers of paths and cycles. In 2007, D.D.F. Liu [1] obtained lower bounds for the radio number of trees, and characterized the trees achieving this bound. The results of D.D.F. Liu [1] generalizes the radio number for paths obtained by D.D.F. Liu and X. Zhu [4]. Further in [3], D.D.F. Liu and M. Xie obtained radio labeling of square path. The radio labeling of cube and fourth power of cycles have been discussed by B. Sooryanarayana and P. Raghunath in [9, 10]. The radio number of cube and fourth power of a path was obtained by B. Sooryanarayana et al in [12] and [5] respectively. The radio labeling of k^{th} power of a path is discussed by B. Sooryanarayana et al in [16].

2 Radiatic dimension of a graph

Definition 2.1. *Let $G(V, E)$ be a graph on n vertices. We define the Radiatic dimension of G to be the smallest positive integer k such that the sequence of injective functions, $f_i : V(G) \to \{1, 2, 3, \ldots, n\}$, $1 \leq i \leq k$, satisfy the condition that for every $u, v \in V(G)$,*

$$|f_i(u) - f_i(v)| \geq diam(G) + 1 - d(u, v)$$

for some i and denote it as $rd(G)$.

Remark 2.2. *By the Definition of radiatic dimension, it follows that a graph G is radio graceful if and only if $rd(G) = 1$. The graph shown in the Figure 2 is a radio graceful graph as there exists a single function $f_1 : V(G) \to \{1, 2, 3, 4, 5\}$ satisfying the condition :*

$$|f_1(u) - f_1(v)| \geq diam(G) + 1 - d(u, v)$$

for every $u, v \in V(G)$. In this paper we determine the radiatic dimension of some standard graphs and characterize graphs of diameter two that are radio graceful.

We recall the Definition of Distance graphs introduced by B.Sooryanarayana in [8]. Let G be a graph on n vertices and D be the set of all distances between any two vertices in G. Then $D = \{1, 2, 3, ..., diam(G)\}$, where $diam(G)$ is the

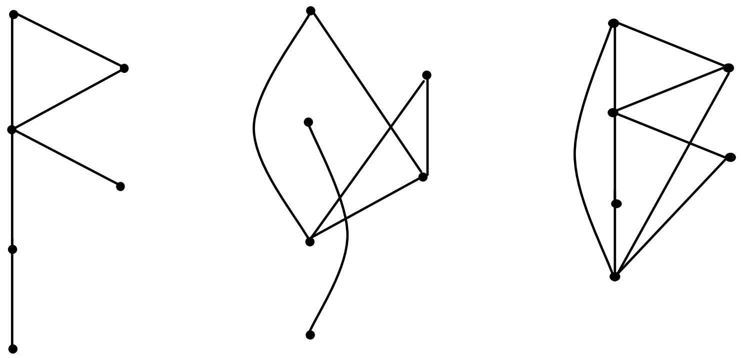

Figure 3: A graph G and its distance graph $D(G,\{2\})$ and $D(G,\{1,3\})$

diameter of G. Let T be a subset of D. The set T is called *distance subset of* G. The *distance graph of* G associated with the distance subset T, denoted by $D(G,T)$, is defined on the vertices of G with the relation that two vertices u and v are adjacent in $D(G,T)$ whenever $d(u,v) \in T$ in G. The Figure 3 shows a graph G and its distance graphs $D(G,\{2\})$ and $D(G,\{1,3\})$. A graph is said to be semi-hamiltonian if it contains a hamiltonian path.

3 Radiatic dimension of some standard graphs

Theorem 3.1. *All complete graphs are radio graceful.*

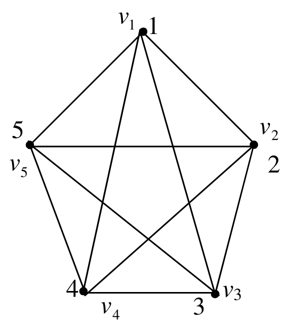

Figure 4: A labeling to show $rd(K_5) = 1$

Proof. Let K_n be the complete graph on the n vertices $\{v_1, v_2, \ldots, v_n\}$. Then, as every pair of distinct vertices in K_n are adjacent, we have $diam(K_n) = 1$.

Define a function $f_1 : V(K_n) \to \{1, 2, \ldots, n\}$ as;

$$f_1(v_i) = i, \qquad \forall i, 1 \leq i \leq n$$

Obviously, f_1 is injective. Also, for $1 \leq i, j \leq n$, $i \neq j$, we get;

$$\begin{aligned} |f_1(v_i) - f_1(v_j)| &= |i - j| \geq 1 \\ &= 1 + 1 - 1 = diam(K_n) + 1 - d(v_i, v_j) \end{aligned}$$

□

Observation 3.2. *Let G be a graph of diameter 2 and*

$$f_1 : V(G) \to \{1, 2, \ldots, n\}$$

be any injective function.
If $d(u, v) = 2$ then

$$|f_1(u) - f_1(v)| \geq diam(G) + 1 - d(u, v) = 2 + 1 - 2 = 1$$

always holds.
If $d(u, v) = 1$ then

$$|f_1(u) - f_1(v)| \geq diam(G) + 1 - d(u, v) = 2 + 1 - 1 = 2$$

must hold for G to be radio graceful. Hence to show that a graph G of diameter 2 is radio graceful it suffices to check if

$$|f_1(u) - f_1(v)| \geq 2$$

for every $uv \in E(G)$.

Theorem 3.3. *For any two positive integers $m \geq 1$, $n \geq 1$,*

$$rd(K_{m,n}) = \begin{cases} 1, & if\ m = 1\ and\ n = 1 \\ 2, & otherwise \end{cases}$$

Proof. The case $m = n = 1$ follows by the Theorem 3.1. We first take the case $n > m \geq 1$. Let $V(K_{m,n}) = V_1 \bigcup V_2$ with $V_1 = \{u_1, u_2, \ldots u_m\}$ and $V_2 = \{v_1, v_2, \ldots v_n\}$.

Necessity: If possible, suppose $rd(K_{m,n}) = 1$, then by Observation 3.2 there exists an injective function

$$f_1 : V(K_{m,n}) \to \{1, 2, \ldots m + n\}$$

such that

$$|f_1(u_i) - f_1(v_j)| \geq diam(K_{m,n}) + 1 - d(u_i, v_j) = 2 + 1 - 1 = 2 \qquad (1)$$

for all $i, j, \quad 1 \leq i \leq m$ and $1 \leq j \leq n$.

Without loss of generality we may assume $f_1(u_1) = 1$ (similar argument holds for the case $f_1(v_1) = 1$ also). Then $f_1(v_j) \neq 2$ for any j, $1 \leq j \leq n$. But as f_1 is injective $f_1(u_i) = 2$ must hold for some i, $2 \leq i \leq m$.

Again without loss of generality we may assume $f_1(u_2) = 2$. Then $f_1(v_j) \neq 3$ for any j, $1 \leq j \leq n$. Proceeding this way we see that $f_1(u_i) = i$ for all i, $1 \leq i \leq m$.

Now since f_1 is injective $m + 1$ must be assigned to some vertex, say v_j in V_2. Then $|f_1(u_m) - f_1(v_j)| = 1$, a contradiction to the Equation (1). Hence $rd(K_{m,n}) \geq 2$.

Sufficiency: We now prove that $rd(K_{m,n}) = 2$.

Define:

$$f_1 : V(K_{m,n}) \to \{1, 2, \ldots, m + n\}$$

as

$$\begin{aligned} f_1(u_i) &= i, & 1 \leq i \leq m \\ f_1(v_j) &= m + j, & 1 \leq j \leq n \end{aligned}$$

and

$$f_2 : V(K_{m,n}) \to \{1, 2, \ldots, m + n\}$$

as

$$f_2(u_i) = i, \quad 1 \leq i \leq m$$

and

$$f_2(v_j) = \begin{cases} m + 2, & \text{if } j = 1 \\ m + 1, & \text{if } j = 2 \\ m + j, & 3 \leq j \leq n \end{cases}$$

By the Definition it is obvious that both f_1 and f_2 are injective functions. Further, for all $1 \leq i \leq m-1, 1 \leq j \leq n$, and for $i = m$ and $2 \leq j \leq n$ we get;

$$|f_1(v_j) - f_1(u_i)| = |m + j - i| \geq 2$$

Also,

$$|f_2(u_m) - f_2(v_1)| = |m - (m + 2)| = 2.$$

Hence in all cases we get; $|f_k(u_i) - f_k(v_j)| \geq 2$ for all $i, j, 1 \leq i \leq m, 1 \leq j \leq n$ either for $k = 1$ or $k = 2$. The proof for the case of $m \geq n$ follows similarly. Hence $rd(K_{m,n}) = 2$ for all positive integers $m \geq 1, n \geq 2$. □

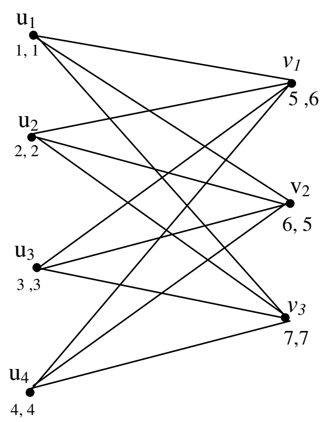

Figure 5: A labeling to show $rd(K_{4,3}) = 2$

As a result of the above Theorem 3.3 we have the following Corollary :

Corollary 3.4. *The radiatic dimension of the star graph $K_{1,n}$ where $n \geq 2$ is 2.*

Remark 3.5. *Theorem 3.1, Theorem 3.3 and Corollary 3.4 show that the complete bipartite graph $K_{m,n}$ is radio graceful if and only if $m = 1$ and $n = 1$.*

Theorem 3.6. *For any integer $n \geq 3$,*

$$rd(W_{1,n}) = \begin{cases} 1, & if\ n = 3 \\ 2, & otherwise \end{cases}$$

Proof. The case of $n = 3$ has been proved in Theorem 3.1. We now consider the case of $n \geq 4$. Let v_1 be the central vertex and v_i, $2 \leq i \leq n+1$ be the rim vertices of the wheel.
Necessity: If possible, suppose $rd(W_{1,n}) = 1$, then, by Observation 3.2, there exists an injective function

$$f_1 : V(W_{1,n}) \to \{1, 2, \ldots n+1\}$$

such that $|f_1(v_i) - f_1(v_j)| \geq diam(W_{1,n}) + 1 - 1 = 2 + 1 - 1 = 2$ for all $v_iv_j \in E(W_{1,n})$.
As v_1 is the central vertex it is adjacent to v_i, for all i, $2 \leq i \leq n+1$.
Suppose $f_1(v_1) = 1$, then $f_1(v_i) \neq 2$ for any i, $2 \leq i \leq n+1$.
Suppose $f_1(v_1) = n+1$, then $f_1(v_i) \neq n$ for any i, $2 \leq i \leq n+1$.
Suppose $f_1(v_1) = l$ where $2 \leq l \leq n$, then $f_1(v_i) \neq l \pm 1$ for any i, $2 \leq i \leq n+1$.
This means at least two vertices of the graph must be assigned the same label a contradiction to the fact that f_1 is injective. Hence $rd(W_{1,n}) \geq 2$.

Sufficiency: We next prove that $rd(W_{1,n}) = 2$ in two cases.
Case(1): n is even, $n \geq 4$.
For $n = 4$ define
$f_1(v_1) = 1, f_1(v_2) = 2, f_1(v_3) = 4, f_1(v_4) = 3, f_1(v_5) = 5$ and
$f_2(v_1) = 1, f_2(v_2) = 3, f_2(v_3) = 2, f_2(v_4) = 4, f_2(v_5) = 5$. From the Figure 6

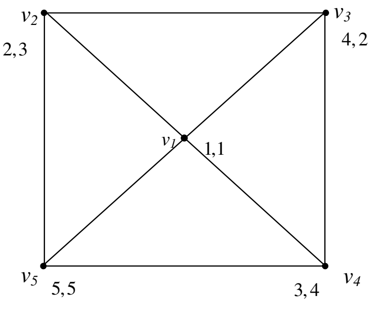

Figure 6: A labeling to show $rd(W_{1,4}) = 2$

it can be easily seen that $|f_k(v_i) - f_k(v_j)| \geq 2$ whenever $k = 1$ or $k = 2$ and $v_iv_j \in E(W_{1,4})$.Hence by Observation 3.2 $rd(W_{1,4}) = 2$.

For $n \geq 6$, define

$$f_1 : V(W_{1,n}) \to \{1, 2, \ldots n+1\}$$

as

$$f_1(v_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2i-2, & \text{if } 2 \leq i \leq \frac{n+2}{2} \\ 2i-n-1, & \text{if } \frac{n+4}{2} \leq i \leq n+1 \end{cases}$$

and

$$f_2 : V(W_{1,n}) \to \{1, 2, \ldots n+1\}$$

as

$$f_2(v_i) = \begin{cases} 1, & \text{if } i = 1 \\ 3, & \text{if } i = 2 \\ 2i-2, & \text{if } 3 \leq i \leq \frac{n+2}{2} \\ 2, & \text{if } i = \frac{n+4}{2} \\ 2i-n-1, & \text{if } \frac{n+6}{2} \leq i \leq n+1 \end{cases}$$

It can be easily verified that f_1 and f_2 are injective functions. We now make the following simple observations :

1. For $2 \leq i \leq \frac{n}{2}$,

$$|f_1(v_i) - f_1(v_{i+1})| = |(2i-2) - (2i+2-2)| = 2.$$

2. If $i = \frac{n+2}{2}$ then $i+1 = \frac{n+4}{2}$. So,

$$\begin{aligned} |f_1(v_i) - f_1(v_{i+1})| &= |[(2(\tfrac{n+2}{2}) - 2] - [2(\tfrac{n+4}{2}) - n - 1]| \\ &= |n-3| \geq 3 \ , \ as\ n \ \geq \ 6 \end{aligned}$$

3. For $\frac{n+4}{2} \leq i \leq n$,

$$|f_1(v_i) - f_1(v_{i+1})| = |(2i-n-1) - (2i+2-n-1)| = 2.$$

4. $|f_1(v_{n+1}) - f_1(v_2)| = |(2n+2-n-1) - 2| = n-1 \geq 5.$

5. For all i, $3 \le i \le \frac{n+2}{2}$,

   $|f_1(v_1) - f_1(v_i)| = |3 - 2i| \;\; \ge \;\; |n-1| \;\ge\; 5$

6. For all i, $\frac{n+4}{2} \le i \le n+1$,

   $|f_1(v_1) - f_1(v_i)| = |2 - 2i + n| \;\; \ge \;\; n \ge \;\; 6$

7. Also $|f_2(v_1) - f_2(v_2)| = |1 - 3| = 2$.

Therefore $|f_k(v_i) - f_k(v_j)| \ge 2$ for $k = 1$ or $k = 2$ whenever $v_iv_j \in E(W_{1,n})$. Hence by Observation 3.2, $rd(W_{1,n}) = 2$ when n is even and $n \ge 6$.

Case (2): When n is odd and $n \ge 3$.
The case of $n = 3$ has been dealt in Theorem 3.1 .
We now consider the case of $n \ge 5$.
Define $f_1 : V(W_{1,n}) \to \{1, 2, \ldots n+1\}$

$$f_1(v_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2i - 2, & \text{if } 2 \le i \le \frac{n+3}{2} \\ 2i - n - 2, & \text{if } \frac{n+5}{2} \le i \le n+1 \end{cases}$$

and

$$f_2 : V(W_{1,n}) \to \{1, 2, \ldots n+1\}$$

as

$$f_2(v_i) = \begin{cases} 1, & \text{if } i = 1 \\ 3, & \text{if } i = 2 \\ 2i - 2, & \text{if } 3 \le i \le \frac{n+3}{2} \\ 2, & \text{if } i = \frac{n+5}{2} \\ 2i - n - 2, & \text{if } \frac{n+7}{2} \le i \le n+1 \end{cases}$$

It can be easily verifed that f_1 and f_2 are injective functions.
We now make the following simple observations :

1. For $2 \le i \le \frac{n+1}{2}$,

$$|f_1(v_i) - f_1(v_{i+1})| = |(2i - 2) - (2i + 2 - 2)| = 2$$

2. If $i = \frac{n+3}{2}$ then $i+1 = \frac{n+5}{2}$
   As $n \geq 5$,
   $$|f_1(v_i) - f_1(v_{i+1})| = |(2(\frac{n+3}{2}) - 2) - (2(\frac{n+5}{2}) - n - 2)| = n - 2 \geq 3$$

3. For $\frac{n+5}{2} \leq i \leq n$,
   $$|f_1(v_i) - f_1(v_{i+1})| = |(2i - n - 2) - (2i + 2 - n - 2)| = 2$$

4. As $n \geq 5$,
   $$|f_1(v_{n+1}) - f_1(v_2)| = |(2n + 2 - n - 2) - 2| = n - 2 \geq 3$$

5. For all i, $3 \leq i \leq \frac{n+3}{2}$,
   $$|f_1(v_1) - f_1(v_i)| = |3 - 2i| \quad \geq \quad n \quad \geq \quad 5$$

6. Also $|f_2(v_1) - f_2(v_2)| = |1 - 3| = 2$

So $|f_k(v_i) - f_k(v_j)| \geq 2$ for $k = 1$ or $k = 2$ whenever $v_iv_j \in E(W_{1,n})$.Therefore by Observation 3.2, $rd(W_{1,n}) = 2$ for all odd integers $n \geq 5$. Hence from both cases we see that for all positive integers $n \geq 4$, $rd(W_{1,n}) = 2$. □

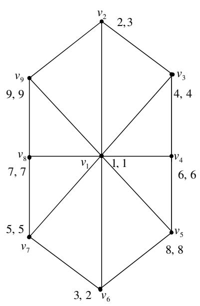

Figure 7: A labeling to show $rd(W_{1,8}) = 2$

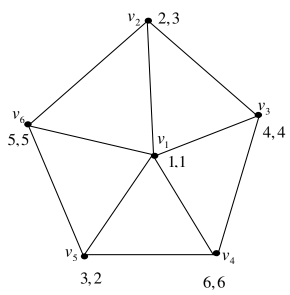

Figure 8: A labeling to show $rd(W_{1,5}) = 2$

Remark 3.7. *Theorem 3.1 and Theorem 3.6 show that the Wheel graph $W_{1,n}$ is radio graceful if and only if $n = 3$.*

Theorem 3.8. *For any positive integer* $n \geq 1$,

$$rd(P_n) = \begin{cases} n, & for\ n\ =\ 1,2 \\ n-1, & for\ n\ \geq\ 3 \end{cases}$$

Proof. The cases of $n = 1$ and $n = 2$ have already been proved in Theorem 3.1 and the case of $n = 3$ has been dealt in Theorem 3.3 and its Corollary 3.4. We now consider the case of $n \geq 4$.
Let P_n be a path on n vertices, $n \geq 4$, with $V(P_n) = \{v_1, v_2, \dots v_n\}$ and v_i is adjacent to v_j if and only if $|i - j| = 1$.
Necessity: We now prove that $rd(P_n) \geq n - 1$.
Suppose $rd(P_n) = n - 2$. Then there exists a sequence of injective functions

$$f_k : V(P_n) \to \{1, 2, \dots n\}, 1 \leq k \leq n - 2$$

such that

$$|f_k(v_i) - f_k(v_j)| \geq diam(P_n) + 1 - d(v_i, v_j) \tag{2}$$

holds for all $i, j, 1 \leq i, j \leq n$, $i \neq j$ for some k, $1 \leq k \leq n - 2$.
As $diam(P_n) = n - 1$, for all $v_iv_j \in E(P_n)$ Equation (2) implies

$$|f_k(v_i) - f_k(v_j)| \geq n - 1 + 1 - 1 = n - 1 \tag{3}$$

for some k, $1 \leq k \leq n-2$. But as the set of labels is $\{1, 2, \dots n\}$ the maximum label difference that can be achieved is $n - 1$. Therefore equation (3) implies $|f_k(v_i) - f_k(v_j)| = n - 1$ for some k, $1 \leq k \leq n - 2$, for all $v_iv_j \in E(P_n)$.
This possible if either

$$f_k(v_i) = n \text{ and } f_k(v_j) = 1$$
$$\text{or}$$
$$f_k(v_i) = 1 \text{ and } f_k(v_j) = n$$

holds for some k, $1 \leq k \leq n - 2$, for all $v_iv_j \in E(P_n)$.
Without loss of generality we define $f_k(v_k) = 1$ and $f_k(v_{k+1}) = n$ for all k, $1 \leq k \leq n - 2$. Now consider the edge $v_{n-1}v_n$. Suppose there exists some k, $1 \leq k \leq n - 2$, such that $f_k(v_{n-1}) = 1$ and $f_k(v_n) = n$, then as each f_k is injective

$$f_k(v_k) = f_k(v_{n-1}) = 1 \Rightarrow v_k = v_{n-1} \Rightarrow k = n - 1$$

Similarly,

$$f_k(v_{k+1}) = f_k(v_n) = n \Rightarrow v_{k+1} = v_n \Rightarrow k + 1 = n \Rightarrow k = n - 1$$

a contradiction.
For the same reason we cannot have $f_k(v_{n-1}) = n$ and $f_k(v_n) = 1$ for any k, $1 \leq k \leq n-2$. Hence $rd(P_n) \geq n-1$ when $n \geq 4$.

Sufficiency: Next we show that $rd(P_n) = n-1$.
Define a sequence of functions $f_k : V(G) \to \{1, 2, \ldots n\}$ as follows :

1. For all k, $1 \leq k \leq n-1$, $f_k(v_k) = 1$

2. For all k, $1 \leq k \leq n-1$ and for all i, $1 \leq i \leq n-k$
   define $f_i(v_{i+k}) = n-k+1$

3. For all k, $1 \leq k \leq n-2$ and for all i, $n-k \leq i \leq n-1$
   define $f_i(v_{i-n+k+1}) = n-k$.

It can be easily verified that for all k, $1 \leq k \leq n-1$, f_k is an injective function.
Now for all k, $1 \leq k \leq n-1$, $d(v_i, v_{i+k}) = k$ in P_n.
By the Definition of f_k given above it is easy to observe that
for all k, $1 \leq k \leq n-1$ and for all i, $1 \leq i \leq n-k$,

$$\begin{aligned} |f_i(v_i) - f_i(v_{i+k})| &= |1 - (n-k+1)| \\ &= n-k \\ &= (n-1) + 1 - d(v_i, v_k) \\ &= diam(P_n) + 1 - d(v_i, v_k) \end{aligned}$$

Hence for every pair of vertices $v_i, v_j \in V(P_n)$ the condition
$|f_k(v_i) - f_k(v_j)| \geq diam(P_n) + 1 - d(v_i, v_j)$ holds for some k, $1 \leq k \leq n-1$.
Hence $rd(P_n) = n-1$. $\square$

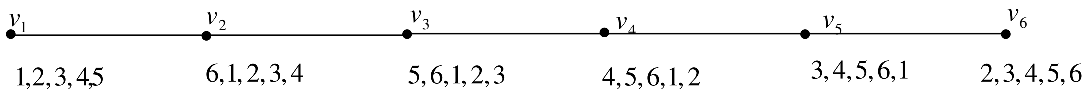

Figure 9: A labeling to show $rd(P_6) = 5$.

Remark 3.9. *Theorem 3.1 and Theorem 3.8 show that the path P_n is radio graceful if and only if $n \leq 2$.*

Theorem 3.10. *For any positive integer $n \geq 1$,*

$$rd(P_n + K_1) = \begin{cases} 1, & for\ n\ =\ 1,\ 2 \\ 2, & for\ n\ \geq\ 3 \end{cases}$$

Proof. The case of $n = 1, 2$ has been proved in Theorem 3.1. We now consider the cases of $n \geq 3$. Let $v_1 \in V(K_1)$ and $v_i \in V(P_n)$, $n \geq 3$ where $2 \leq i \leq n+1$.

Necessity : We now prove that $rd(P_n + K_1) \geq 2$. Suppose $rd(P_n + K_1) = 1$. Then by Observation 3.2 there exists an injective function
$f_1 : V(P_n + K_1) \to \{1, 2, \ldots n+1\}$ such that

$$|f_1(v_i) - f_1(v_j)| \geq diam(P_n + K_1) + 1 - d(v_i, v_j) = 2 + 1 - 1 = 2$$

for all $v_iv_j \in E(P_n + K_1)$. The vertex v_1 is adjacent to all v_i for all i,
$2 \leq i \leq n+1$.
Suppose $f_1(v_1) = 1$ then $f_1(v_i) \neq 2$ for any $i, 2 \leq i \leq n$.
Suppose $f_1(v_1) = n+1$ then $f_1(v_i) \neq n$ for any i, $2 \leq i \leq n$.
Suppose $f_1(v_1) = l$, where $2 \leq l \leq n$ then $f_1(v_i) \neq l \pm 1$, for any i, $2 \leq i \leq n$. This means at least two vertices have to be assigned the same label, a contradiction to the fact that f_1 is injective. Hence $rd(P_n + K_1) \geq 2$ for $n \geq 3$.
Sufficiency : For $n = 3$ define

$$\begin{aligned} f_1(v_1) &= 1, & f_1(v_2) &= 3, \\ f_1(v_3) &= 2, & f_1(v_4) &= 4, \\ f_2(v_1) &= 4, & f_2(v_2) &= 3, \\ f_2(v_3) &= 1, & f_2(v_4) &= 2. \end{aligned}$$

as shown in the Figure 10. Therefore,$|f_k(v_i) - f_k(v_j)| \geq 2$ holds, for all $v_iv_j \in E(P_3 + K_1)$ whenever $k = 1$ or $k = 2$. Hence $rd(P_3 + k_1) = 2$.
For $n \geq 4$, define $f_1 : V(P_n + K_1) \to \{1, 2, \ldots n+1\}$ as

$$f_1(v_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2i - 1, & \text{if } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ 2i - 2\lfloor \frac{n}{2} \rfloor - 2, & \text{if } \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n+1 \end{cases}$$

and

$$f_2 : V(P_n + K_1) \to \{1, 2, \ldots n+1\}$$

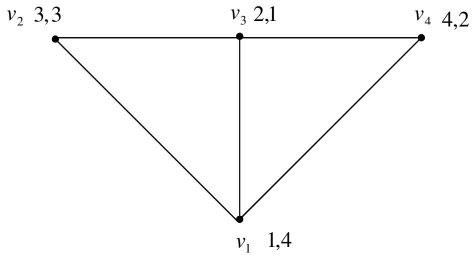

Figure 10: A labeling to show $rd(P_3 + K_1) = 2$

as

$$f_2(v_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } i = 2 \\ 2i-1, & \text{if } 3 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ 3, & \text{if } i = \lfloor \frac{n}{2} \rfloor + 2 \\ 2i - 2\lfloor \frac{n}{2} \rfloor - 2, & \text{if } \lfloor \frac{n}{2} \rfloor + 3 \leq i \leq n+1 \end{cases}$$

It can be easily verified that f_1 and f_2 are injective functions.
We now make the following simple observations :

1. For $2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$, we have $|f_1(v_i) - f_1(v_1)| = |2i - 2| \geq 2$

2. For $\lfloor \frac{n}{2} \rfloor + 3 \leq i \leq n+1$,

$$|f_1(v_i) - f_1(v_1)| = |2i - 2\lfloor \frac{n}{2} \rfloor - 3| \tag{4}$$

$$i \geq \lfloor \frac{n}{2} \rfloor + 3 \Rightarrow |2i - 2\lfloor \frac{n}{2} \rfloor - 3| \geq |2\lfloor \frac{n}{2} \rfloor + 6 - 2\lfloor \frac{n}{2} \rfloor - 3| = 3 \tag{5}$$

   Equations (4) and (5) imply $|f_1(v_i) - f_1(v_1)| \geq 3$.

3. For $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$|f_1(v_i) - f_1(v_{i+1})| = |(2i-1) - (2(i+1) - 1)| = 2.$$

4. For $\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n$

$$|f_1(v_i) - f_1(v_{i+1})| = |(2i - 2\lfloor \frac{n}{2} \rfloor - 2) - (2(i+1) - 2\lfloor \frac{n}{2} \rfloor - 2)| = 2.$$

5. For $i = \lfloor \frac{n}{2} \rfloor + 1$, we have $|f_1(v_i) - f_1(v_{i+1})| = |2\lfloor \frac{n}{2} \rfloor - 1|$

   Now, $n \geq 4 \implies \lfloor \frac{n}{2} \rfloor \geq 2 \implies |2\lfloor \frac{n}{2} \rfloor - 1| \geq 3$.
   Hence $|f_1(v_i) - f_1(v_{i+1})| \geq 3$.

6. $|f_2(v_1) - f_2(v_{\lfloor \frac{n}{2} \rfloor + 2})| = |1 - 3| = 2$.

Therefore

$$\begin{aligned} |f_k(v_i) - f_k(v_j)| &\geq 2 \\ &= 2 + 1 - 1 \\ &= diam(P_n + K_1) + 1 - d(v_i, v_j) \end{aligned}$$

for $k = 1$ or $k = 2$, whenever $v_iv_j \in E(P_n + K_1)$.
Hence by Corollary 3.2 $rd(P_n + K_1) = 2$ for all $n \geq 3$. □

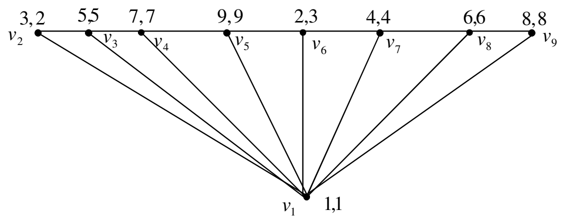

Figure 11: A labeling to show $rd(P_8 + K_1) = 2$

Remark 3.11. *Theorem 3.1 and Theorem 3.10 show that the Fan $P_n + K_1$ is radio graceful if and only if $n \leq 2$.*

Theorem 3.12. *For any positive integer $n \geq 3$, the radiatic dimension of a cycle on n vertices $rd(C_n) = 1$ if and only if $n = 3$ or $n = 5$.*

Proof. The case of $n = 3$ has already been proved in Theorem 3.1. Labeling the vertices of C_5 as shown in the Figure 12 it is obvious that $rd(C_5) = 1$. We now prove that $rd(C_n) > 1$, when $n \geq 4, n \neq 5$. Let C_n be a cycle on n vertices, $n \geq 4$, $n \neq 5$ with $V(C_n) = \{v_1, v_2, \dots v_n\}$ and $E(C_n) = \{v_1v_n, v_iv_{i+1}\}$ where $1 \leq i \leq n - 1$.

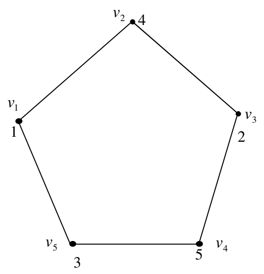

Figure 12: A labeling to show $rd(C_5) = 1$

Suppose $rd(C_n) = 1$ for $n \geq 4, n \neq 5$. Then, as $diam(C_n) = \lfloor n/2 \rfloor$, there exists an injective function

$$f_1 : V(G) \to \{1, 2, ...n\}$$

such that

$$|f_1(v_i) - f_1(v_j)| \geq \lfloor \frac{n}{2} \rfloor + 1 - d(v_i, v_j) \tag{6}$$

holds for all $v_i, v_j \in V(C_n)$.

Case(1): n is even and $n \geq 4$.
As n is even the condition (6) becomes

$$|f_1(v_i) - f_1(v_j)| \geq \frac{n}{2} + 1 - d(v_i, v_j). \tag{7}$$

As f_1 is injective and $\frac{n}{2} \in \{1, 2 \dots n\}$, without loss of generality we may take $f_1(v_1) = \frac{n}{2}$.
As $d(v_1, v_2) = 1$, the inequality (7) implies

$$\begin{array}{rccccccc} & |f_1(v_2) - f_1(v_1)| & = & |f_1(v_2) - \frac{n}{2}| & \geq & \frac{n}{2} & & \\ \Rightarrow & (f_1(v_2) - \frac{n}{2}) & \leq & -\frac{n}{2} & \text{or} & (f_1(v_2) - \frac{n}{2}) & \geq & \frac{n}{2} \\ \Rightarrow & f_1(v_2) & \leq & 0 & \text{or} & f_1(v_2) & \geq & n \end{array}$$

By Definition of f_1 we get;

$$f_1(v_2) = n \tag{8}$$

As $d(v_1, v_n) = 1$, the inequality (7) implies

$$\begin{array}{rccccccc} & |f_1(v_n) - f_1(v_1)| & = & |f_1(v_n) - \frac{n}{2}| & \geq & \frac{n}{2} & & \\ \Rightarrow & (f_1(v_n) - \frac{n}{2}) & \leq & -\frac{n}{2} & \text{or} & (f_1(v_n) - \frac{n}{2}) & \geq & \frac{n}{2} \\ \Rightarrow & f_1(v_n) & \leq & 0 & \text{or} & f_1(v_n) & \geq & n \end{array}$$

By Definition of f_1 we get;

$$f_1(v_n) = n \tag{9}$$

Equations (8)and (9) $\Rightarrow f_1(v_2) = f_1(v_n) = n$, a contradiction to the fact that f_1 is injective.
Hence $rd(C_n) \geq 2$ when n is even and $n \geq 4$.

Case(2): n is odd and $n \geq 7$.
As f_1 is injective and $\lceil \frac{n}{2} \rceil \in \{1, 2, \ldots n\}$, without loss of generality, we may assume $f_1(v_1) = \lceil \frac{n}{2} \rceil$.
As $d(v_1, v_2) = 1$, the inequality (6) implies

$$\begin{array}{rcccccc} & |f_1(v_2) - f_1(v_1)| & = & |f_1(v_2) - \lceil \frac{n}{2} \rceil| & \geq & \lfloor \frac{n}{2} \rfloor & \\ \Rightarrow & (f_1(v_2) - \lceil \frac{n}{2} \rceil) & \leq & -\lfloor \frac{n}{2} \rfloor \ \text{or} & (f_1(v_2) - \lceil \frac{n}{2} \rceil) & \geq & \lfloor \frac{n}{2} \rfloor \\ \Rightarrow & f_1(v_2) & \leq & 1 \quad \text{or} & f_1(v_2) & \geq & n \end{array}$$

By Definition of f_1 we get;

$$f_1(v_2) = 1 \quad \text{or} \quad f_1(v_2) = n \tag{10}$$

As $d(v_1, v_n) = 1$, the inequality (6) implies

$$\begin{array}{rcccccc} & |f_1(v_n) - f_1(v_1)| & = & |f_1(v_n) - \lceil \frac{n}{2} \rceil| & \geq & \lfloor \frac{n}{2} \rfloor & \\ \Rightarrow & (f_1(v_n) - \lceil \frac{n}{2} \rceil) & \leq & -\lfloor \frac{n}{2} \rfloor \ \text{or} & (f_1(v_n) - \lceil \frac{n}{2} \rceil) & \geq & \lfloor \frac{n}{2} \rfloor \\ \Rightarrow & f_1(v_n) & \leq & 1 \quad \text{or} & f_1(v_n) & \geq & n \end{array}$$

By Definition of f_1 we get;

$$f_1(v_n) = 1 \quad \text{or} \quad f_1(v_n) = n \tag{11}$$

By Equations (10) and (11) without loss of generality we may take;

$$f_1(v_2) = 1 \text{ and } f_1(v_n) = n.$$

Now $f_1(v_3)$ must satisfy the following:
As f_1 is injective and $f_1(v_1) = \lceil \frac{n}{2} \rceil$,

$$f_1(v_3) \neq \lceil \frac{n}{2} \rceil \tag{12}$$

and as $f_1(v_2) = 1$, $f_1(v_n) = n$

$$2 \quad \leq \quad f_1(v_3) \quad \leq (n-1) \tag{13}$$

As $d(v_1, v_3) = 2$, by inequality (6)

$$\begin{array}{rcl} |f_1(v_3) - f_1(v_1)| & \geq & \lfloor \frac{n}{2} \rfloor + 1 - d(v_1, v_3) \\ \Rightarrow \quad |f_1(v_3) - \lceil \frac{n}{2} \rceil| & \geq & \lfloor \frac{n}{2} \rfloor - 1 \\ \Rightarrow (f_1(v_3) - \lceil \frac{n}{2} \rceil) \leq & -(\lfloor \frac{n}{2} \rfloor - 1) \quad \text{or} \quad (f_1(v_3) - \lceil \frac{n}{2} \rceil) & \geq \lfloor \frac{n}{2} \rfloor - 1 \end{array}$$

$$\Rightarrow \quad f_1(v_3) \leq 2 \quad \text{or} \quad f_1(v_3) \geq n - 1 \tag{14}$$

contradicting the Equation (13). Hence the result. □

Remark 3.13. *Theorem 3.1 and Theorem 3.12 show that the cycle C_n, $n \geq 3$ is radio graceful if and only if $n = 3$ or $n = 5$.*

4 Radiatic dimension of graphs of diameter 2

Theorem 4.1. *If a graph G on $n \geq 3$ vertices with diameter 2 is radio graceful then the maximum degree of a vertex in G cannot exceed $n - 2$ and there exists at most 2 vertices in G whose degree is $n - 2$.*

Proof. Let G be a radio graceful graph on $n \geq 3$ vertices with $diam(G) = 2$. Then by Observation 3.2 there exists an injective mapping

$$f : V(G) \to \{1, 2, \ldots n\}$$

such that $|f(u) - f(v)| \geq 2$ for all $uv \in E(G)$.

If $f(u) = 1$ then $f(v) \neq 1, 2$

If $f(u) = n$ then $f(v) \neq n, n - 1$.

If $f(u) = l$, where $2 \leq l \leq n - 1$ then $f(v) \neq l, l - 1, l + 1$.

Suppose $deg(u) \geq (n - 1)$.
If $f(u) = 1$, we may possibly assign the $n - 2$ labels $\{3, 4, \ldots n\}$ to $n - 2$ neighbors of u. But there exists at least one more neighbor (say) v of u to which no label can be assigned.
If $f(u) = n$, we may possibly assign the $n - 2$ labels $\{1, 2, \ldots n - 2\}$ to $n - 2$

neighbors of u. But there exists at least one more neighbor (say) v of u to which no label can be assigned.
Similarly, if $f(u) = l$, where $2 \leq l \leq n-1$ then we may possibly assign the $n-3$ labels $\{1, 2, \ldots l-2, l+2, \ldots n\}$ to $n-3$ neighbors of u. But there exists at least one more neighbor (say) v of u to which no label can be assigned. Hence $deg(u) \leq (n-2)$.
Further, suppose $deg(u) = n-2$ and $f(u) = l$ where $2 \leq l \leq n-1$, then we may possibly assign the $n-3$ labels $\{1, 2, \ldots l-2, l+2, \ldots n\}$ to $n-3$ neighbors of u. But there exists at least one more neighbor (say) v of u to which no label can be assigned. Hence if $u \in V(G)$ is such that $deg(u) = n-2$, then $f(u) = 1$ or $f(u) = n$.
Suppose $u_1, u_2, u_3 \in V(G)$ are such that

$$deg(u_1) = deg(u_2) = deg(u_3) = (n-2).$$

Then $f(u_1), f(u_2), f(u_3) \in \{1, n\}$ a contradiction to the fact that f is injective. Hence at most 2 vertices of G can have degree $n-2$. □

Corollary 4.2. *If a graph G on $n \geq 3$ vertices with $diam(G) = 2$ is radio graceful then G has at most $\frac{(n-2)(n-1)}{2}$ edges.*

Proof. By the previous Theorem 4.1, the maximum degree of a vertex in G is $n-2$ and at most two vertices (say) $v_1, v_2 \in V(G)$ can attain the maximum degree of $n-2$. The remaining $n-2$ vertices have degree $n-3$ or less. Hence,

$$\begin{aligned} \sum_{i=1}^{n} d(v_i) &\leq 2(n-2) + (n-2)(n-3) \\ &\leq (n-2)(n-1) \end{aligned}$$

As $\sum_{i=1}^{n} d(v_i) = 2|E(G)|$, we get $|E(G)| \leq \frac{(n-2)(n-1)}{2}$. □

Remark 4.3. *The converse of the above Corollary 4.2 is not true. The graph $K_{2,3}$ serves as a counter example.*

Theorem 4.4. *A graph G on $n \geq 3$ vertices and diameter 2 is radio graceful if and only if $D(G, \{2\})$ is semi-hamiltonian.*

Proof. Suppose $D(G, \{2\})$ is semi-hamiltonian. Then there exists a hamiltonian path say $\{v_1, v_2, \ldots v_n\}$ in $D(G, \{2\})$.
Define $f(v_i) = i$ for all i, $1 \leq i \leq n$. Then

$$|f(v_i) - f(v_j)| \geq 2, \text{ for all } 1 \leq i, j \leq n, \ |i - j| \geq 2.$$

Let $v_i, v_j \in D(G, \{2\})$ where $1 \leq i, j \leq n$, $|i - j| \geq 2$ and $d(v_i, v_j) \neq 1$, in $D(G, \{2\})$. Then by Definition of $D(G, \{2\})$, $d(v_i, v_j) \neq 2$, in G.
Since $diam(G) = 2$, it is obvious that $d(v_i, v_j) = 1$ in G and by the Definition of f, $|f(v_i) - f(v_j)| \geq 2$, as required. Therefore G is radio graceful. Conversely, suppose G is radio graceful. Then there exists a function

$$f : V(G) \rightarrow \{1, 2, \ldots n\}$$

such that $|f(u) - f(v)| \geq 2$ whenever $uv \in E(G)$.
Relabel the vertices of G such that $f(v_i) = i$, for all i, $1 \leq i \leq n$.
Now for $1 \leq i \leq n - 1$ we have, $|f(v_i) - f(v_{i+1})| = 1$. Hence $v_i v_{i+1}$ is not an edge in G or $d(v_i, v_{i+1}) \neq 1$ in G.
Since $diam(G) = 2$, $d(v_i, v_{i+1}) = 2$ in G for all i, $1 \leq i \leq n - 1$, we have $d(v_i, v_{i+1}) = 1$ in $D(G, \{2\})$ and hence there exists a path with edges $v_1v_2, v_2v_3, \ldots v_{n-1}v_n$ in $D(G, \{2\})$ or $D(G, \{2\})$ is semi hamiltonian. □

5 Conclusion

The radio gracefulness of several standard graphs have been investigated in this paper and a characterization of graphs with at least three vertices and diameter two has been found. We are now working towards finding whether there exists a radio graceful graph with a given number of vertices and a given diameter.

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